Exercise Sheet 5 — Chaos and Fractals (MTH6107)

due: Thursday, 6 November 2008, 5pm

1. Show that the function $f(x) = x + \frac{1}{2}\sin x$ is a diffeomorphism $\mathbf{R} \to \mathbf{R}$. Locate the fixed points and find whether they are attracting or repelling. Show that f has no periodic orbits of length $L > 1$.

2. Let some arbitrary continuous mapping f be given. Assume that you have proved that the mapping has a periodic orbit of period length L_1. Does this imply that the mapping also has a periodic orbit of period length L_2 ?

   a) $L_1 = 2, L_2 = 3$

   b) $L_1 = 38, L_2 = 84$

   c) $L_1 = 16, L_2 = 24$

   d) $L_1 = 17408, L_2 = 61440$

   e) $L_1 = 835674463, L_2 = 296303332$

   Give reasons.

3. *(numerical exercise)*

   a) Iterate the logistic map $f(x) = 1 - \mu x^2$ for two different initial values that are very close to each other, for example $x_0^{(1)} = 0.2000000$ and $x_0^{(2)} = 0.2000001$. The orbit of $x_0^{(1)}$ is denoted by $x_n^{(1)}$, that of $x_0^{(2)}$ by $x_n^{(2)}$. Plot $\log|x_n^{(1)} - x_n^{(2)}|$ versus n for $n = 0, 1, 2, \ldots, n_{max}$, where $n_{max} \approx 40$. Do this for $\mu = 2$, $\mu = 1.7$, $\mu = 1.2$ and a fourth parameter value μ of your choice. Briefly describe your findings. For which values of μ do you find sensitive dependence on initial conditions?

   b) Numerically determine the Liapunov exponent λ of the logistic map for the above four values of μ. You should iterate some arbitrary initial value x_0 N times, where N is a large but finite number, and average the quantity $\log|f'(x_n)|$ along the orbit.