MAS205 Complex Variables 2005-2006

Exercises 5

Exercise 17: For each of the following functions $f(x+iy) = u(x,y) + iv(x,y)$, find the set of all points (x,y) at which u and v satisfy the Cauchy-Riemann differential equations ($\partial u/\partial x = \partial v/\partial y$ and ($\partial u/\partial y = -\partial v/\partial x$).

(a) $f(x+iy) = y^2 + ixy^2$

(b) $f(x+iy) = 2xy + 2ixy + y^3/3$.

Exercise 18: Let $f(z) = ze^z$. Write $f(z)$ as $u(x,y) + iv(x,y)$ and show that u and v satisfy the Cauchy-Riemann differential equations. Write

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and use this to express $f'(z)$ as a function of z.

Exercise 19: Use the Cauchy-Riemann differential equations to find at which values of z the following functions are differentiable. Find the derivative of the functions at these points.

(a) $f(x+iy) = 3xy^2 - x^3 + i(y^3 - 3x^2y)$

(b) $f(x+iy) = 3x^2y - x^3 + i(y^3 - 3x^2y)$

(c) $f(z) = z(z - \overline{z})^2$.

Exercise 20: Let f and g denote functions $\mathbb{C} \to \mathbb{C}$. For each question below, give either a proof or a counterexample to justify your answer.

(a) If f and g are both differentiable at z_0, does it follow that gf is continuous at z_0?

(b) If f and g are both non-differentiable at z_0, does it follow that $f+g$ is non-differentiable at z_0?

(c) If f is differentiable for all $z \in \mathbb{C}$ and g is differentiable at z_0, does it follow that $g \circ f$ is differentiable at z_0?

(d) Suppose f is discontinuous at $3+4i$, but continuous everywhere else, and g is discontinuous at $2+i$, but continuous everywhere else. Is $f-g$ differentiable at $1+3i$?

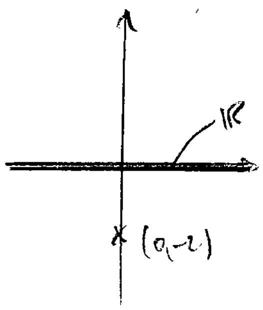
Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 16th November

Thomas Prellberg, November 2005

17)

(a) $u(x,y) = y^2$ $\qquad \frac{\partial u}{\partial x} = 0$ $\qquad \frac{\partial u}{\partial y} = 2y$

$v(x,y) = xy^2$ $\qquad \frac{\partial v}{\partial x} = y^2$ $\qquad \frac{\partial v}{\partial y} = 2xy$



CR : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \leadsto 0 = 2xy \leadsto x=0$ or $y=0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \leadsto 2y = -y^2 \leadsto y=0$ or $y=-2$

satisfied on real axis $(y=0)$ or at $(0,-2)$ (13)

(b) $u(x,y) = 2xy + y^3/3$ $\qquad \frac{\partial u}{\partial x} = 2y$ $\qquad \frac{\partial u}{\partial y} = 2x + y^2$

$v(x,y) = 2xy$ $\qquad \frac{\partial v}{\partial x} = 2y$ $\qquad \frac{\partial v}{\partial y} = 2x$

CR : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \leadsto 2y = 2x \leadsto x = y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \leadsto 2x + y^2 = -2y \leadsto x = -y - \frac{y^2}{2}$

$y = -y - \frac{y^2}{2} \leadsto y = 0$ or $y = -4$

satisfied for $(0,0)$ or $(-4,-4)$ (12)

(8) $f(z) = z e^{z}$ $\qquad z = x+iy$

$= (x+iy) e^{x+iy}$

$= e^{x}(x \cos y - y \sin y) + i e^{x}(y \cos y + x \sin y)$

$u(x,y) = e^{x}(x \cos y - y \sin y)$

$v(x,y) = e^{x}(y \cos y + x \sin y)$ $\qquad$ 7

$\frac{\partial u}{\partial x} = e^{x}(x \cos y - y \sin y + \cos y)$

$\frac{\partial u}{\partial y} = e^{x}(-x \sin y - \sin y - y \cos y)$

$\frac{\partial v}{\partial x} = e^{x}(y \cos y + x \sin y + \sin y) = -\frac{\partial u}{\partial y}$ ✓

$\frac{\partial v}{\partial y} = e^{x}(\cos y - y \sin y + x \cos y) = \frac{\partial u}{\partial x}$ ✓ $\qquad$ 4

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^{x}(x \cos y - y \sin y + \cos y$
$+ iy \cos y + ix \sin y + i \sin y)$

$= e^{x}(\cos y + i \sin y)(1 + x + iy)$

$= e^{x+iy}(1 + x + iy) = e^{z}(1+z)$

as to be expected $\qquad$ 7

19) (a) $u(x,y) = 3xy^2 - x^3$ $\quad \frac{\partial u}{\partial x} = 3y^2 - 3x^2 \quad \frac{\partial u}{\partial y} = 6xy$

$v(x,y) = y^3 - 3x^2y \quad \frac{\partial v}{\partial x} = -6xy \quad \frac{\partial v}{\partial y} = 3y^2 - 3x^2$

CR: $\frac{\partial u}{\partial x} = 3y^2 - 3x^2 = \frac{\partial v}{\partial y}$ ✓

$\frac{\partial u}{\partial y} = 6xy = -\frac{\partial v}{\partial x}$ ✓

holds for all $z \in \mathbb{C}$

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f is differentiable for all $z \in \mathbb{C}$, $f'(z) = 3y^2 - 3x^2 + i(-6xy)$

[$f(z) = -z^3$, $f'(z) = -3z^2$ in disguise]

(b) $u(x,y) = 3x^2y - x^3 \quad \frac{\partial u}{\partial x} = 6xy - 3x^2 \quad \frac{\partial u}{\partial y} = 3x^2$

$v(x,y) = y^3 - 3x^2y \quad \frac{\partial v}{\partial x} = -6xy \quad \frac{\partial v}{\partial y} = 3y^2 - 3x^2$

CR: $6xy - \cancel{3x^2} = 3y^2 - \cancel{3x^2} \leadsto y = 0$ or $y = 2x$

$3x^2 = 6xy \leadsto x = 0$ or $x = 2y$

f is differentiable only at $z = 0$, $f'(0) = 0$

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(c) $f(z) = z(z - \bar{z})^2 = (x+iy)(2iy)^2 = -4xy^2 - 4iy^3$

$u = -4xy^2 \quad \frac{\partial u}{\partial x} = -4y^2 \quad \frac{\partial u}{\partial y} = -8xy$

$v = -4y^3 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = -48y^2$

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CR: $-4y^2 = -8y^2 \leadsto y = 0 \qquad 0 = 8xy \leadsto$ ~~x=0~~ or $y = 0$

f is differentiable at $z = x \in \mathbb{R}$, $f'(x) = 0$

20)

(a) f, g diff'able at $z_0 \leadsto$ fg diff'able at z_0

$\leadsto$ fg continuous at z_0 yes! 6

(b) $f(z) = \operatorname{Re}(z)$ nowhere diff'able

$g(z) = i \operatorname{Im}(z)$ nowhere diff'able no!

but $f(z) + g(z) = z$ diff'able $\forall z \in \mathbb{C}$ 7

(c) g may not be diff'able at $f(z_0)$ no!

e.g. $f(z) = 0$, $z_0 = 2$ 6

and $g(z) = \begin{cases} z & \text{for } |z| > 1 \\ \bar{z} & \text{for } |z| \leq 1 \end{cases}$

(d) we cannot conclude differentiability from continuity! no!

example: $f(z) = \begin{cases} \bar{z} & z \neq 3+4i \\ 0 & z = 3+4i \end{cases}$ 6

$g(z) = \begin{cases} -\bar{z} & z \neq 2+i \\ 0 & z = 2+i \end{cases}$

$f(z) - g(z) = \begin{cases} 2\bar{z} & z \notin \{3+4i, 2+i\} \\ 0 & z \in \{3+4i, 2+i\} \end{cases}$