MAS205 Complex Variables 2005-2006

Exercises 6

Exercise 21: Find the radius of convergence of the following power series

$$(a)\quad \sum_{n=1}^{\infty} \frac{z^n}{(2+2i)^n}\,, \quad (b)\quad \sum_{n=0}^{\infty} (n+1)^7 z^n\,, \quad (c)\quad \sum_{n=0}^{\infty} z^n \exp(n)\,,$$

$$(d)\quad \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}\,, \quad (e)\quad \sum_{n=0}^{\infty} (-1)^n n! z^n\,.$$

Exercise 22: Give an example, if possible, of power series with the following properties:

(a) centred at $z_0 = i$, with radius of convergence $R = 0$

(b) centred at $z_0 = -2 + 2i$, with radius of convergence $R = 2$

(c) centred at $z_0 = 1$ and convergent for all z with $\Re(z) < 2$ but divergent for all z with $\Re(z) > 4$

(d) centred at $z_0 = (1+i)/\sqrt{2}$, with radius of convergence $R = \infty$

(e) centred at $z_0 = 0$ and convergent for all z with $\Im(z) = 2$ but divergent for all other $z \in \mathbb{C}$

(Proofs are not necessary, but if you can't find an example you should explain why.)

Exercise 23: Let $f(z) = (2+z)/(1-z)$. Determine the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ for

$$(a)\quad f(z) = \frac{1}{1+z} \quad \text{around } z_0 = 0,$$

$$(b)\quad f(z) = \frac{1}{1+z} \quad \text{around } z_0 = 1,$$

$$(c)\quad f(z) = \frac{1}{(1+z)(1-z)} \quad \text{around } z_0 = 0.$$

In each of the cases, give the radius of convergence of the Taylor series.

Exercise 24: Let $D = \{z : |z+3i| < 2\}$. Suppose that $f : D \to \mathbb{C}$ is defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{(z+3i)^n}{(2i)^n}\,.$$

Calculate the Taylor series for f at the point $z_0 = 0$ and determine its radius of convergence.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 23rd November

Thomas Prellberg, November 2005

21)

(a) $$\left| \frac{z^{n+1}}{(2+2i)^{n+1}} \bigg/ \frac{z^n}{(2+2i)^n} \right| = \frac{|z|}{2\sqrt{2}}$$ 5

$\rightsquigarrow$ $R = 2\sqrt{2}$

(b) $$\left|(n+2)^7 z^{n+1} \big/ (n+1)^7 z^n\right| = \frac{\left|\left(1+\frac{2}{n}\right)^7\right|}{\left|\left(1+\frac{1}{n}\right)^7\right|} |z|$$ 5

$\to |z|$ as $n \to \infty$

$\rightsquigarrow$ $R = 1$

(c) $$\left| z^{n+1} e^{n+1} \big/ z^n e^n \right| = |z| e$$ 5

$\rightsquigarrow$ $R = e^{-1}$

(d) $$\left| \frac{z^{n+1}}{(n+1)!^2} \bigg/ \frac{z^n}{n!^2} \right| = \frac{|z|}{(n+1)^2}$$ 5

$\to 0$ as $n \to \infty$

$\rightsquigarrow$ $R = \infty$

(d) $$\left| (-1)^{n+1} (n+1)! \, z^{n+1} \big/ (-1)^n n! \, z^n \right| = (n+1)|z|$$ 5

$\to \infty$ as $n \to \infty$

$\rightsquigarrow$ $R = 0$

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22)

(a) $\sum_{n=0}^{\infty} n!\,(z-i)^n$ (5)

(b) $\sum_{n=0}^{\infty} \left(\frac{z-(-2+2i)}{2}\right)^n$ (5)

(c) impossible, domain of convergence needs to be disk (5)

(d) $\sum_{n=0}^{\infty} \frac{1}{n!}\left(z-\frac{1+i}{\sqrt{2}}\right)^n$ (5)

(d) impossible, domain of convergence needs to be disk (5)

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23)

(a) $f(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} \underset{|z|<1}{=} \sum_{n=0}^{\infty}(-z)^n$

$= \sum_{n=0}^{\infty}(-1)^n z^n$ for $|z|<1$, i.e. $R=1$ (8)

(b) $$f(z) = \frac{1}{1+z} = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{1-\frac{1-z}{2}}$$ (9)

$$\underset{\left|\frac{1-z}{2}\right|<1}{=} \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1-z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(1-z)^n$$

for $|z-1| < 2$ i.e. $R=2$

(c) $$f(z) = \frac{1}{(1+z)(1-z)} = \frac{1}{1-z^2} \underset{|z^2|<1}{=} \sum_{n=0}^{\infty}(z^2)^n$$ (8)

$$= \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + \dots$$ for $|z|<1$ i.e. $R=1$

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24)

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{z+3i}{2i}\right)^n$$ geom series for $\left|\frac{z+3i}{2i}\right|<1$

$$\leadsto f(z) = \frac{1}{1-\frac{z+3i}{2i}} = \frac{2i}{2i-z-3i} = \frac{2i}{-i-z} = \frac{-2}{1-iz}$$ (7) (7)

$$= -2\frac{1}{1-iz} \underset{|z|<1}{=} -2\sum_{n=0}^{\infty}(iz)^n = \sum_{n=0}^{\infty} 2(-1)^{n+1} i^n z^n$$ (7)

for $|z|<1$

i.e. $R=1$ (4)

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