MAS205 Complex Variables 2005-2006

Exercises 7

Exercise 25: Let

$$f(z) = \frac{1}{(z-1)(z+2)} \, .$$

(a) Compute the partial fraction decomposition of f.

(b) Write down the Laurent series for f on $\{z : |z| > 2\}$.

(c) Write down the Laurent series for f on a punctured disk centred at $z_0 = 1$. For what values of z does this series converge?

Exercise 26: Find the Laurent series of the function

$$f(z) = \frac{1}{(z-1)(z+2)^2}$$

on a punctured disk centred at the point $z_0 = -2$.
Where is this Laurent series valid (i.e. absolutely convergent)?
What is the principal part of this Laurent series?
What type of singularity does f have at $z_0 = -2$?
What is the residue of f at $z_0 = -2$?

Exercise 27: Locate the singularities for each of the following functions, and determine the nature of each singularity:

$$(a) \quad \frac{1}{z^4+4} \qquad (b) \quad \frac{1}{(z+1)^3} - e^{-1/z} \qquad (c) \quad z(e^{-1/z}+1) \qquad (d) \quad \frac{\sin(z^3)}{z^3}$$

Exercise 28: (a) List the singularities of the function $f(z) = e^z/(z^2+\pi^2)$ and determine the nature of each singularity. Compute the residue of f at each singularity.

(b) List the singularities of the function $f(z) = z^5 e^{1/z}$ and determine the nature of each singularity. Compute the residue of f at each singularity.

Note: determining the type of singularity means finding out whether it is a pole (if so, which order?), an essential singularity, or a removable singularity.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 30th November

Thomas Prellberg, November 2005

25) (a) $$f(z) = \frac{1}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$\rightsquigarrow \quad 1 = A(z+2) + B(z-1) = (A+B)z + (2A-B)$$

$$\rightsquigarrow \quad A+B=0, \quad 2A-B=1 \quad \rightsquigarrow \quad A=\frac{1}{3}, \; B=-\frac{1}{3}$$

$$f(z) = \frac{1}{3}\cdot\frac{1}{z-1} - \frac{1}{3}\,\frac{1}{z+2}$$ (5)

(b) $$|z|>2: \quad f(z) = \frac{1}{3z}\left(\frac{1}{1-1/z} - \frac{1}{1-\left(\frac{-2}{z}\right)}\right)$$

$$= \frac{1}{3z}\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \text{(3)} - \sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^n \text{(3)}\right)$$

$$= \sum_{n=\cancel{0}\,1}^{\infty} \frac{1^n - (-2)^n}{3z^{n+1}} = \sum_{n=2}^{\infty} \frac{1-(-2)^{n-1}}{3} z^{-n}$$ (4)

$$\left[= \frac{1-(-2)}{3} z^{-2} + \frac{1-(-2)^2}{3} z^{-3} + \frac{1-(-2)^3}{3} z^{-4} + \ldots \right.$$

$$\left. = z^{-2} - z^{-3} + 3z^{-4} + \ldots \right]$$ (5) for solution like this (instead of 10)

(c) $$z_0 = 1: \quad f(z) = \frac{1}{3}\,\frac{1}{z-1} - \frac{1}{3}\,\frac{1}{3+(z-1)}$$

$$= \frac{1}{3}\,\frac{1}{z-1} - \frac{1}{9}\,\frac{1}{1-\left(\frac{1-z}{3}\right)} = \frac{1}{3(z-1)} - \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{1-z}{3}\right)^n$$

$$= \frac{1}{3}\,\frac{1}{z-1} \text{(3)} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{3^{n+2}} (z-1)^n \text{(4)}$$

$$\left[= \frac{1}{3}\,\frac{1}{z-1} - \frac{1}{9} + \frac{1}{27}(z-1) - \frac{1}{81}(z-1)^2 + \ldots \right]$$

converges for $1 < |z-1| < 3$ (3)

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2c)

$$f(z) = \frac{1}{(z-1)(z+2)^2} = \frac{1}{(z+2)^2} \frac{1}{(z+2)-3}$$

$$= \frac{1}{(z+2)^2}\left(-\frac{1}{3}\right)\frac{1}{1-\frac{z+2}{3}}$$

$$= -\frac{1}{3}\frac{1}{(z+2)^2}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{(z+2)^{n-2}}{3^{n+1}}$$

$$= -\frac{1}{3(z+2)^2} - \frac{1}{9(z+2)} - \sum_{n=2}^{\infty}\frac{(z+2)^{n-2}}{3^{n+1}}$$

$$= -\frac{1}{3}(z+2)^{-2} - \frac{1}{9}(z+2)^{-1} + \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+3}}\right)(z+2)^n$$

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convergent for $0<|z+2|<3$ 4

principal part $-\frac{1}{3}(z+2)^{-2} - \frac{1}{9}(z+2)^{-1}$ 4

pole of order 2 at $z_0 = -2$ 4

residue $-\frac{1}{9}$ at $z_0 = -2$ 4

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27)

(a) simple poles at $1+i,\ 1-i,\ -1+i,\ -1-i$ 8

(b) pole of order 3 at -1 7

ess. sing at 0

(c) ess. sing at 0 5

(d) remov. sing at 0 $\left(f(z) = \frac{1}{z^3}\left(z^3 - \frac{z^9}{3!} + \dots\right)\right)$ 5

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28) (a) $f(z) = \dfrac{e^z}{z^2+\pi^2} = \dfrac{1}{2\pi i}\left(\dfrac{1}{z-i\pi} - \dfrac{1}{z+i\pi}\right)e^z$

simple poles at $z = \pm i\pi$ 5

residue at $i\pi$: $\dfrac{1}{2\pi i} e^{i\pi} = \dfrac{i}{2\pi}$ 5

residue at $-i\pi$: $\dfrac{1}{2\pi i} e^{-i\pi} = -\dfrac{i}{2\pi}$ 5

(b) $f(z) = z^5 e^{1/z} = z^5 \sum_{n=0}^{\infty} \dfrac{1}{n!}\dfrac{1}{z^n}$

ess sing at 0 5

residue $\dfrac{1}{4!} = \dfrac{1}{24}$ 5

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