MAS115 Calculus I 2006-2007

Problem sheet for exercise class 4

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1: Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or $x = -1$? Give reasons for your answers.

Problem 2: **Limits and continuity.** Which of the following statements are true and which false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

(*) a. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}(f(x) + g(x))$ does not exist.

(*) b. If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}(f(x) + g(x))$ does not exist.

c. If f is continuous at a, then so is $|f|$.

d. If $|f|$ is continuous at a, then so is f.

(*) Problem 3: **Stretching a rubber band.** Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

(*) Problem 4: Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Extra: **A function continuous at only one point.** Let

$$f(x) = \begin{cases} x\,, & \text{if } x \text{ is rational} \\ 0\,, & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

Thomas Prellberg, October 2006

Problem 1

At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x(x^2-1)}{|x^2-1|}$

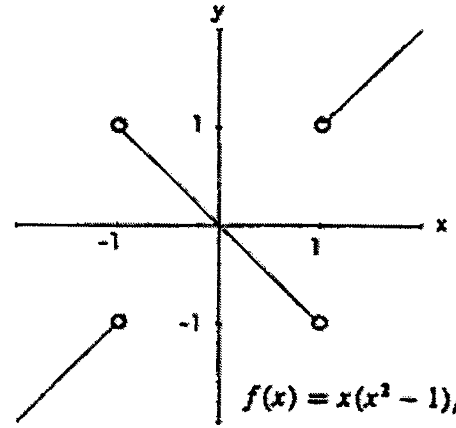
$= \lim_{x \to -1^-} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to -1^-} x = -1$, and

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to -1^+} \frac{x(x^2-1)}{-(x^2-1)}$

$= \lim_{x \to -1} (-x) = -(-1) = 1$. Since

$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

$\Rightarrow \lim_{x \to -1} f(x)$ does not exist, the function f <u>cannot</u> be extended to a continuous function at $x = -1$.



$f(x) = x(x^2 - 1)/|x^2 - 1|$

At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^-} \frac{x(x^2-1)}{-(x^2-1)} = \lim_{x \to 1^-} (-x) = -1$, and

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^+} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to 1^+} x = 1$. Again $\lim_{x \to 1} f(x)$ does not exist so f <u>cannot</u> be extended to a continuous function at $x = 1$ either.

Problem 2

(a) True, because if $\lim_{x \to a} (f(x) + g(x))$ exists then $\lim_{x \to a} (f(x) + g(x)) - \lim_{x \to a} f(x) = \lim_{x \to a} [(f(x) + g(x)) - f(x)]$ $= \lim_{x \to a} g(x)$ exists, contrary to assumption.

(b) False; for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists, but $\lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x \to 0} 0 = 0$ exists.

(c) True, because $g(x) = |x|$ is continuous $\Rightarrow$ $g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions).

(d) False; for example let $f(x) = \begin{cases} -1, & x \leq 0 \\ 1, & x > 0 \end{cases}$ $\Rightarrow$ $f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

Problem 3

Let $f(x)$ be the new position of point x and let $d(x) = f(x) - x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive if x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x) = 0$ for some point in between. That is, $f(x) = x$ for some point x, which is then in its original position.

Problem 4

Let $f(x) = x^3 - 15x + 1$ which is continuous on $[-4, 4]$. Then $f(-4) = -3$, $f(-1) = 15$, $f(1) = -13$, and $f(4) = 5$. By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. That is, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since a polynomial of degree 3 can have at most 3 solutions, these are the only solutions.

Extra

(a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon \Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at $x = 0$.

(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0| = |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.