MAS115 Calculus I 2006-2007

Problem sheet for exercise class 5

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Questionnaire: Please fill in the student questionnaire. The exercise class helper will collect the questionnaires, or you may return them directly to Prof Thomas Müller (Maths 115), who will process them for MAS115. Note that the questionnaires will be processed anonymously.

Problem 1: Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

(*) Problem 2: Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for the answer.

Extra: Suppose that a function f satisfies the following conditions for all real values of x and y:

i. $f(x + y) = f(x)f(y)$.

ii. $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

Thomas Prellberg, October 2006

Extra

From the given conditions we have $f(x+h) = f(x)\,f(h)$, $f(h) - 1 = hg(h)$ and $\lim_{h \to 0} g(h) = 1$. Therefore,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)\,f(h) - f(x)}{h} = \lim_{h \to 0} f(x) \left[\frac{f(h) - 1}{h}\right] = f(x) \left[\lim_{h \to 0} g(h)\right] = f(x) \cdot 1 = f(x)$$

$\Rightarrow f'(x) = f(x)$ and $f'(x)$exists at every value of x.

Problem 1

(they can use $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ by now)

For the curve $y = \sqrt{x}$, we have $y' = \lim\limits_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim\limits_{h \to 0} \frac{(x+h) - x}{\left(\sqrt{x+h} + \sqrt{x}\right) h}$ $= \lim\limits_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $\left(a, \sqrt{a}\right)$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal $\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$ $\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

Problem 2

$2x^2 + 3y^2 = 5 \Rightarrow 4x + 6yy' = 0 \Rightarrow y' = -\frac{2x}{3y} \Rightarrow y'\big|_{(1,1)} = -\frac{2x}{3y}\Big|_{(1,1)} = -\frac{2}{3}$ and $y'\big|_{(1,-1)} = -\frac{2x}{3y}\Big|_{(1,-1)} = \frac{2}{3}$; also, $y^2 = x^3 \Rightarrow 2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y} \Rightarrow y'\big|_{(1,1)} = \frac{3x^2}{2y}\Big|_{(1,1)} = \frac{3}{2}$ and $y'\big|_{(1,-1)} = \frac{3x^2}{2y}\Big|_{(1,-1)} = -\frac{3}{2}$. Therefore the tangents to the curves are perpendicular at $(1, 1)$ and $(1, -1)$ (i.e., the curves are orthogonal at these two points of intersection).