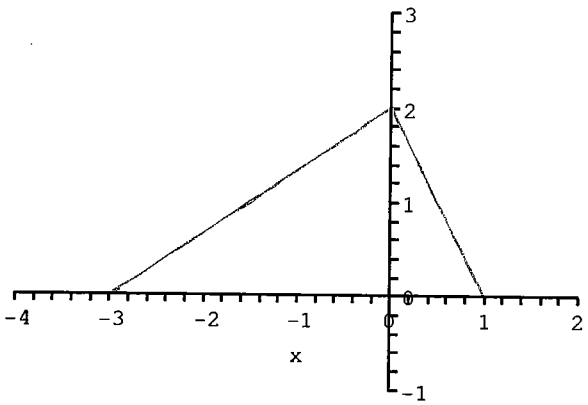
MAS115 Calculus I 2006-2007

Problem sheet for exercise class 2

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.



(*) Problem 1: The graph of f is shown. Draw the graph of each function.
(a) $y = f(-x)$, (b) $y = -f(x)$, (c) $y = -2f(x+1)+1$, (d) $y = 3f(x-2)-2$.

(*) Problem 2: Prove the following identities.

$$(a) \quad \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$
$$(b) \quad \frac{1-\cos x}{1+\cos x} = \tan^2 \frac{x}{2}$$

Problem 3: Find a formula for $f \circ g$ and $g \circ f$ and find the domain and range of each.

$$(a) \quad f(x) = 2 - x^2\,, \qquad g(x) = \sqrt{x+2}$$
$$(b) \quad f(x) = \sqrt{x}\,, \qquad g(x) = \sqrt{1-x}$$

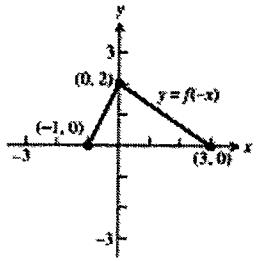
Problem 4: Evaluate $\sin \frac{7\pi}{12}$ and $\cos \frac{\pi}{12}$.

Extra: Graph the equations (a) $|x| + |y| = 1 + x$ and (b) $y + |y| = x + |x|$.

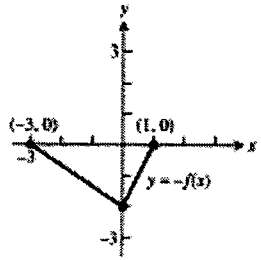
Thomas Prellberg, October 2006

Problem 1

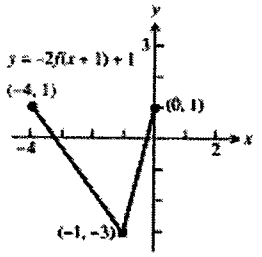
(a) The given graph is reflected about the y-axis.



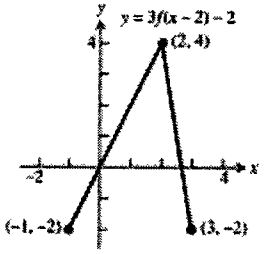
(b) The given graph is reflected about the x-axis.



(c) The given graph is shifted left 1 unit, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted upward 1 unit.



(d) The given graph is shifted right 2 units, stretched vertically by a factor of 3, and then shifted downward 2 units.



Problem 2

(a) $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x) \Rightarrow (1 - \cos x) = \frac{\sin^2 x}{1+\cos x}$

$\Rightarrow \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

(b) Using the definition of the tangent function and the double angle formulas, we have

$$\tan^2\left(\frac{x}{2}\right) = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{\frac{1-\cos\left(2\left(\frac{x}{2}\right)\right)}{2}}{\frac{1+\cos\left(2\left(\frac{x}{2}\right)\right)}{2}} = \frac{1-\cos x}{1+\cos x}.$$

Problem 3

(a) $(fog)(x) = f(g(x)) = f\left(\sqrt{x+2}\right) = 2 - \left(\sqrt{x+2}\right)^2 = -x,\ x \geq -2.$

$(gof)(x) = f(g(x)) = g(2 - x^2) = \sqrt{(2 - x^2) + 2} = \sqrt{4 - x^2}$

Domain of fog: $[-2, \infty)$.

Domain of gof: $[-2, 2]$.

Range of fog: $(-\infty, 2]$.

Range of gof: $[0, 2]$.

(b) $(fog)(x) = f(g(x)) = f\left(\sqrt{1 - x}\right) = \sqrt{\sqrt{1 - x}} = \sqrt[4]{1 - x}.$

$(gof)(x) = f(g(x)) = g\left(\sqrt{x}\right) = \sqrt{1 - \sqrt{x}}$

Domain of fog: $(-\infty, 1]$.

Domain of gof: $[0, 1]$.

Range of fog: $[0, \infty)$.

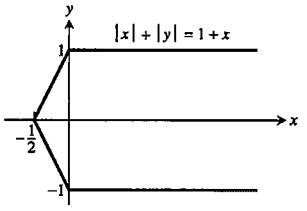
Range of gof: $[0, 1]$.

Problem 4

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\cos\frac{11\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

(a) For (x, y) in the 1st quadrant, $|x| + |y| = 1 + x$ $\Leftrightarrow$ $x + y = 1 + x$ $\Leftrightarrow$ $y = 1$. For (x, y) in the 2nd quadrant, $|x| + |y| = x + 1$ $\Leftrightarrow$ $-x + y = x + 1$ $\Leftrightarrow$ $y = 2x + 1$. In the 3rd quadrant, $|x| + |y| = x + 1$ $\Leftrightarrow$ $-x - y = x + 1$ $\Leftrightarrow$ $y = -2x - 1$. In the 4th quadrant, $|x| + |y| = x + 1$ $\Leftrightarrow$ $x + (-y) = x + 1$ $\Leftrightarrow$ $y = -1$. The graph is given at the right.



(b) We use reasoning similar to the example

(1) 1st quadrant: $y + |y| = x + |x|$ $\Leftrightarrow$ $2y = 2x$ $\Leftrightarrow$ $y = x$.

(2) 2nd quadrant: $y + |y| = x + |x|$ $\Leftrightarrow$ $2y = x + (-x) = 0$ $\Leftrightarrow$ $y = 0$.

(3) 3rd quadrant: $y + |y| = x + |x|$ $\Leftrightarrow$ $y + (-y) = x + (-x)$ $\Leftrightarrow$ $0 = 0$ $\Rightarrow$ <u>all</u> points in the 3rd quadrant satisfy the equation.

(4) 4th quadrant: $y + |y| = x + |x|$ $\Leftrightarrow$ $y + (-y) = 2x$ $\Leftrightarrow$ $0 = x$. Combining these results we have the graph given at the right: