MAS115 Calculus I 2006-2007

Problem sheet for exercise class 7

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: The average value of an integrable function on the interval $[a, b]$ is defined as

$$\mathrm{av}(f) = \frac{1}{b-a} \int_a^b f(x)dx \,.$$

(i) If $\mathrm{av}(f)$ really is a typical value of the function $f(x)$ on $[a, b]$, then

$$\int_a^b \mathrm{av}(f)dx = \int_a^b f(x)dx$$

should hold. Does it?

(ii) It would be nice if average values obeyed the following rules on an interval $[a, b]$.

a. $\mathrm{av}(f+g) = \mathrm{av}(f) + \mathrm{av}(g)$

b. $\mathrm{av}(kf) = k\ \mathrm{av}(f)$ (any number k)

c. $\mathrm{av}(f) \le \mathrm{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold?

Give reasons for your answers.

Problem 2: Which formula is not equivalent to the other two?

a. $\sum_{j=2}^{4} \frac{(-1)^{j-1}}{j-1}$

b. $\sum_{k=0}^{2} \frac{(-1)^k}{k+1}$

c. $\sum_{l=-1}^{1} \frac{(-1)^l}{l+2}$

Problem 3: L'Hopital's rule does not help with the following limits. Find them some other way:

a. $\lim_{x\to\infty} \frac{\sqrt{x+5}}{\sqrt{x}+5}$

b. $\lim_{x\to\infty} \frac{2x}{x+7\sqrt{x}}$

Extra: Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, fing $h(10)$.

Thomas Prellberg, November 2006

Problem 1 a

Yes, for the following reasons: $\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is a constant K. Thus $\int_a^b \mathrm{av}(f)\,dx = \int_a^b K\,dx$ $= K(b-a) \Rightarrow \int_a^b \mathrm{av}(f)\,dx = (b-a)K = (b-a)\cdot\frac{1}{b-a}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx.$

Problem 15

All three rules hold. The reasons: On any interval $[a, b]$ on which f and g are integrable, we have:

(a) $\text{av}(f+g) = \frac{1}{b-a}\int_a^b [f(x)+g(x)]\,dx = \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right] = \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$
$= \text{av}(f) + \text{av}(g)$

(b) $\text{av}(kf) = \frac{1}{b-a}\int_a^b kf(x)\,dx = \frac{1}{b-a}\left[k\int_a^b f(x)\,dx\right] = k\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] = k\,\text{av}(f)$

(c) $\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{b-a}\int_a^b g(x)\,dx$ since $f(x) \le g(x)$ on $[a, b]$, and $\frac{1}{b-a}\int_a^b g(x)\,dx = \text{av}(g)$.
Therefore, $\text{av}(f) \le \text{av}(g)$.

Problem 2

(a) $$\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$$

(b) $$\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1} = \frac{(-1)^{0}}{0+1} + \frac{(-1)^{1}}{1+1} + \frac{(-1)^{2}}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$$

(c) $$\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^{0}}{0+2} + \frac{(-1)^{1}}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$$

(a) and (c) are equivalent; (b) is not equivalent to the other two.

Problem 3

(a) $$\lim_{x \to \infty} \frac{\sqrt{x+5}}{\sqrt{x}+5} = \lim_{x \to \infty} \frac{\frac{\sqrt{x+5}}{\sqrt{x}}}{\frac{\sqrt{x}+5}{\sqrt{x}}} = \lim_{x \to \infty} \frac{\sqrt{1+\frac{5}{x}}}{1+\frac{5}{\sqrt{x}}} = \frac{1}{1} = 1$$

(b) $$\lim_{x \to \infty} \frac{2x}{x+7\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{2x}{x}}{\frac{x+7\sqrt{x}}{x}} = \lim_{x \to \infty} \frac{2}{1+7\sqrt{\frac{1}{x}}} = \frac{2}{1+0} = 2$$

Extra

$h(x) = f^2(x) + g^2(x) \Rightarrow h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2\left[f(x)f'(x) + g(x)g'(x)\right] = 2\left[f(x)g(x) + g(x)(-f(x))\right]$
$= 2 \cdot 0 = 0$. Thus $h(x) = c$, a constant. Since $h(0) = 5$, $h(x) = 5$ for all x in the domain of h. Thus $h(10) = 5$.