MAS115 Calculus I 2006-2007

Problem sheet for exercise class 3

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1: Compute the following limits:

$$\text{(a)} \quad \lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}, \qquad \text{(b)} \quad \lim_{u\to 1} \frac{u^4-1}{u^3-1}.$$

Problem 2: **Two wrong statements about limits.** Show by example that the following statements are wrong.

(*a) The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0.

(b) The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the functions in your examples do not have the given value of L as a limit as $x \to x_0$.

Problem 3: Use the graph of the greatest integer function $y = \lfloor x \rfloor$ to determine the limits

$$\text{(*a)} \quad \lim_{\theta\to 3^+} \frac{\lfloor \theta \rfloor}{\theta}, \quad \lim_{\theta\to 3^-} \frac{\lfloor \theta \rfloor}{\theta}, \qquad \text{(b)} \quad \lim_{t\to 4^+} (t - \lfloor t \rfloor), \qquad \lim_{t\to 4^-} (t - \lfloor t \rfloor).$$

Problem 4: Compute the following limits:

$$\text{(*a)} \quad \lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}, \qquad \text{(b)} \quad \lim_{x\to\infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + cos^2 x}.$$

Extra: **Roots of a quadratic equation that is almost linear.** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1+\sqrt{1+a}}{a}, \quad r_-(a) = \frac{-1-\sqrt{1+a}}{a}.$$

(a) What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?

(b) What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?

(c) Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.

Thomas Prellberg, October 2006

Problem 1(a)

$$\lim_{x \to 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{\left(5-\sqrt{x^2+9}\right)\left(5+\sqrt{x^2+9}\right)} = \lim_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-(x^2+9)}$$

$$= \lim_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2} = \lim_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)} = \lim_{x \to 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5+\sqrt{25}}{8} = \frac{5}{4}$$

Problem 1 (5)

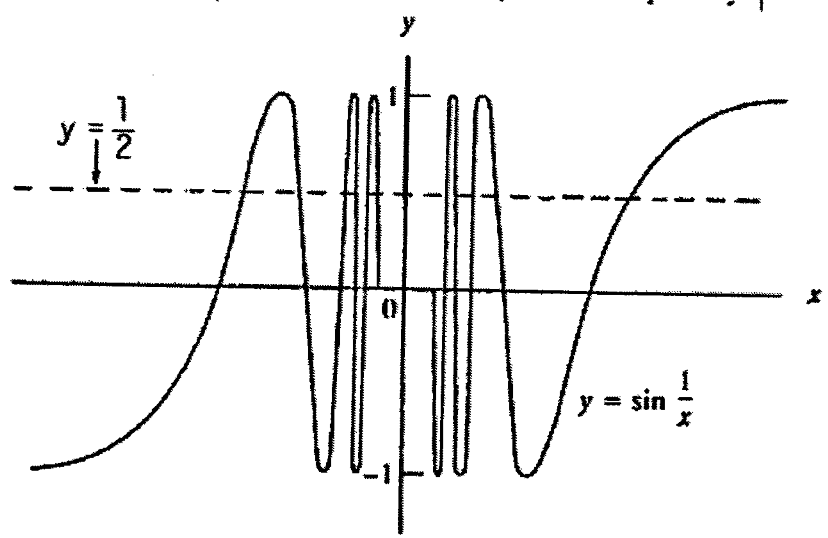
$$\lim_{u \to 1} \frac{u^4-1}{u^3-1} = \lim_{u \to 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$$

Problem 2(a)

Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim_{x \to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily close</u> to -1 for x near 0.

Problem 2(b)

Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely, $x = \frac{\pi}{6}$) for which $\left|\sin x - \frac{1}{2}\right| < \epsilon$ for any given $\epsilon > 0$. However, $\lim_{x \to 0} \sin x = 0$, not $\frac{1}{2}$. The wrong statement does not require x to be arbitrarily close to x_0. As another example, let $g(x) = \sin \frac{1}{x}$, $L = \frac{1}{2}$, and $x_0 = 0$. We can choose infinitely many values of x near 0 such that $\sin \frac{1}{x} = \frac{1}{2}$ as you can see from the accompanying figure. However, $\lim_{x \to 0} \sin \frac{1}{x}$ fails to exist. The wrong statement does not require <u>all</u> values of x arbitrarily close to $x_0 = 0$ to lie within $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are also infinitely many values of x near 0 such that $\sin \frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$ we cannot satisfy the inequality $\left|\sin \frac{1}{x} - \frac{1}{2}\right| < \epsilon$ for all values of x sufficiently near $x_0 = 0$.



Problem 3

(a) $\lim\limits_{\theta \to 3^+} \frac{\lfloor\theta\rfloor}{\theta} = \frac{3}{3} = 1$ $\qquad$ $\lim\limits_{\theta \to 3^-} \frac{\lfloor\theta\rfloor}{\theta} = \frac{2}{3}$

(b) $\lim\limits_{t \to 4^+} (t - \lfloor t\rfloor) = 4 - 4 = 0$ $\qquad$ $\lim\limits_{t \to 4^-} (t - \lfloor t\rfloor) = 4 - 3 = 1$

Problem 4

(a) $$\lim_{x \to \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x} + \frac{2}{\sqrt{x}}}{1 + \frac{\sin x}{x}} = \frac{1+0+0}{1+0} = 1$$

(b) $$\lim_{x \to \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x} = \lim_{x \to \infty} \left(\frac{1 + x^{-5/3}}{1 + \frac{\cos^2 x}{x^{2/3}}} \right) = \frac{1+0}{1+0} = 1$$

Extra:

(a) At x = 0: $\lim\limits_{a \to 0} r_+(a) = \lim\limits_{a \to 0} \frac{-1+\sqrt{1+a}}{a} = \lim\limits_{a \to 0} \left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)$

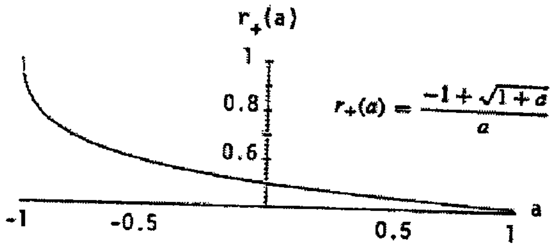
$= \lim\limits_{a \to 0} \frac{1-(1+a)}{a(-1-\sqrt{1+a})} = \frac{-1}{-1-\sqrt{1+0}} = \frac{1}{2}$

At x = −1: $\lim\limits_{a \to -1^+} r_+(a) = \lim\limits_{a \to -1^+} \frac{1-(1+a)}{a(-1-\sqrt{1+a})} = \lim\limits_{a \to -1} \frac{-a}{a(-1-\sqrt{1+a})} = \frac{-1}{-1-\sqrt{0}} = 1$

(b) At x = 0: $\lim\limits_{a \to 0^-} r_-(a) = \lim\limits_{a \to 0^-} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a \to 0^-} \left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$

$= \lim\limits_{a \to 0^-} \frac{1-(1+a)}{a(-1+\sqrt{1+a})} = \lim\limits_{a \to 0^-} \frac{-a}{a(-1+\sqrt{1+a})} = \lim\limits_{a \to 0^-} \frac{-1}{-1+\sqrt{1+a}} = \infty$ (because the denominator is always negative); $\lim\limits_{a \to 0^+} r_-(a) = \lim\limits_{a \to 0^+} \frac{-1}{-1+\sqrt{1+a}} = -\infty$ (because the denominator is always positive). Therefore, $\lim\limits_{a \to 0} r_-(a)$ does not exist.

At x = −1: $\lim\limits_{a \to -1^+} r_-(a) = \lim\limits_{a \to -1^+} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a \to -1^+} \frac{-1}{-1+\sqrt{1+a}} = 1$



Graph not to scale