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Examiner: Dr T Prellberg

MAS205 COMPLEX VARIABLES
MID-TERM TEST

Date: 9-11-2005 *Time:* 16.00-17.00

Complete the following information:

Name	
Student Number (9 digit code)	

The test has FOUR questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish marked.

Question	Marks
A	
B	
C	
D	
Total Marks	

Nothing on this page will be marked!

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Question A. [*25 marks*]

a Show that $z + \bar{z} = 2\Re(z)$ and $z - \bar{z} = 2i\Im(z)$ for any $z \in \mathbb{C}$. [*10 marks*]

b Describe the set of points $z \in \mathbb{C}$ satisfying:
(i) $|z + i| = 1$, (ii) $\Re(z + i) = 1$, and (iii) $z + i = 1$. [*15 marks*]

Answer A.

(a) let $z = x + iy$, so $Re(z) = x$, $Im(z) = y$

$z + \bar{z} = x + iy + \overline{x+iy} = x + iy + x - iy$

$= 2x = 2 Re(z)$ (5)

$z - \bar{z} = x + iy - \overline{x-iy} = x + iy - x + iy$

$= 2iy = 2i\, Im(z)$ (5)

(b) let $z = x + iy$

(i) $|z + i| = 1 \Leftrightarrow |x + iy + i| = 1 \Leftrightarrow x^2 + (y+1)^2 = 1$

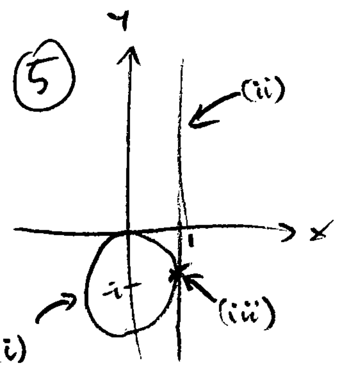
circle with radius 1 about $-i$ (5)

(ii) $Re(z + i) = 1 \Leftrightarrow Re(x + iy + i) = x = 1$

line through 1 parallel to imaginary axis (5)

(iii) $z + i = 1 \Leftrightarrow x = 1, y = -1$

point $(1, -1)$ (5)



Answer A. (*Continue*)

Question B. [*25 marks*]

a Find all complex solutions of $e^z = -i$. [*10 marks*]

b Show that under the map $w = z^2$, the vertical line $\Re(z) = 1$ is mapped to the parabola given by $u = 1 - v^2/4$, where $w = u + iv$. [*15 marks*]

Answer B.

(a) $e^z = -i = e^{\frac{3}{2}\pi i + 2k\pi i}$ $\quad k \in \mathbb{Z}$

thus $z = \left(\frac{3}{2} + 2k\right)\pi i$ $\quad k \in \mathbb{Z}$ (10)

(b) $w = z^2$ $\quad z = x + iy$

$w = x^2 - y^2 + 2ixy = u + iv$

$u = x^2 - y^2$, $\quad v = 2xy$ (5)

$\mathrm{Re}(z) = 1 \rightsquigarrow x = 1$

$u = 1 - y^2$, $\quad v = 2y$ (5)

eliminate $y \rightsquigarrow y = \frac{v}{2}$

$u = 1 - \left(\frac{v}{2}\right)^2 = 1 - v^2/4$ (5)

Answer B. (*Continue*)

Question C. [*25 marks*]

a Find the Möbius transformation $f(z) = (az+b)/(cz+d)$ which maps $0 \to 0$, $-1 \to -2$, and $1 \to 2i$. [*15 marks*]

b Which circle does the real axis get mapped to under the transformation f? [*10 marks*]

Answer C.

(a) $0 = \frac{a0+b}{c0+d} = \frac{b}{d} \leadsto b = 0$

$-2 = \frac{a(-1)+\cancel{b}}{c(-1)+d} = \frac{a}{c-d} \leadsto a = 2d - 2c$

$2i = \frac{a1+\cancel{b}}{c1+d} = \frac{a}{c+d} \leadsto a = 2id + 2ic$

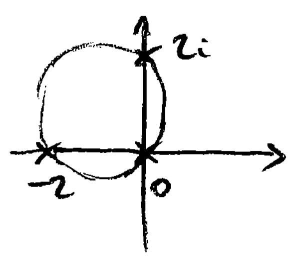
$\leadsto \cancel{2}d - \cancel{2}c = \cancel{2}id + \cancel{2}ic \qquad (1-i)d = (1+i)c \leadsto d = \frac{1+i}{1-i}c$

$a = 2d - 2c = 2\left(\frac{1+i}{1-i} - 1\right)c = \frac{4i}{1-i}c$

$f(z) = \frac{\frac{4i}{1-i}cz}{cz + \frac{1+i}{1-i}} = \frac{4iz}{(1-i)z + 1+i}$ (not unique) (15)

test: $0 \to 0$ ✓ $\quad 1 \to \frac{4i}{1-i+1+i} = 2i$ ✓ $\quad -1 \to \frac{-4i}{-(1-i)+1+i} = \frac{-4i}{2i} = -2$ ✓

(b)



unique circle through $0, -2, 2i$

circle with radius $\sqrt{2}$ about $-1+i$ (10)

Answer C. (*Continue*)

Question D. [*25 marks*]

a Show that $f(z) = \Re(z)$ is continuous at $z = 0$. [*10 marks*]

b Show that $f(z) = |z|^2$ is differentiable at $z = 0$ and that $f'(0) = 0$. [*15 marks*]

Answer D.

(a) $$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \Re(x+iy) = \lim_{\substack{x \to 0 \\ y \to 0}} x$$

$$= \lim_{x \to 0} x = 0 = f(0)$$

(10)

Thus $f(z)$ continuous at $z=0$

(b) $$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - 0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z \overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$

(15)

Thus $f(z)$ differentiable at $z=0$ and $f'(0)=0$

Answer D. (*Continue*)