MAS115 Calculus I 2006-2007

Problem sheet for exercise class 8

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)dt \ ?$$

a. g is a differentiable function of x.
b. g is a continuous function of x.
c. The graph of g has a horizontal tangent at $x = 1$.
d. g has a local maximum at $x = 1$.
e. g has a local minimum at $x = 1$.
f. The graph of g has an inflection point at $x = 1$.
g. The graph of dg/dx crosses the x-axis at $x = 1$.

Problem 2: Sometimes it helps to reduce the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. Practice this on

$$\int \sqrt{1 + \sin^2(x-1)} \sin(x-1)\cos(x-1)dx \ .$$

a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$
b. $u = \sin(x-1)$, followed by $v = 1 + v^2$
c. $u = 1 + \sin^2(x-1)$

Problem 3: Determine conditions on the constants a, b, c, and d so that the rational function

$$f(x) = \frac{ax+b}{cx+d}$$

has an inverse.

Extra: Prove that

$$\int_0^x \left(\int_0^u f(t)dt \right) du = \int_0^x f(u)(x-u)du \ .$$

(Hint: Express the integral on the right hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

Thomas Prellberg, November 2006

Problem 1

(a) True: since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.

(b) True: g is continuous because it is differentiable.

(c) True, since $g'(1) = f(1) = 0$.

(d) False, since $g''(1) = f'(1) > 0$.

(e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.

(f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.

(g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

Problem 2

(a) Let $u = x - 1 \Rightarrow du = dx$; $v = \sin u \Rightarrow dv = \cos u\, du$; $w = 1 + v^2 \Rightarrow dw = 2v\, dv \Rightarrow \frac{1}{2}\, dw = v\, dv$

$$\int \sqrt{1 + \sin^2(x-1)}\, \sin(x-1) \cos(x-1)\, dx = \int \sqrt{1 + \sin^2 u}\, \sin u \cos u\, du = \int v\sqrt{1 + v^2}\, dv$$

$$= \int \tfrac{1}{2} \sqrt{w}\, dw = \tfrac{1}{3} w^{3/2} + C = \tfrac{1}{3}\left(1 + v^2\right)^{3/2} + C = \tfrac{1}{3}\left(1 + \sin^2 u\right)^{3/2} + C = \tfrac{1}{3}\left(1 + \sin^2(x-1)\right)^{3/2} + C$$

(b) Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\, dx$; $v = 1 + u^2 \Rightarrow dv = 2u\, du \Rightarrow \frac{1}{2}\, dv = u\, du$

$$\int \sqrt{1 + \sin^2(x-1)}\, \sin(x-1) \cos(x-1)\, dx = \int u \sqrt{1 + u^2}\, du = \int \tfrac{1}{2} \sqrt{v}\, dv = \int \tfrac{1}{2} v^{1/2}\, dv$$

$$= \left(\tfrac{1}{2}\left(\tfrac{2}{3}\right) v^{3/2}\right) + C = \tfrac{1}{3} v^{3/2} + C = \tfrac{1}{3}\left(1 + u^2\right)^{3/2} + C = \tfrac{1}{3}\left(1 + \sin^2(x-1)\right)^{3/2} + C$$

(c) Let $u = 1 + \sin^2(x-1) \Rightarrow du = 2\sin(x-1)\cos(x-1)\, dx \Rightarrow \frac{1}{2}\, du = \sin(x-1)\cos(x-1)\, dx$

$$\int \sqrt{1 + \sin^2(x-1)}\, \sin(x-1) \cos(x-1)\, dx = \int \tfrac{1}{2} \sqrt{u}\, du = \int \tfrac{1}{2} u^{1/2}\, du = \tfrac{1}{2}\left(\tfrac{2}{3} u^{3/2}\right) + C$$

$$= \tfrac{1}{3}\left(1 + \sin^2(x-1)\right)^{3/2} + C$$

Problem 3

$f'(x) = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$. Thus if $ad - bc \neq 0$, $f'(x)$ is either always positive or always negative. Hence $f(x)$ is either always increasing or always decreasing. If follows that $f(x)$ is one-to-one if $ad - bc \neq 0$.

Extra

The derivative of the left side of the equation is: $\frac{d}{dx}\left[\int_0^x\left[\int_0^u f(t)\,dt\right]du\right] = \int_0^x f(t)\,dt$; the derivative of the right side of the equation is: $\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\int_0^x f(u)\,x\,du - \frac{d}{dx}\int_0^x u\,f(u)\,du$

$= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\int_0^x u\,f(u)\,du = \int_0^x f(u)\,du + x\left[\frac{d}{dx}\int_0^x f(u)\,du\right] - xf(x) = \int_0^x f(u)\,du + xf(x) - xf(x)$

$= \int_0^x f(u)\,du$. Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x = 0$, the constant must be 0. Therefore, $\int_0^x\left[\int_0^u f(t)\,dt\right]du = \int_0^x f(u)(x-u)\,du$.