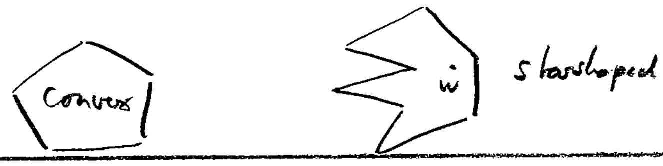
## Cauchy's Theorem

This is the key theorem of complex analysis, so we shall prove it in detail. Afterwards many consequences will follow easily.

**Definition** We say $U \subset \mathbb{C}$ is <u>convex</u> if $\forall z_1, z_2 \in U$ the line segment $\{t z_2 + (1-t) z_1 : 0 \le t \le 1\} \subseteq U$

notation: $[z_1, z_2]$

We say $U$ is <u>star-shaped about $w$</u> if $\forall z \in U$ the line segment $[w, z] \subseteq U$

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**Theorem 4.6** (Cauchy's Theorem for a star-shaped region)

Let $f$ be a function holomorphic on an open star-shaped region $U \subset \mathbb{C}$. Then for every closed contour $C$ in $U$ $\int_C f(z)\,dz = 0$

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Cauchy 1789-1857 announced this in 1813 & published a proof in 1825. Gauß knew of the result in 1811. Cauchy's original proof was via Green's Theorem. The proof we look at is essentially due to Goursat, and avoids having to assume continuity of $f'$, which is actually a <u>consequence</u> of the theorem.

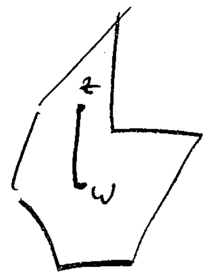
**Proof** It will suffice to show: $\boxed{f \text{ has an antiderivative } F \text{ on } U}$ (+)

since it will follow by (4.4) that for a closed curve $\int_C f(z)\,dz = 0$

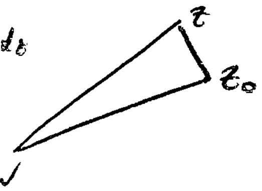
[We can't use Taylor's Theorem since we will need Cauchy's Theorem to <u>prove</u> Taylor's Theorem]

We start by defining $F(z) = \int_{[w,z]} f(z)\, dz$

where $w$ is the "centre" of the (starshaped) region and $[w,z]$ is the straight line segment from $w$ to $z$. It now just remains to prove that $F$ is differentiable and that $F'(z) = f(z)$.
But this will take some work.




$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{\int_{[w,z]} f(z)dz - \int_{[w,z_0]} f(z)dz}{z - z_0}$$




Suppose we could show

$$\int_{[w,z]} f(z)dz - \int_{[w,z_0]} f(z)dz = \int_{[z_0,z]} f(z)dz \qquad (*)$$

then $$\frac{F(z) - F(z_0)}{z - z_0} = \frac{\int_{[z_0,z]} f(z)dz}{z - z_0}$$ and, by continuity of $f$,

given $\varepsilon > 0$ $\exists\, \delta > 0$ with $|f(s) - f(z_0)| < \varepsilon$ $\forall\, |s - z_0| < \delta$

Thus, for $|s - z_0| < \delta$ we have $\left| \int_{[z_0,z]} f(s)\, ds - \underbrace{\int_{[z_0,z]} f(z_0) ds}_{f(z_0)(z - z_0)} \right| < \varepsilon |z - z_0|$

Therefore,

$$\left| \frac{\int_{[z,z_0]} f(s)\, ds}{z - z_0} - f(z_0) \right| < \varepsilon \quad \text{for } |z - z_0| < \delta$$

i.e. $\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0)$, $F$ is differentiable at $z_0$ with

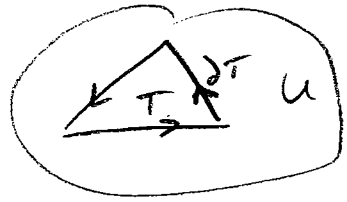
derivative $F'(z_0) = f(z_0)$, proving $(+)$ and Cauchy's theorem.

All we need to do now is prove $(*)$, Cauchy's Theorem for a $\Delta$.

## Cauchy's Theorem for a Triangle

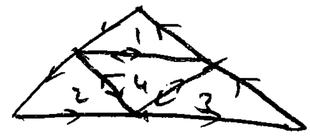
Let $f$ be holomorphic on a domain $U$ containing a triangle $T$ and its interior.

Let $\partial T$ denote the perimeter of the triangle $T$.




Then $\displaystyle\int_{\partial T} f(z)\, dz = 0$

**Proof** Let $\eta(T) = \displaystyle\int_{\partial T} f(z)\, dz$ and let the length of $\partial T$ be $L$. Divide $T$ into four triangles by bisecting the sides of $T$




$$\eta(T) = \sum_{j=1}^{4} \eta(T^{(j)}) \qquad \text{(internal edges cancel)}$$

Each $T^{(j)}$ has length $(\partial T^{(j)}) = L/2$. At least one of the $T^{(j)}$ must have $|\eta(T^{(j)})| \geq \frac{1}{4}|\eta(T)|$. Call this triangle $T_1$. Now repeat this process to get $T_2$ with $\text{length}(\partial T_2) = L/4$ and $|\eta(T_2)| \geq \frac{1}{4}|\eta(T_1)| \geq \frac{1}{16}|\eta(T)|$

Repeat to get $T \supset T_1 \supset T_2 \supset T_3 \supset \dots$ with $\text{length}(\partial T_n) = L/2^n$ and $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$. Let $z_0$ be the point $\displaystyle\bigcap_{n\geq 1} T_n$.

Since $f$ is differentiable at $z_0$, given any $\varepsilon > 0$ we know that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall |z - z_0| < \delta \ : \ |f(z) - f(z_0) - (z - z_0) f'(z_0)| \leq |z - z_0|\,\varepsilon$$

Take $n$ such that $L/2^n < \delta$. Then $|z - z_0|\varepsilon < \frac{L}{2^n}\varepsilon \quad \forall z \in \partial T_n$

Thus, $$\left| \int_{\partial T_n} \Big( f(z) \underbrace{- f(z_0) - (z - z_0) f'(z_0)}\Big)\, dz \right| \leq \frac{L}{2^n}\varepsilon \frac{L}{2^n} = \varepsilon \frac{L^2}{4^n}$$

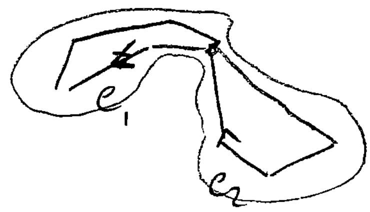
Integral vanishes since antiderivative $-z f(z_0) - \left(\frac{z^2}{2} - z_0 z\right) f'(z_0)$ exists & closed curve $\partial T_n$

$\leadsto \left| \displaystyle\int_{\partial T_n} f(z)\, dz \right| \leq \varepsilon \frac{L^2}{4^n} \quad \leadsto \quad \frac{\varepsilon L^2}{4^n} \geq |\eta(T_n)| \geq \frac{1}{4^n} \eta(T)$

$\leadsto |\eta(T)| \leq \varepsilon L^2$ but $\varepsilon > 0$ arbitrary. $\Rightarrow \eta(T) = 0$ $\square$

**Notes.** 1) If we assume $f'$ continuous, then Cauchy's Theorem follows from Stokes'. Theorem (Calculus III). But the point here is $f$ differentiable once $\rightarrow$ $f$ differentiable many times. Thus, continuity of $f'$ is a <u>consequence</u> of Cauchy's Theorem and there is no need to assume it as a hypothesis.

2) From the statement of Cauchy's Theorem for a star-shaped region one can deduce it for more general regions made up of star-shaped pieces



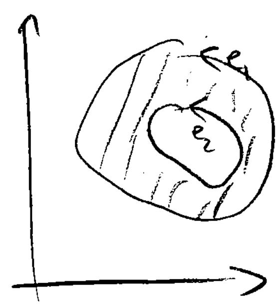

$$\int_C = \int_{C_1} + \int_{C_2}$$

In particular, one can prove that if $C$ is any simple closed contour in $\mathbb{C}$ & $f$ is holomorphic <u>everywhere on and inside $C$</u> then $\int_C f(z)\,dz = 0$ (the <u>general form</u> of Cauchy's Theorem)

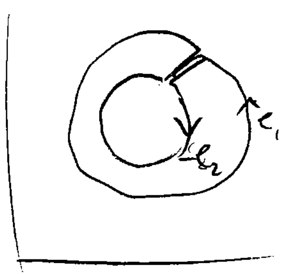
**<u>Corollary 4.7 (Deformation Principle)</u>**

If $f$ is holomorphic on the region between two simple closed contours, disjoint from each other, then
(and same orientation!)

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$




<u>Proof</u> (Idea)




$$0 = \int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

(using the general form of Cauchy's Theorem)