MAS205 Complex Variables 2005-2006

Exercises 3

Exercise 9: Find the Möbius transformation $f(z) = (az+b)/(cz+d)$ which maps $1 \mapsto 0$, $i \mapsto 1$, and $0 \mapsto i$.

(a) What is the image of $z = -1$?

(b) Which point is mapped by f to $2i$?

(c) What is the image of the unit disk under f?

Exercise 10: Evaluate the following limits:

$$\text{(a)} \quad \lim_{z\to\infty} \frac{(iz-1)(z+3)^2}{(z+i)^2(z-1)} \quad \text{(b)} \quad \lim_{z\to 2-2i} \frac{z^5+1}{z^2+8i} \quad \text{(c)} \quad \lim_{z\to i} \frac{z^2}{z^3-1}$$

Exercise 11: (a) Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z\to i} f(z) = \infty \qquad \text{and} \qquad \lim_{z\to 1} f(z) = 0\,.$$

(b) Suppose

$$f(z) = \frac{p(z)}{z^2-4}\,, \quad \text{where } p(z) = az+b \text{ for some } a, b \in \mathbb{C}.$$

If $\lim_{z\to -2} f(z) = 1$, what is $p(z)$?

(c) Suppose

$$f(z) = \frac{p(z)}{z^2+1}\,, \quad \text{where } p(z) \text{ is a quadratic polynomial.}$$

If $\lim_{z\to -i} f(z) = 1$ and $\lim_{z\to\infty} f(z) = 1$, what is $p(z)$?

Exercise 12: For each of the following functions, decide at which values of z the function is continuous and at which values it is not continuous. Give reasons, but detailed proofs are not expected.

(a) $f(z) = z + i\Re(z)$

(b) $f(z) = (z/\overline{z})^3$ for all non-zero z, and $f(0) = 1$.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 26th October

Thomas Prellberg, October 2005

9) (i) $\dfrac{a+b}{c+d} = 0 \quad \rightsquigarrow \quad a+b = 0$

(ii) $\dfrac{ai+b}{ci+d} = 1 \quad \rightsquigarrow \quad ai+b = ci+d$

(iii) $\dfrac{b}{d} = i \quad \rightsquigarrow \quad b = id$

Thus $b = id$, $a = -b = -id$, and $(-id)\,i + id = ci + d$

implies $c = d$, thus

$$f(z) = \frac{-id\,z + id}{dz + d} = \frac{-iz + i}{z+1} \qquad (10)$$

check: $f(1) = \dfrac{-i+i}{1+1} = 0$ ✓ $\quad f(i) = \dfrac{1+i}{i+1} = 1$ ✓

$f(0) = \dfrac{i}{1} = i$ ✓

(a) $f(-1) = \dfrac{i+i}{-1+1} = \infty \qquad (4)$

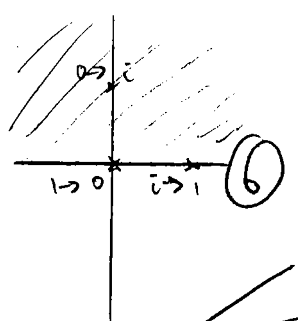
(b) $\dfrac{-iz+i}{z+1} = 2i \quad \rightsquigarrow \quad -\cancel{i}z + \cancel{i} = 2\cancel{i}(z+1)$

$\rightsquigarrow \quad z = -\dfrac{1}{3}$. check: $f(-\tfrac{1}{3}) = \dfrac{+i\frac{1}{3} + i}{-\frac{1}{3}+1} = \dfrac{\frac{4}{3}i}{\frac{2}{3}} = 2i$ ✓ (5)

(c) need 3 points, have $f(1) = 0$, $f(i) = 1$, $f(-1) = \infty$

and interior point: $f(0) = i$

Thus, the image is the upper half plane



(6)

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10) (a) $$\lim_{z\to\infty} \frac{(iz-1)(z+3)^2}{(z+i)^2(z-1)} = \lim_{s\to 0} \frac{(i/s-1)(1/s+3)^2}{(1/s+i)^2(1/s-1)}$$

$$= \lim_{s\to 0} \frac{(i-s)(1+3s)^2}{(1+is)^2(1-is)} = \frac{i\cdot 1^2}{1^2\cdot 1} = i$$ 7

(b) $$\lim_{z\to 2-2i} \frac{z^5+1}{z^2+8i} = \infty$$

as $\lim_{z\to 2-2i} (z^5+1) = (2-2i)^5+1 \neq 0$

and $\lim_{z\to 2-2i} (z^2+8i) = (2-2i)^2+8i = 4-4-8i+8i = 0$ 7

(c) $$\lim_{z\to i} \frac{z^2}{z^3-1} = \frac{i^2}{i^3-1} = \frac{-1}{-i-1} = \frac{1}{1+i} = \frac{1-i}{2}$$ 6

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11) (a) e.g. $f(z) = \dfrac{z-1}{z-i}$

check: rational function, denom. zero at $z_0 = i$

numerator zero at $z_1 = 1$ 8

(b) $$f(z) = \frac{p(z)}{q(z)} \quad \text{with } p(z) = az+b, \; q(z) = z^2-4$$

$$\lim_{z \to -2} \frac{p(z)}{q(z)} = 1 \text{ possible only if } p(-2) = 0, \text{ as } q(-2)=0$$

$$\rightsquigarrow b = 2a \quad \text{and} \quad f(z) = \frac{az+2a}{z^2-4} = \frac{a}{z-2}$$

$$1 = \lim_{z \to -2} \frac{a}{z-2} = -\frac{a}{4} \rightsquigarrow a = -4 \text{ and then}$$

$$p(z) = -4z-8$$

(10)

(c) $$f(z) = \frac{p(z)}{z^2+1} \quad \text{with } p(z) \text{ quadratic}$$

$$\lim_{z \to -i} f(z) = 1 \text{ possible only if } p(-i) = 0 \text{ as } (-i)^2+1=0$$

$$\rightsquigarrow p(z) = (z+i)(az+b) \quad \text{and} \quad f(z) = \frac{az+b}{z-i}$$

$$\left.\begin{aligned} 1 &= \lim_{z \to -i} f(z) = \frac{-ai+b}{-2i} \rightsquigarrow b = ai-2i \\ 1 &= \lim_{z \to \infty} f(z) = \lim_{s \to 0} \frac{a/s+b}{1/s-i} = a \rightsquigarrow a=1 \end{aligned}\right\} b = -i$$

$$p(z) = (z+i)(z-i) = z^2+1, \quad f(z) = 1$$

(12)

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12)

(a) $f(z) = z + i\,\mathrm{Re}(z)$

continuous for all $z \in \mathbb{C}$

by proposition 1.10 and the fact

that $z \mapsto \mathrm{Re}(z)$ is continuous:

$$\lim_{z \to z_0} \mathrm{Re}(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} y = y_0 = \mathrm{Re}(z_0)$$

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(b) $f(z) = \left(z/\bar{z}\right)^3$ is continuous for all $z \neq 0$

by proposition 1.10 and the fact that $z \to \bar{z}$ is continuous

but is not continuous for $z = 0$, as

$f(re^{i\varphi}) = e^{6i\varphi}$ well-defined for $r > 0$,

but for $r \to 0$ φ-dependent values.

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