# MAS205 Complex Variables 2005-2006

Exercises 1

Exercise 1: Let $z_1 = \frac{3}{2} - 2i$ and $z_2 = 4 + 3i$. Compute (in standard $x + iy$ form):

$$(a) \quad z_1 z_2 \qquad (b) \quad \frac{1}{z_1} \qquad (c) \quad \frac{z_2}{z_1} \qquad (d) \quad \frac{1}{z_1} + \frac{1}{z_2}$$

Compute the moduli:

$$(a) \quad |z_1| \qquad (b) \quad \left|\frac{z_1}{z_2}\right| \qquad (c) \quad |z_1 z_2|$$

Exercise 2: Express the following complex numbers in polar exponential form:

$$(a) \quad -1 \qquad (b) \quad 2i \qquad (c) \quad 1+i \qquad (d) \quad 1-\sqrt{3}i \qquad (e) \quad 1/(1-i)$$

Exercise 3: Solve for the roots of the equation

$$z^3 + 8i = 0$$

Express all the roots in standard and polar form, and draw a diagram showing their location in the complex plane.

Exercise 4: Describe graphically the sets of points in the complex plane defined by the following equations and inequalities:

(a) $|z - 2i| < 2$

(b) $\Im(z^2) = 0$

(c) $1 \leq \Im(z+1) < 2$

(d) $z^2 = -2$

Notation: $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of $z$, respectively.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 12th October

Thomas Prellberg, September 2005

1) $z_1 = \frac{3}{2} - 2i$ , $z_2 = 4+3i$

(a) $z_1 z_2 = \left(\frac{3}{2} - 2i\right)\left(4+3i\right) = 6 + 6 - 8i + \frac{9}{2}i = 12 - \frac{7}{2}i$ 4

(b) $\frac{1}{z_1} = \frac{1}{\frac{3}{2}-2i} = \frac{\frac{3}{2}+2i}{\frac{9}{4}+4} = \frac{6+8i}{9+16} = \frac{6}{25} + \frac{8}{25}i$ 4

(c) $\frac{z_2}{z_1} = z_2 \frac{1}{z_1} = (4+3i)\left(\frac{6}{25} + \frac{8}{25}i\right) =$ 4

$= \frac{24-24}{25} + \frac{18+32}{25}i = 2i$

(d) $\frac{1}{z_2} = \frac{1}{4+3i} = \frac{4-3i}{16+9} = \frac{4}{25} - \frac{3}{25}i$ 4

$\rightsquigarrow \frac{1}{z_1} + \frac{1}{z_2} = \frac{6}{25} + \frac{8}{25}i + \frac{4}{25} - \frac{3}{25}i = \frac{2}{5} + \frac{1}{5}i$

(a) $|z_1| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \frac{5}{2}$ 3 (b) $|z_2| = \sqrt{4^2+3^2} = 5 \rightsquigarrow \left|\frac{z_1}{z_2}\right| = \frac{1}{2}$ 3

(c) $|z_1 z_2| = \frac{25}{2}$ as $|z_2| = 5$ 3 /25

2) (a) $-1 = 1e^{i\pi}$ 5

(b) $2i = 2e^{i\frac{\pi}{2}}$ 5

(c) $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ 5

(d) $1-\sqrt{3}i = 2e^{-i\frac{\pi}{6}}$ 5


(e) $\frac{1}{1-i} = \frac{1+i}{2} = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}$ 5

/25

3)

(a) $z^3 = -8i = 2^3 e^{i\frac{3}{2}\pi + 2k\pi i}$ $k \in \mathbb{Z}$

$\rightsquigarrow z_k = 2 e^{i\frac{\pi}{2} + \frac{2}{3}k\pi i}$ $k = 0,1,2$

$z_1 = 2 e^{i\frac{\pi}{2}}$ 3 $= 2i$ 3

$z_2 = 2 e^{i(\frac{\pi}{2} + \frac{2}{3}\pi)} = 2 e^{i\frac{7}{6}\pi}$ 3

$= 2\left(\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi\right) = 2\left(-\frac{\sqrt{3}}{2}\right) + 2i\left(-\frac{1}{2}\right)$ 3

$= -\sqrt{3} - i$

$z_3 = 2 e^{i(\frac{\pi}{2} + \frac{4}{3}\pi)} = 2 e^{i\frac{4}{6}\pi}$ 3

$= 2\left(\cos\frac{4}{6}\pi + i\sin\frac{4}{6}\pi\right) = 2\frac{\sqrt{3}}{2} + 2i\left(-\frac{1}{2}\right)$ 3

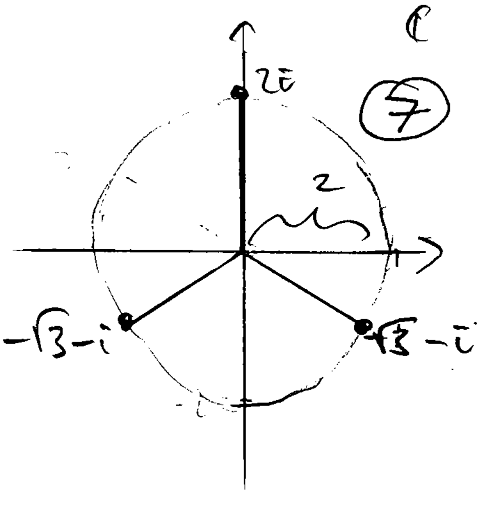
$= +\sqrt{3} - i$

check: $(-\sqrt{3} - i)^3 = (-\sqrt{3} - i)\underbrace{(+\sqrt{3} + i)(+\sqrt{3} + i)}$

$(-\sqrt{3} - i)(\cancel{3}^{2} - 1 + 2\sqrt{3}i)$

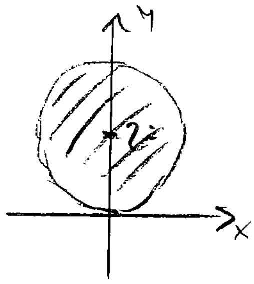
$= -\cancel{2\sqrt{3}} - 2i + \cancel{2\sqrt{3}} - 6i = -8i$ ✓

$(+\sqrt{3} - i)^3$ similarly



4) (a) $|z-2i|<2$

open disk centred at $2i$ with radius 2

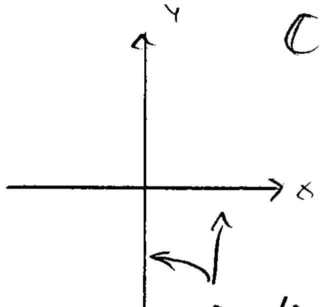



7

(b) $\mathrm{Im}(z^2)=0$

$z=x+iy$

$z^2=(x+iy)^2 = x^2-y^2+2ixy$




coordinate axes

7

$\rightsquigarrow \mathrm{Im}(z^2)=2xy=0$ implies $x=0$ or $y=0$

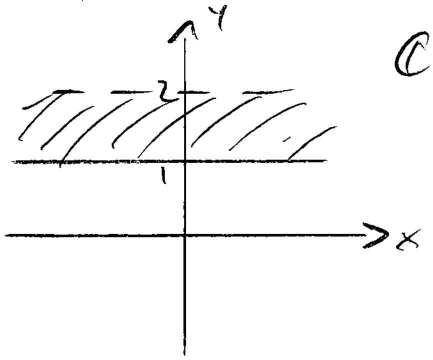
(c) $1\le \mathrm{Im}(z+1)<2$

$z=x+iy \rightsquigarrow \mathrm{Im}(z+1)=y \rightsquigarrow 1\le y<2$

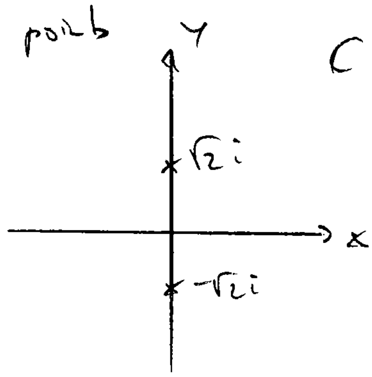
6

horizontal strip between 1 and 2,

including 1 but excluding 2




(d) $z^2=-2 \rightsquigarrow z=\pm\sqrt{2}\,i$ two points




5