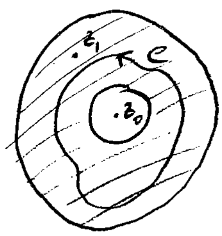
Filling in the gaps: the missing proofs

Laurent's Theorem (viz 3.7) Let f be holomorphic on the annulus

$A = \{ z : R_1 < |z_1 - z_0| < R_2 \}$ (where R_1 can be zero and R_2 can be ∞)

Let C be a simple closed positively oriented contour around z_0 in A.

Let z_1 be any point of A. Then



$$f(z_1) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C f(z) (z - z_0)^{n-1} \, dz$$

Proof Take circles C_1, C_2 centred at z_0, both inside A, and with z_1 and C lying in between C_1 and C_2. Take B a small circle around z_1 not meeting C_1 or C_2 (may meet C). We have

$$(1) \quad f(z_1) \underset{\text{CIF}}{=} \frac{1}{2\pi i} \int_B \frac{f(z)}{z - z_1} \, dz \underset{\text{Cauchy Theorem}}{=} \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1} \, dz - \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_1} \, dz$$

Let $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

and b_n similarly. If we can show that

$$(2) \quad \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1} \, dz, \qquad (3) \quad \sum_{n=1}^{\infty} b_n (z_1 - z_0)^{-n} = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_1} \, dz$$

then (2) and (3) prove the theorem.

To prove (2), write

$$\sum_{n=0}^{N}{}' a_n (z_1 - z_0)^n = \sum_{n=0}^{N}{}' \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \, (z_1 - z_0)^n$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0} \underbrace{\sum_{n=0}^{N}{}' \left(\frac{z_1 - z_0}{z - z_0} \right)^n}_{\frac{1 - \left(\frac{z_1 - z_0}{z - z_0}\right)^{N+1}}{1 - \frac{z_1 - z_0}{z - z_0}}} dz = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1} \left(1 - \left(\frac{z_1 - z_0}{z - z_0} \right)^{N+1} \right) dz$$

now $\left| \frac{z_1 - z_0}{z - z_0} \right| \leq \rho < 1$ for $z \in C_2$, and $\left| \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1} \left(\frac{z_1 - z_0}{z - z_0} \right)^{N+1} dz \right|$

$\leq \frac{1}{2\pi} 2\pi R_2 \rho^N \max_{z \in C_2} \left| \frac{f(z)}{z - z_1} \right| \to 0$ as $N \to \infty$, thus proving (2).

A similar argument holds for (3). □

Corollary I: Taylor's Theorem (viz 3.6) Let f be holomorphic everywhere on $D = \{ z : |z - z_0| < R \}$. Then $f^{(n)}(z_0)$ exists for all $n \geq 0$ and for any z_1 in D, $f(z_1) = \sum_{n=0}^{\infty}{}' \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$

Proof Let C be a circle in D centred at z_0 and containing z_1.

By Laurent's Theorem

$$f(z_1) = \sum_{n=0}^{\infty}{}' \underbrace{\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz}_{\substack{\frac{f^{(n)}(z_0)}{n!} \\ \text{(extended CIF)}}} (z_1 - z_0)^n + \sum_{n=1}^{\infty}{}' \underbrace{\frac{1}{2\pi i} \int_C f(z) (z - z_0)^{n-1} dz}_{\substack{0 \\ \text{(Cauchy Theorem)}}} (z_1 - z_0)^{-n}$$

□

Corollary II : The Residue Theorem (viz 4.10) (general proof)

Let f be holomorphic on and inside a simple closed contour C except at a finite number of singularities $z_1, \ldots, z_n$ inside (but not on) C. Then, if C is positively oriented,
$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j}(f)$$

Proof By the deformation principle, $\displaystyle\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$, but this is just $2\pi i$ times the coefficient b_1 of the respective Laurent series, i.e. the residue of f at z_j. $\square$