Exercise Sheet 4 — Chaos and Fractals (MTH6107)

due: Thursday, 23 October 2008, 5pm

1. Using the Intermediate Value Theorem show that the equation $x^3+x-1=0$ has a solution α between $x=0$ and $x=1$. Show that α is a fixed point of the dynamical system $x_{n+1}=(1-x_n)^{1/3}$. Find the numerical value of α by iterating this dynamical system for an arbitrary initial point $x_0 \in (0,1)$.

   Show that the same point $x=\alpha$ is also a fixed point of the dynamical system $x_{n+1}=1-x_n^3$. Can one compute α by iterating this new dynamical system starting from an arbitrary initial value $x_0 \in (0,1)$? Investigate the stability of the fixed point in both cases.

2. Decide which of the following are C^1-diffeomorphisms or homeomorphisms (or neither) $\mathbf{R} \to \mathbf{R}$.

   a) $f(x) = -7x + 17$

   b) $f(x) = x^7$

   c) $f(x) = e^x - 7$

   d) $f(x) = 2x + \sin x$

   e) $f(x) = 1/x$

   In case f is a diffeomorphism $\mathbf{R} \to \mathbf{R}$, decide which periodic orbits are possible for this system.

3. Consider the map $y = f(x) = 1 - x^2$, where x takes on values in $\mathbf{R}$.

   a) Is f a diffeomorphism?

   b) Show that each value y has two different preimages x, except when $y = 1$. Determine the two branches of the inverse function $x = f^{-1}(y)$.

   c) Given some value y, determine the set S_k of all values x that are mapped onto y after k iteration steps, $k = 1, 2, 3$. Write down these sets for the special example $y = 0$.