MAS205 Complex Variables 2005-2006

Exercises 9

Exercise 33: Let $f(z) = \Re(z)$. Find the values of $\int_{C_k} f(z)dz$ where

(a) C_1 denotes the straight line from $z_0 = 3$ to $z_1 = 3i$,

(b) C_2 denotes the arc from $z_0 = 3$ to $z_1 = 3i$ along a circle of radius 3 about the origin.

Find a simple closed contour C for which $\int_C f(z)dz \neq 0$.

Exercise 34: By applying Cauchy's theorem (or otherwise) show that $\int_C f(z)dz = 0$ where C is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and

$$(a) \quad f(z) = \frac{1}{z^2+5} \qquad (b) \quad f(z) = \frac{1}{z^2 - 2iz - 5} \qquad (c) \quad f(z) = \tanh z$$

Exercise 35: Use the Cauchy integral formula to evaluate each of the following integrals, where C is the positively oriented circle $\{z \in \mathbb{C} : |z-3| = 1/2\}$:

(a)

$$\int_C \frac{(z+2)^3}{(z-3)z^2} dz$$

(b)

$$\int_C \frac{\exp z}{(z-\pi)\cos z} dz$$

Exercise 36: Use the residue theorem to calculate

(a)

$$\int_C \frac{1}{(z^2+4)(z+2+2i)} dz$$

for $C = C_1$, the positively oriented circle of radius 3 centred at 1, and
for $C = C_2$, the positively oriented square with corners $-3-3i$ and $1+i$;

(b)

$$\int_C \frac{1}{z(z^2-4)(z-2)} dz$$

where C is the positively oriented circle of radius 2 centred at 1.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 10:30am Wednesday 14th December

Thomas Prellberg, December 2005

33) $f(z) = \operatorname{Re}(z) = x$

(a) $\gamma_1(t) = 3 + (3i-3)t = 3(1-t) + 3it \qquad t\in[0,1]$

$\gamma_1'(t) = 3i-3$ (5)

$$\int_{C_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt = \int_0^1 3(1-t)(3i-3)\,dt$$

$$= 9(i-1)\int_0^1 (1-t)\,dt = \frac{9}{2}(i-1) = -\frac{9}{2} + \frac{9}{2}i$$ (5)

(b) $\gamma_2(t) = 3e^{it} = 3\cos t + 3i\sin t \qquad t\in[0,1]$

$\gamma_2'(t) = 3ie^{it} = -3\sin t + 3i\cos t$ (5)

$$\int_{C_2} f(z)\,dz = \int_0^{\pi/2} f(\gamma_2(t))\gamma_2'(t)\,dt = \int_0^{\pi/2} 3\cos t\,(-3\sin t + 3i\cos t)\,dt$$

$$= -9\int_0^{\pi/2} \sin t\cos t\,dt + 9i\int_0^{\pi/2} \cos^2 t\,dt$$

$$= -9\,\frac{1}{2}\sin^2 t\Big|_0^{\pi/2} + 9i\left(\frac{1}{2}\sin t\cos t - \frac{t}{2}\right)\Big|_0^{\pi/2}$$

$$= -\frac{9}{2} + \frac{9\pi}{2}i$$ (5)

let $C = C_1 - C_2$ then $\int_C \operatorname{Re}(z)\,dz = \int_{C_1}\operatorname{Re}(z)\,dz - \int_{C_2}\operatorname{Re}(z)\,dz$

$$= \frac{9}{2}(1-\pi)i \neq 0$$ (5) /25

34)

(a) $f(z)$ has isolated singularities at $\pm i\sqrt{5}$ (outside C); $f(z)$ is holomorphic on and inside C $\leadsto$ $\int_C f(z)\,dz = 0$ 6

(b) $f(z)$ has isolated singularities at $z_{1,2} = i \mp \sqrt{i^2+5} = \mp 2 + i$, $f(z)$ is holomorphic on and inside C $\leadsto$ $\int_C f(z)\,dz = 0$ 7

(c) $f(z)$ has isolated singularities at $z = \left(\frac{1}{2} + k\pi\right)i$ $k \in \mathbb{Z}$

$f(z)$ is holomorphic on and inside C $\leadsto$ $\int_C f(z)\,dz = 0$ 7

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35) (a) $f(z) = \dfrac{(z+2)^2}{(z-3)z^2} = \dfrac{\phi(z)}{z-3}$ with $\phi(z) = \dfrac{(z+2)^2}{z^2}$ 4

and $\phi(z)$ holomorphic on and inside C (pole at zero) 3

$$\int_C f(z)\,dz = 2\pi i\,\phi(3) = 2\pi i\,\frac{(3+2)^2}{2^2} = \frac{25}{2}\pi i$$ 5

(b) $f(z) = \dfrac{e^z}{(z-\pi)\cos z} = \dfrac{\phi(z)}{z-\pi}$ with $\phi(z)$ $\dfrac{e^z}{\cos z}$ (poles at $(k+\frac{1}{2})\pi$ $k \in \mathbb{Z}$) 4

and $\phi(z)$ holomorphic on and inside C 3

$$\int_C f(z)\,dz = 2\pi i\,\phi(\pi) = 2\pi i\,\frac{e^\pi}{\cos\pi} = 2\pi e^\pi i$$ 6

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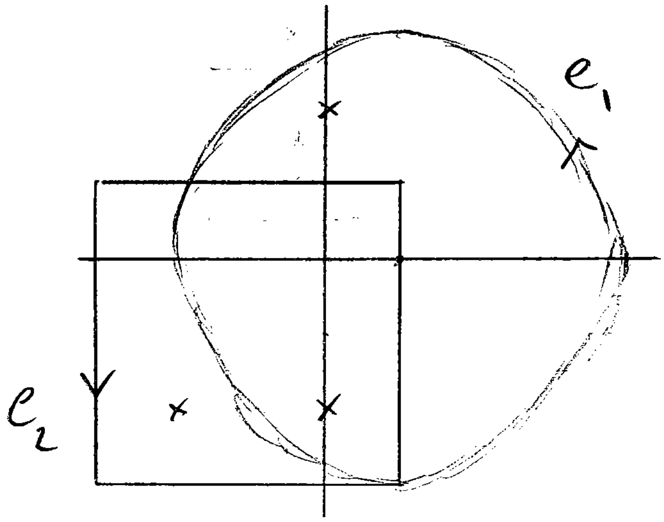
36) (a) $f(z) = \dfrac{1}{(z^2+4)(z+2+2i)}$ holomorphic except for simple poles at $\pm 2i$, $-2-2i$ (3)



$$f(z) = \frac{1}{(z-2i)(z+2i)(z+2+2i)}$$

$$\mathrm{Res}_{-2i} f = \frac{1}{(-2i-2i)(-2i+2+2i)} = \frac{i}{8} \quad (3)$$

$$\mathrm{Res}_{2i} f = \frac{1}{(2i+2i)(2i+2+2i)} = -\frac{i}{8}\,\frac{1}{1+2i} = -\frac{1}{20} + \frac{i}{40} \quad (3)$$

$$\mathrm{Res}_{-2-2i} f = \frac{1}{(-2-2i-2i)(-2-2i+2i)} = \frac{1}{4}\,\frac{1}{1+2i} = \frac{1}{20} - \frac{i}{10} \quad (3)$$

$$\int_{\mathcal{C}_1} f(z)\,dz = 2\pi i \left(\mathrm{Res}_{-2i} f + \mathrm{Res}_{2i} f \right) = 2\pi i \left(-\frac{1}{20} + \frac{i}{10} \right) \quad (3)$$

$$\int_{\mathcal{C}_2} f(z)\,dz = 2\pi i \left(\mathrm{Res}_{-2i} f + \mathrm{Res}_{-2-2i} f \right) = 2\pi i \left(\frac{1}{20} - \frac{i}{40} \right) \quad (3)$$

(b) $f(z) = \dfrac{1}{z(z^2-4)(z-2)} = \dfrac{1}{z(z-2)^2(z+2)}$ holom except

for simple poles at $0, -2$ and double pole at 2. (3)

$$\int_{\mathcal{C}} f(z)\,dz = \mathrm{Res}_0 f + \mathrm{Res}_2 f$$

$z=0$: $\quad f(z) = \dfrac{\phi(z)}{z}$ with $\phi(z) = \dfrac{1}{(z-2)^2(z+2)}$

$$\operatorname{res}_0 f = \phi(0) = \frac{1}{(-2)^2 2} = \frac{1}{8}$$ (3)

$z=2$: $\quad f(z) = \dfrac{\phi(z)}{(z-2)^2}$ with $\phi(z) = \dfrac{1}{z(z+2)} = \dfrac{1}{z^2+2z}$

$$\operatorname{res}_2 f = \frac{\phi'(2)}{1!} = -\frac{2\cdot 2+2}{(2^2+2)^2} = -\frac{6}{8^2} = -\frac{3}{32}$$ (3)

$\left[\ \phi'(z) = \dfrac{-(2z+2)}{(z^2+2z)^2}\ \right]$

$\leadsto \displaystyle\int_{\mathcal{C}} f(z)\,dz = 2\pi i\left(\frac{1}{8} - \frac{3}{32}\right) = \frac{\pi i}{16}$ (3)

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