MAS205 Complex Variables 2005-2006

Exercises 4

Exercise 13: For each of the following functions, decide at which values of z the function is continuous and at which values it is not continuous. Give reasons, but detailed proofs are not expected.

(a) $f(z) = |z|$

(b) $f(z) = z^3/\overline{z}$ for all non-zero z, and $f(0) = 0$.

Exercise 14: Let f and g denote functions $\mathbb{C} \to \mathbb{C}$. For each question below, give either a proof or a counterexample to justify your answer.

(a) If f and g are both continuous at z_0, does it follow that $g - f$ is continuous at z_0?

(b) If f and g are both discontinuous at z_0, does it follow that fg is discontinuous at z_0?

(c) If f and g are both continuous at z_0, does it follow that $f \circ g$ is continuous at z_0?

(d) Suppose f is discontinuous at $3 + i$, but continuous everywhere else, and g is discontinuous at $2 + i$, but continuous everywhere else. Is $f + g$ continuous at $4 + 2i$?

Exercise 15: Starting from the definition of the derivative of a complex function as a limit,

(a) find the derivative of $f(z) = iz(1 - 2z)$ at $z = i$;

(b) find the derivative of $f(z) = z^3 + z$ for all $z \in \mathbb{C}$;

(c) prove that $f(z) = |z|^2 + z^2$ does not have a derivative at z_0 unless $z_0 = 0$. What is the value of $f'(0)$?

Exercise 16: For each of the following functions, decide at which values of z the function is differentiable and at which values it is not differentiable. Give reasons, but detailed proofs are not expected.

(a) $f(z) = |z|$

(b) $f(z) = z^3/\overline{z}$ for all non-zero z, and $f(0) = 0$.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 2nd November

Thomas Prellberg, October 2004

13) (a) $f(z) = |z| = \sqrt{x^2+y^2}$

continuous for all $z \in \mathbb{C}$, as (10)

$u(x,y) = \sqrt{x^2+y^2}$ continuous for all $(x,y) \in \mathbb{R}^2$

and $v(x,y) = 0$ — " —

(b) $f(z) = z^3/\bar{z}$, $z \neq 0$; $= 0$, $z = 0$

continuous for all $z \in \mathbb{C}$, as (8)

(i) $z \neq 0$: $z \to \bar{z}$ continuous

$z \to z^3$ continuous

& $z \to z^3/\bar{z}$ continuous for $\bar{z} \neq 0$

(ii) $z = 0$: $f(z) = \underbrace{z^2}_{\to 0} \underbrace{(z/\bar{z})}_{\text{bounded}} \to 0$ as $z \to 0$ (7)

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14) (a) yes: g continuous, $-f$ continuous (6)

$\rightsquigarrow g + (-f)$ continuous

(b) no: $f(z) = z/\bar{z}$, $z \neq 0$, $g(z) = \bar{z}/z$, $z \neq 0$, $fg(z) = 1$ (6)

$f(0) = 1$ $g(0) = 1$

$\rightsquigarrow fg$ is continuous at $z_0 = 0$ even though

f and g aren't

(c) no, need continuity of f at $g(z_0)$, not z_0!

example: $g(z) = 1-z$

$$f(z) = z/\bar{z} \quad z \neq 0 \quad g(0) = 0$$

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$f(z), g(z)$ continuous at $z_0 = 1$ but

$$f(g(z)) = \frac{1-z}{1-\bar{z}} \text{ is not.}$$

(d) yes, f and g are both continuous at $4+2i$, so $f+g$ is, as well.

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15) (a)

$$f'(z) = \lim_{\Delta z \to 0} \frac{i(z+\Delta z)(1+z+\Delta z) - iz(1+z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{iz(1+z)} + i\Delta z(1+z) + iz\Delta z + i\Delta z^2 - \cancel{iz(1+z)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(i(1+2z) + i\Delta z\right) = i(1+2z)$$

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$$f'(i) = i(1+2i) = -2+i$$

$$\left(\text{or } f'(i) = \lim_{\Delta z \to 0} \frac{i(i+\Delta z)(1+i+\Delta z) - i\,i(1+i)}{\Delta z} = \ldots\right)$$

(b) $$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 + z + \Delta z - z^3 - z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + 3z^2\Delta z + 3z\Delta z^2 + \Delta z^3 \cancel{- z^3} + \Delta z}{\Delta z}$$

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$$= \lim_{\Delta z \to 0} \left(3z^2 + 3z\Delta z + \Delta z^2 + 1\right) = 3z^2 + 1$$

$$\leadsto f'(z) = 3z^2 + 1$$

(c) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) + (z_0 + \Delta z)^2 - z_0\bar{z}_0 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0\bar{z}_0} + \Delta z\,\bar{z}_0 + z_0\overline{\Delta z} + \Delta z\overline{\Delta z} + \cancel{z_0^2} + 2z_0\Delta z + \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} + 2z_0 + \Delta z\right)$$

$\lim\limits_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist $\leadsto$ $f'(z_0)$ does not exist for $z_0 \neq 0$

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for $z_0 = 0$, $f'(0) = 0$

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1b) a) $f(z) = |z| = \sqrt{x^2+y^2}$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \frac{\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2}}{\Delta x + i\Delta y}$$

(i) $\Delta y = 0$: $\quad = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$

(ii) $\Delta x = 0$: $\quad = \frac{1}{i}\frac{\partial}{\partial y}\sqrt{x^2+y^2} = -i\,\frac{y}{\sqrt{x^2+y^2}}$

(i) is real, (ii) is imaginary, cannot be equal (unless $z=0$)

$\rightsquigarrow$ not differentiable for $z \neq 0$. 7

$z=0$, however, gives $\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|}{\Delta z}$ does not exist, either.

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$\rightsquigarrow$ nowhere diff'able.

(b) $f(z) = \begin{cases} z^3/\bar{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$

z^3 diff'able, $\bar{z}$ not $\rightsquigarrow$ $f(z)$ not diff'able for $z \neq 0$

(need $z^3 \neq 0$) 6

$z=0$ gives

$$f'(0) = \lim_{\Delta z \to 0} \frac{(\Delta z)^3/\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\Delta z)^2}{\overline{\Delta z}} = \lim_{\Delta z \to 0} \underbrace{\Delta z}_{\to 0} \underbrace{\left(\frac{\Delta z}{\overline{\Delta z}}\right)}_{\text{bounded}} = 0.$$

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