MAS205 Complex Variables 2005-2006

Exercises 8

Exercise 29: Let f and g be holomorphic on a disk D centred at z_0, and let h be holomorphic on the punctured disk $D' = D \setminus \{z_0\}$. Suppose f and g both have zeros of order $m \geq 1$ at z_0 and h has a pole of order $n \geq 1$ at z_0.

(a) Does fh have a zero or pole at z_0? If so, what is its order?

(b) Does $f + g$ have a zero or pole at z_0? If so, what is its order?

Exercise 30: Let the curve C be given by the graph of the function $y = f(x)$ with

$$f(x) = \cosh(x) \qquad (-1 \leq x \leq 1)$$

embedded in $\mathbb{C}$ via $z = x + iy$.

(a) Give a path $\gamma : [a, b] \to \mathbb{C}$ which has the curve C as its image. Draw a sketch of the curve and indicate the parametrisation.

(b) Compute the length $L(C)$. Evaluate the result numerically and discuss it in view of your sketch (i.e. does your result make sense and why).

Exercise 31: Let C be the unit circle described counterclockwise. Show that

$$\left| \int_C \frac{e^z}{z^3} dz \right| \leq 2\pi e \, .$$

Exercise 32: Using the definition of the integral of a complex function f along a contour $\gamma : [a, b] \to \mathbb{C}$ as

$$\int_a^b f(\gamma(t))\gamma'(t)dt \, ,$$

compute the integral of $f(z) = (z - 1)^2$ along

(a) the straight line segment from 0 to i,

(b) the straight line segment from i to $1 + i$,

(c) [**10 bonus marks**] an arc from 0 to $1 + i$ on a circle of radius 1 about 1.

Check your answers by finding an antiderivative F for f and evaluating F at the points $z = 0, i, 1 + i$.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 7th December

Thomas Prellberg, November 2005

29)

$f(z) = F(z)\,(z-z_0)^m \quad F(z_0) \neq 0 \quad F$ holom on D ③

$g(z) = G(z)\,(z-z_0)^m \quad G(z_0) \neq 0 \quad G$ holo on D ③

$h(z) = H(z)\,(z-z_0)^{-n} \quad H(z_0) \neq 0 \quad H$ holom on D ③

(a) $f(z)\,h(z) = F(z)\,H(z)\,(z-z_0)^{m-n}$

$F(z_0)\,H(z_0) \neq 0$, FH holom on D ③

$m>n$: zero of order $m-n$ ②

$m=n$: neither zero nor pole ②

$m<n$: pole of order $n-m$ ②

(b) $f(z)+g(z) = [F(z)+G(z)]\,(z-z_0)^m$

$F(z_0)+G(z_0)$ <u>may be</u> zero, $F+G$ holom on D ④

zero of <u>at least</u> order m ③

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30) a) $\gamma(t) = t + i\cosh t \qquad t \in [-1,1]$

(different parametrisation possible) (6)

b) $\gamma'(t) = 1 + i\sinh t$

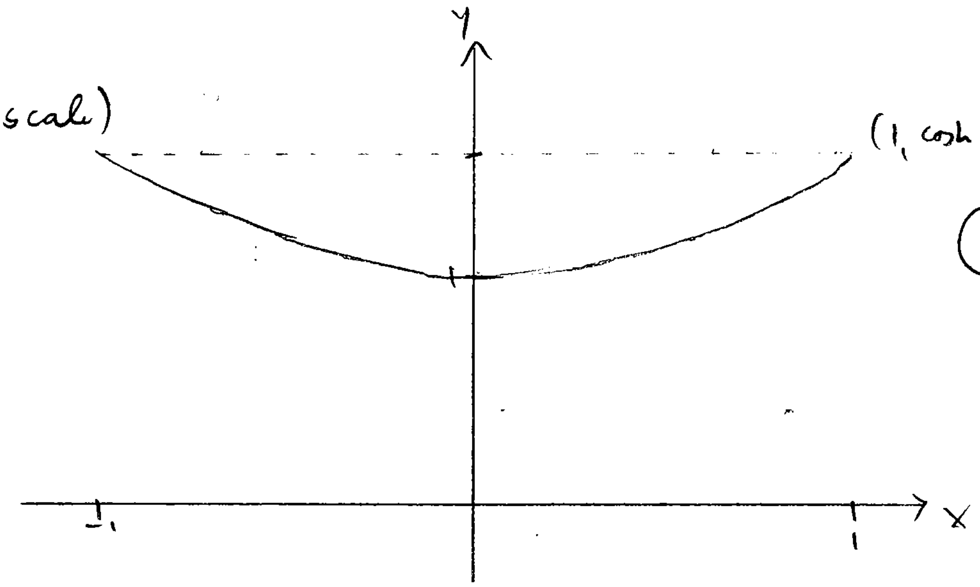
$|\gamma'(t)|^2 = 1 + \sinh^2 t = \cosh^2 t$ (6)

$\rightsquigarrow |\gamma'(t)| = \cosh t$

$$L(\mathcal{C}) = \int_{-1}^{1} \cosh t \, dt = \sinh t \Big|_{-1}^{1} = 2\sinh 1$$ (4)

$$= e - e^{-1} \approx 2.35$$

Sketch: (not to scale)



(5)

we should find $\quad 2 \leq L(\mathcal{C}) \leq 2\cosh 1 \approx 3.08$ (4)

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31) $\gamma(t) = e^{it} \qquad 0 \leq t \leq 2\pi$

$\gamma'(t) = i e^{it}$

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$$\left| \int_{C} \frac{e^{e^z}}{z^3}\, dz \right| = \left| \int_0^{2\pi} e^{e^{it}} \frac{i e^{it}}{e^{3it}}\, dt \right|$$

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$$\leq \underbrace{\int_0^{2\pi} dt}_{L} \times \underbrace{\max_{0 \leq t \leq 2\pi} \left| e^{e^{it}} \cancel{ie^{-2it}} \right|}_{M}$$

$L = 2\pi$ and

$$M = \max_{0 \leq t \leq 2\pi} \left| e^{e^{it}} \right|$$

now $\left| e^{e^{it}} \right| = \left| e^{\cos t + i \sin t} \right| = \left| e^{\cos t} \right|$

so that $M = \max_{0 \leq t \leq 2\pi} \left| e^{\cos t} \right| = e$

and therefore

$$\left| \int_C \frac{e^z}{z^3}\, dz \right| \leq LM = 2\pi e$$

10

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32) $f(z) = (z-1)^2$

(a) $\gamma(t) = it \qquad 0 \le t \le 1 \qquad \gamma'(t) = i$ 5

$$\int_{[0,i]} f(z)\,dz = \int_0^1 (it-1)^2\, i\,dt = \frac{(it-1)^3}{3}\Bigg|_0^1 = \frac{(i-1)^3}{3} - \frac{(-1)^3}{3}$$

$$= \frac{\overset{-i}{\cancel{i^3}} - 3i^2 + 3i \cancel{-1} \cancel{+1}}{3} = 1 + \frac{2}{3} i$$ 5

(b) $\gamma(t) = i+t \qquad 0 \le t \le 1 \qquad \gamma'(t) = 1$ 5

$$\int_{[i,i+1]} f(z)\,dz = \int_0^1 (i+t-1)^2\, 1\,dt = \frac{(i+t-1)^3}{3}\Bigg|_0^1 = \frac{i^3}{3} - \frac{(i-1)^3}{3}$$

$$= \frac{\cancel{i^3} - \cancel{i^3} + 3i^2 - 3i + 1}{3} = -\frac{2}{3} - i$$ 5

(c) $\gamma(t) = 1 - e^{-it} \qquad 0 \le t \le \frac{\pi}{2} \qquad \gamma'(t) = +i e^{-it}$ 5

$$\int_{arc} f(z)\,dz = \int_0^{\pi/2} \left(\cancel{1} - e^{-it} \cancel{-1}\right)^2 \left(+i e^{-it}\right) dt$$

Bonus!

$$= \int_0^{\pi/2} e^{-3it} (+i)\,dt = -\frac{1}{3} e^{-3it}\Bigg|_0^{\pi/2} = -\frac{1}{3} e^{-\frac{3}{2} i\pi} + \frac{1}{3} e^0$$

$$= \frac{1}{3} - \frac{1}{3} i$$ 5

Antiderivative $F(z) = \dfrac{(z-1)^3}{3}$ $\rightsquigarrow$ $F'(z) = f(z) = (z-1)^2$

$F(0) = -\frac{1}{3}$

$F(i) = \dfrac{(i-1)^3}{3} = \frac{2}{3} + \frac{2}{3} i$

$F(1+i) = \dfrac{i^3}{3} = -\frac{1}{3} i$

(a) $F(i) - F(0) = 1 + \frac{2}{3} i$ ✓

(b) $F(1+i) - F(i) = -\frac{2}{3} - i$ ✓

(c) $F(1+i) - F(0) = \frac{1}{3} - \frac{1}{3} i$ ✓

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