MAS205 Complex Variables 2005-2006

Exercises 2

Exercise 5: Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$, show that

(a) $e^{i\theta} = e^{i(\theta+2n\pi)}$ for $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$

(b) $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$ for $\theta, \phi \in \mathbb{R}$

(c) $1/e^{i\theta} = e^{-i\theta}$ for $\theta \in \mathbb{R}$

Using (b) and mathematical induction, show that

(d) $(e^{i\theta})^n = e^{in\theta}$ for $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$

Exercise 6: Find all complex solutions of the following equations:

$$(a) \quad e^z = i \qquad (b) \quad e^{2z} = 1 \qquad (c) \quad \sinh z = 0 \qquad (d) \quad \cos z = 0$$

Exercise 7: Consider the transformation

$$z \mapsto w = (z-1)^2 .$$

(a) Find the equation of the image of the line $\Re(z) = 0$ and sketch the image.

(b) What is the image of the upper half plane?

Exercise 8: (a) Find the region in the w-plane which is the image of the upper half of the z-plane under the transformation

$$w = 1 + 1/z$$

(b) Find the regions in the z-plane which map to the left half of the w-plane under the transformation

$$w = z^4$$

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 19th October

Thomas Prellberg, September 2005

5) (a) $e^{i(\theta+2n\pi)} = \cos(\theta+2n\pi) + i\sin(\theta+2n\pi)$

$= \cos\theta + i\sin\theta = e^{i\theta}$ 4

(b) $e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= \cos(\theta+\phi) + i\sin(\theta+\phi) = e^{i(\theta+\phi)}$ 6

(c) $\dfrac{1}{e^{i\theta}} = \dfrac{1}{\cos\theta + i\sin\theta} = \dfrac{\cos\theta}{\cos^2\theta+\sin^2\theta} + i\dfrac{-\sin\theta}{\cos^2\theta+\sin^2\theta}$

$= \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$

$= e^{-i\theta}$ 6

(d) $n=1$: $(e^{i\theta})^1 = e^{i\theta}$

$n \to n+1$: $(e^{i\theta})^{n+1} = (e^{i\theta})^n e^{i\theta}$

$\underset{\uparrow}{=} e^{in\theta} e^{i\theta} \underset{\uparrow}{=} e^{i(n\theta+\theta)} = e^{i(n+1)\theta}$

Induction assumption (b)

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6) (a) $e^z = i = e^{i\frac{\pi}{2}}$

$\rightsquigarrow z = i\frac{\pi}{2} + 2k\pi i \quad , k \in \mathbb{Z}$ (6)

(b) $e^{2z} = 1 = e^0$

$\rightsquigarrow 2z = 2k\pi i \rightsquigarrow z = k\pi i \quad , k \in \mathbb{Z}$ (6)

(c) $\sinh z = 0 \rightsquigarrow e^z - e^{-z} = 0$

$e^{+z} \neq 0 : \rightsquigarrow e^{2z} - 1 = 0 \rightarrow e^{2z} = 1$ (6)

as in b) : $z = k\pi i \quad , k \in \mathbb{Z}$

(d) $\cos z = 0 \rightsquigarrow e^{iz} + e^{-iz} = 0$

$e^{iz} \neq 0 \rightsquigarrow e^{2iz} + 1 = 0 \quad , e^{2iz} = 1$ (7)

as in b): $iz = k\pi i \rightsquigarrow z = k\pi \quad k \in \mathbb{Z}$

(only the real zeros of $\cos x$)

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7) $w = (z-1)^2 = (x-1+iy)^2$

$= (x-1)^2 - y^2 + 2i(x-1)y$

$u = (x-1)^2 - y^2$, $v = 2(x-1)y$ 5

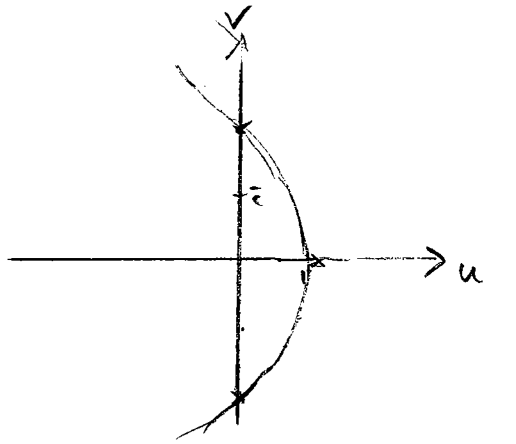
(a) $\mathrm{Re}(z) = 0 \rightsquigarrow z = iy$, $y \in \mathbb{R}$

i.e. $x = 0$ 5

$\rightsquigarrow u = 1 - y^2$, $v = 2(-1)y = -2y$

eliminate y: $u = 1 - \left(\frac{v}{-2}\right)^2 = 1 - \frac{1}{4}v^2$ 5

parabola



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(b) The upper half plane is invariant under $z \to z-1$. Squaring gives all of $\mathbb{C}$.

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Answer: $\mathbb{C}$

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8) (a) $w = 1 + \frac{1}{z}$ transforms upper half plane to lower half plane $(z \to \frac{1}{z})$ and shifts to the right by 1 $(z \to z+1)$

the image is the lower half plane

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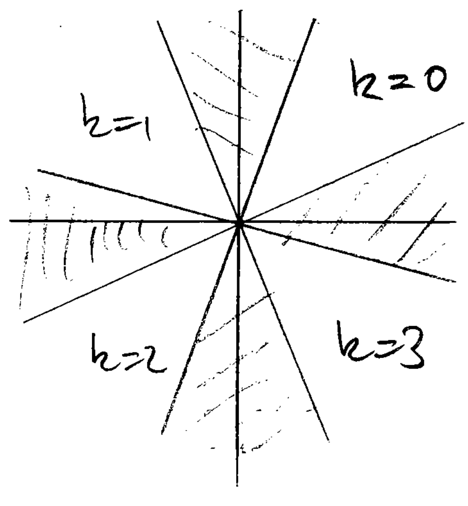
(b) $z^4 = w$ $\quad z = re^{i\theta}$

$\rightsquigarrow w = r^4 e^{i4\theta}$.

w in left half plane : $\frac{\pi}{2} + 2k\pi \leq 4\theta \leq \frac{3}{2}\pi + 2k\pi$

$$\frac{\pi}{8} + \frac{k}{2}\pi \leq \theta \leq \frac{3\pi}{8} + \frac{k}{2}\pi$$



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