Atmospheric turbulence and superstatistics

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In equilibrium statistical mechanics, the inverse temperature β is a constant system parameter—but many nonequilibrium systems actually exhibit spatial or temporal temperature fluctuations on a rather large scale. Think, for example, of the weather: It is unlikely that the temperature in London, New York, and Firenze is the same at the same time. There are spatio-temporal temperature fluctuations on a rather large scale, though locally equilibrium statistical mechanics with a given fixed temperature is certainly valid. A traveller who frequently travels between the three cities sees a 'mixture' of canonical ensembles corresponding to different local temperatures. Such type of macroscopic inhomogenities of an intensive parameter occur not only for the weather but for many other driven nonequilibrium systems as well. There are often certain regions where some system parameter has a rather constant value, which then differs completely from that in another spatial region. In general the fluctuating parameter need not be the inverse temperature but can be any relevant system parameter. In turbulent flows, for example, a very relevant system parameter is the local energy dissipation rate ϵ, which, according to Kolmogorov's theory of 1962 [1], exhibits spatio-temporal fluctuations on all kinds of scales. Nonequilibrium phenomena with macroscopic inhomogenities of an intensive parameter can often be effectively described by a concept recently introduced as 'superstatistics' [2]. This concept is quite general and has been successfully applied to a variety of systems, such as hydrodynamic turbulence, atmospheric turbulence, pattern formation in Rayleigh-Benard flows, cosmic ray statistics, solar flares, networks, and models of share price evolution [3]. For a particular probability distribution of large-scale fluctuations of the relevant system parameter, namely the Gamma-distribution, the corresponding superstatistics reduces to Tsallis statistics [4], thus reproducing the generalized canonical distributions of nonextensive statistical mechanics by a plausible physical mechanism based on fluctuations.

In this article we want to illustrate the general concepts of superstatistics by a recent example: atmospheric turbulence. Rizzo and Rapisarda [5, 6] analysed the statistical properties of turbulent wind velocity fluctuations at Florence Airport. The data were recorded by two head anemometers A and B on two poles 10 m high a distance 900 m apart at a sampling frequency of 5 minutes. Components of spatial wind velocity differences at the two anemometers A and B as well as of temporal wind velocity differences at A were investigated.

Analysing these data, two well separated time scales can be distinguished. On the one hand, the temporal velocity difference $u(t) = v(t + \delta) - v(t)$ (as well as the spatial one)

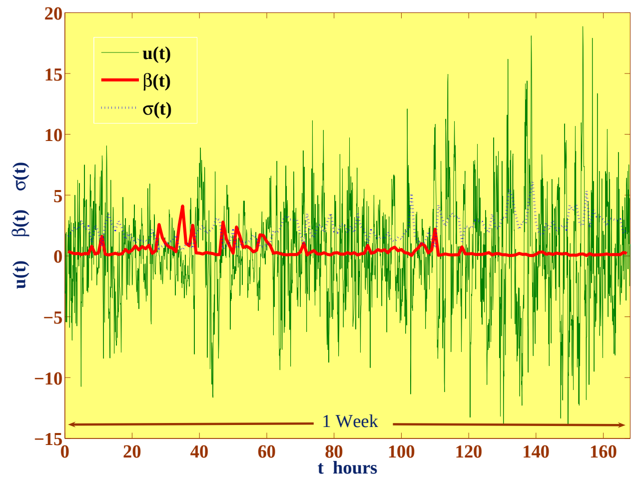

FIG. 1: Time series of a temporal wind velocity difference $u(t)$ ($\delta = 60$ min) recorded by anemometer A every 5 min for one week (green line) and the corresponding parameter $\beta(t)$ (red line), as well as the corresponding standard deviation $\sigma(t)$ (blue dotted line), both for a 1 hour window.

fluctuates on the rather short time scale τ (see Fig. 1). On the other hand, we may also look at a measure of the average activity of the wind bursts in a given longer time interval, say 1 hour, where the signal behaves approximately in a Gaussian way. The variance of the signal $u(t)$ during that time interval is given by $\sigma^2 = \langle u^2 \rangle - \langle u \rangle^2$, where $\langle ... \rangle$ means taking the average over the given time interval. We then define a parameter $\beta(t)$ by the inverse of this local variance (i.e. $\beta = 1/\sigma^2$). β depends on time t, but in a much slower way than the original signal. Both signals are displayed in Fig. 1. One clearly recognizes that the typical time scale T on which β changes is much larger than the typical time scale τ where the velocity (or velocity difference) changes.

Dividing the wind flow region between A and B into spatial cells, so that air flows from one cell to another, one assumes that each cell is characterized by a different value of the local variance parameter β, which plays a similar role as the inverse temperature in Brownian motion and fluctuates on the relatively long spatio-temporal scale T. As mentioned before, one can then distinguish two well separated time scales for the wind through the cells: a short time scale τ which allows velocity differences u to come to local equilibrium described by local Gaussians $\sim \exp[-\beta \frac{1}{2} u^2]$, and a long time scale T, which characterizes the long time secular fluctuations of β over many cells. Similar fluctuations of a local variance parameter are also observed in financial time series, e.g. for share price indices, and come under the heading ‘volatility fluctuations’ [7].

A terrestrial example would be a Brownian particle of mass m moving from cell to cell in an inhomogeneous fluid environment characterized by an inverse temperature β which varies slowly from cell to cell. The two time scales are then the short local time scale τ on which the Brownian particle reaches local equilibrium and a long global time scale over which β changes significantly. If the particle moves for a sufficiently long time through the fluid then it samples, in the cells it passes through, values of β distributed according to a probability density function $f(\beta)$, which leads to a resulting long-term probability distribution $p(v)$ to find the Brownian particle in the fluid with velocity v given by $p(v) \sim \int e^{-\frac{\beta}{2}mv^2} f(\beta) d\beta$. This is like a superposition of two statistics in the sense that $p(v)$ is given by an integral over local statistics given by the local equilibrium Boltzmann statistics convoluted with the statistics $f(\beta)$ of the β occurring in the Boltzmann statistics. In other words, it is a ‘statistics of a statistics’ or a superstatistics.

Returning now to the atmospheric experiment, it is this superstatistics which is employed here to analyse the wind data. However, there is a fundamental difference in the interpretation of the corresponding variables: First of all, the variable v (the velocity of the Brownian particle) corresponds to the longitudinal velocity *difference* in the flow (either spatial or temporal), not the velocity itself. Secondly, since we are analyzing turbulent velocity fluctuations and not thermal ones, the parameter β is a local variance parameter of the macroscopic turbulent fluctuations and hence it does not have the physical meaning of an inverse temperature as given by the actual temperature at the airport. Rather, it is much more related to a suitable power of the local energy dissipation rate ϵ.

The fluctuations of the variance parameter β can be analysed using time windows of different lengths. Rizzo and Rapisarda carried this out for two time series of interest: for the temporal fluctuations of the wind velocity component (in the x-direction) as recorded at the anemometer A and also the spatial fluctuations as given by the longitudinal wind velocity differences between the anemometers A and B. The probability distribution of β as obtained for the temporal case is shown in Fig. 2 for a time window of 1 hour. For comparison, the dashed (blue) line shows a Gamma or χ^2-distribution function, which is of the general form $f(\beta) \sim \beta^{c-1}e^{-\beta/b}$, with b and c appropriate constants. The solid (red) line represents a lognormal distribution function which is of the general form $f(\beta) \sim (1/\beta s) \exp[-(\log(\beta/\mu))^2/(2s^2)]$, with μ and s appropriate constants. Apparently,

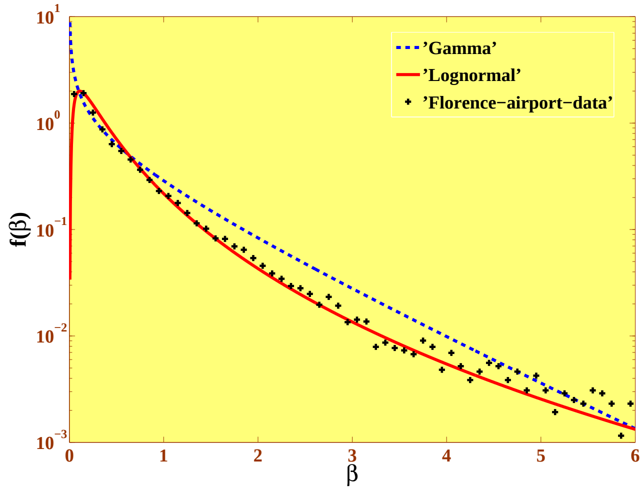

FIG. 2: Rescaled probability density of the fluctuating parameter β, as obtained for the Florence airport data. Also shown is a Gamma distribution (dashed blue line) and a lognormal distribution (solid red line) sharing the same mean and variance as the data. The data are reasonably well fitted by the lognormal distribution.

the data are reasonably well fitted by a lognormal distribution (note that a different conclusion was reached by Rizzo and Rapisarda in [5, 6]). We see that our result for atmospheric turbulence is similar to laboratory turbulence experiments on much smaller space and time scales, such as a turbulent Taylor-Couette flow as generated by two rotating cylinders. For Taylor-Couette flow it has been shown [8] that β is indeed lognormally distributed, see Fig. 3.

In general, for a given nonequilibrium system the probability density of the parameter β is ultimately determined by the underlying spatio-temporal dynamics of the system under consideration.

The Gamma distribution results if β can be represented by a <u>sum</u> of n independent squared Gaussian random variables X_i^2 (with $i = 1, ..., n$) with mean zero, i.e. $\beta = \sum_{i=1}^n X_i^2 > 0$. The constants c and b above are related to n.

The lognormal distribution results if β is due to a multiplicative cascade process, i.e. if it can be represented by a <u>product</u> of n independent positive random variables ξ_i, i.e. $\beta = \Pi_{i=1}^n \xi_i$ or $\log \beta = \sum_{i=1}^n \log \xi_i$. Due the Central Limit Theorem, under suitable rescaling the latter sum will become Gaussian for large n. But if $\log \beta$ is Gaussian this means that β is lognormally distributed.

We notice that the difference between the Gamma distribution and the lognormal distri-

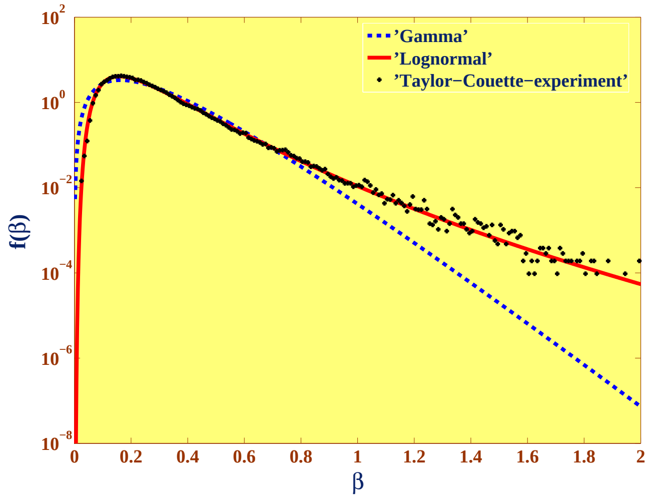

FIG. 3: Probability density of the local variance parameter β as extracted from a time series of longitudinal velocity differences measured in a turbulent Taylor-Couette flow at Reynolds number $Re = 540000$ [8]. A lognormal distribution yields a good fit.

bution is essentially that of an additive versus a multiplicative definition of β. So far there is no theory of turbulence, but following Kolmogorov [1], the mechanism of the turbulent motion of the fluid is critically determined by the transfer mechanism of the energy dissipation between neighboring cells and between different spatial scales in the flow. A multiplicative cascade process is expected to lead to a lognormally distributed β. It seems that the above mentioned transfer mechanism for energy dissipation is similar for turbulent wind fluctuations and laboratory turbulence, which are performed under very different conditions. The spatial scale of environmental turbulence as measured at the airport is much larger than in the laboratory, moreover the Reynolds number fluctuates for the wind measurements, whereas in the laboratory experiments it is controlled.

The probability density $p(u)$ of longitudinal wind velocity differences u (either temporal or spatial) as measured at the airport has strong deviations from a Gaussian distribution and it exhibits prominent ('fat') tails (see Fig. 4). In superstatistical models one can understand these tails simply from a superposition of Gaussian distributions whose inverse variance β fluctuates on a rather large spatio-temporal scale. In the long-term run one has $p(u) \sim \int_0^\infty f(\beta) e^{-\frac{1}{2}\beta u^2} d\beta$, and generically these types of distributions $p(u)$ exhibit broader tails than a Gaussian.

For the special case that $f(\beta)$ is a Gamma distribution the integral can be explicitely

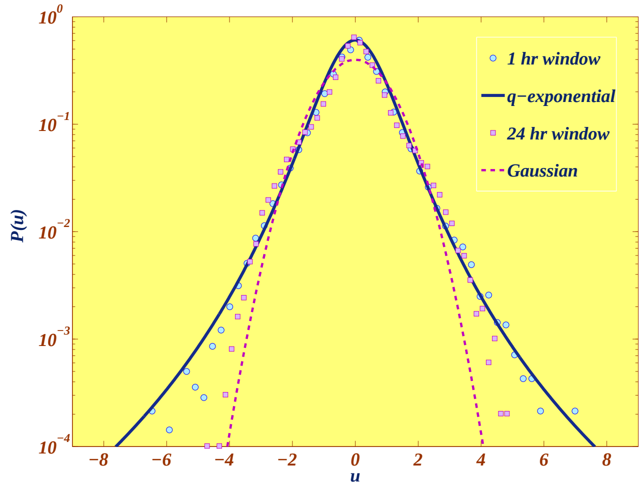

FIG. 4: Probability density $p(u)$ of temporal wind velocity differences as observed at the airport.

evaluated, and one ends up with the generalized canonical distributions (q-exponentials) of nonextensive statistical mechanics, i.e. $p(u) \sim (1+\tilde{\beta}(q-1)\frac{1}{2}u^2)^{-\frac{1}{q-1}}$, where $\tilde{\beta}$ is proportional to the average of β and q is an entropic parameter [2, 4]. These distributions asymptotically decay with a power law. For other $f(\beta)$ (such as the lognormal distributions relevant in our case), the integral cannot be evaluated explicitly, and more complicated behaviour arises. However, it can be shown that for sharply peaked distributions $f(\beta)$ a q-exponential for $p(u)$ is often a good approximation provided $|u|$ is not too large [2].

Quite generally, the superstatistics approach also gives a plausible physical interpretation to the entropic index q. One may generally define

$$q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}, \tag{1}$$

where $\langle \beta \rangle = \int f(\beta)\beta d\beta$ and $\langle \beta^2 \rangle = \int f(\beta)\beta^2 d\beta$ denote the average and second moment of β, respectively. Clearly, if there are no fluctuations in β at all but β is fixed to a constant value β_0, one has $\langle \beta^2 \rangle = \langle \beta \rangle^2 = \beta_0^2$, hence in this case one just obtains $q = 1$ and ordinary statistical mechanics arises. On the other hand, if there are temperature fluctuations (as in most nonequilibrium situations) then those are effectively described by $q > 1$. For the special case that $f(\beta)$ is a Gamma-distribution, the q obtained by eq. (1) coincides with Tsallis' entropic index q (up to some minor correction arising from the local β-dependent normalization constants). But the superstatistics concept is more general in that it also allows for other distribution $f(\beta)$, as for example the lognormal distribution observed in Fig. 2 and 3. General superstatistics can lead to a variety of distributions $p(u)$ with prominent ('fat') tails, i.e. not only power laws but, for example, also stretched exponentials tails and much more.

The atmospheric turbulence data seem roughly consistent with Kolomogorov's general ideas of a lognormally distributed fluctuating energy dissipation rate, as are the laboratory turbulence data. In that connection comparison with long range oceanic measurements of a similar kind as the atmospheric wind experiments discussed here might be instructive, testing yet larger scales.

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