# A closer look at time averages of the logistic map at the edge of chaos

Ugur Tirnakli
*Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey*

Constantino Tsallis
*Centro Brasileiro de Pesquisas Físicas,*
*R. Dr. Xavier Sigaud 150,*
*22290-180 Rio de Janeiro, RJ, Brazil*
*and*
*Santa Fe Institute, 1399 Hyde Park Road,*
*Santa Fe, NM 87501, USA*

Christian Beck
*School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*
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The probability distribution of sums of iterates of the logistic map at the edge of chaos has been recently shown [see U. Tirnakli, C. Beck and C. Tsallis, Phys. Rev. E **75**, 040106(R) (2007)] to be numerically consistent with a $q$-Gaussian, the distribution which, under appropriate constraints, maximizes the nonadditive entropy $S_q$, the basis of nonextensive statistical mechanics. This analysis was based on a study of the tails of the distribution. We now check the entire distribution, in particular its central part. This is important in view of a recent $q$-generalization of the Central Limit Theorem, which states that for certain classes of strongly correlated random variables the rescaled sum approaches a $q$-Gaussian limit distribution. We numerically investigate for the logistic map with a parameter in a small vicinity of the critical point under which conditions there is convergence to a $q$-Gaussian both in the central region and in the tail region, and find a scaling law involving the Feigenbaum constant $\delta$. Our results are consistent with a large number of already available analytical and numerical evidences that the edge of chaos is well described in terms of the entropy $S_q$ and its associated concepts.

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One of the cornerstones of statistical mechanics and of probability theory is the Central Limit Theorem (CLT). It states that the sum of $N$ independent identically distributed random variables, after appropriate centering and rescaling, exhibits a Gaussian distribution as $N \to \infty$. In general, this concept lies at the very heart of the fact that many stochastic processes in nature which consist of a sum of many independent or nearly independent variables converge to a Gaussian [1, 2]. On the other hand, there are also many other occasions in nature for which the limit distribution is not a Gaussian. The common ingredient for such systems is the existence of strong correlations between the random variables, which prevent the attractor of the system to end up being a Gaussian. Recently, for certain classes of strong correlations of this kind, it has been proved that the rescaled sum approaches a $q$-Gaussian, which constitutes a $q$-generalization of the standard CLT [3, 4, 5, 6]. This represents a progress since the $q$-Gaussians are the distributions that optimize the nonadditive entropy $S_q$ (defined to be $S_q \equiv (1 - \sum_i p_i^q)\,/\,(q-1)$), on which nonextensive statistical mechanics is based [7, 8]. A $q$-generalized CLT was expected for several years since the role of $q$-Gaussians in nonextensive statistical mechanics is pretty much the same as that of Gaussians in Boltzmann-Gibbs statistical mechanics. Therefore it is not surprising at all to see $q$-Gaussians replace the usual Gaussian attractor distributions for those systems whose agents exhibit certain types of strong correlations.

Immediately after these achievements, an increasing interest developed for checking these ideas and findings in real and model systems whose dynamical properties make them appropriate candidates to be analyzed along these lines. Cortines and Riera have analysed stock market index changes for a considerable range of time delays using Brazilian financial data [9] and found that the histograms can be well approximated by $q$-Gaussians with $q \approx 1.75$ (see Fig. 6 of [9]). Another interesting study has been done by Caruso *et al.* [10] by using real earthquake data from the World and Northern California catalogs, where they observed that the probability density of energy differences of subsequent earthquakes can also be well fitted by a $q$-Gaussian with roughly the same value of $q$, i.e., $q \approx 1.75$ (see Fig. 2 of [10]). A more very recent contribution along these lines consists in a molecular dynamical test of the $q$-CLT in a long-range-interacting many-body classical Hamiltonian system known as HMF model [11], where it was numerically shown that, in the longstanding quasi-stationary regime (where the system is only weakly chaotic), the relevant densities appear to converge to $q$-Gaussians with $q \approx 1.5$ (see for example Fig. 7 of [11]; see also [12]). Moreover, $q$-Gaussians have also been observed in the motion of Hydra cells in cellular aggregates [13], for defect turbulence [14], silo drainage of granular matter [15], cold atoms in dissipative optical lattices [16], and dissipative 2D dusty plasma [17]. Finally, in a recent paper, we numerically investigated the central limit behavior of deterministic dynamical systems [18], where one of our main purposes was to see what kind of limit distributions emerge for the attractor whenever the dynamical system is not mixing (for example at the edge of chaos, where the Lyapunov exponent vanishes) and thus the standard CLT is not valid anymore. In [18], using the well-known standard example of discrete one-dimensional dissipative dynamical systems, the logistic map, defined as

$$x_{t+1} = 1 - a x_t^2 \quad , \tag{1}$$

(where $0 < a \leq 2$; $|x_t| \leq 1$; $t = 0, 1, 2, \ldots$), we numerically checked that, at the edge of chaos (i.e., taking the parameter of the map as $a_c = 1.401155189092$), the tails of the limit distribution are consistent with a $q$-Gaussian having, once again, a value of $q$ close to 1.75. However, the central part of the distribution was not meticulously studied. In the present manuscript, our aim is to focus on this point, having a closer look on the entire attractor of the logistic map at its chaos threshold.

Although the iterates of a deterministic dynamical system can never be completely independent, one can still prove some standard CLTs for such systems [19, 20, 21], provided that the assumption of independent identically distributed random variables is replaced by the property that the system is sufficiently mixing (i.e., asymptotic statistical independence). As an example one can consider the logistic map at $a = 2$ where it is strongly mixing. For this system, it can be rigorously proved [19, 20] that the distribution of the quantity

$$y := \sum_{i=1}^{N} (x_i - \langle x \rangle) \tag{2}$$

becomes Gaussian for $N \to \infty$ after appropriate rescaling with a factor $1/\sqrt{N}$, regarding the initial value $x_1$ as a random variable with a smooth probability distribution. Here $\langle x \rangle$ denotes the mean of $x$, which happens to vanish for the special case $a = 2$. This is a highly nontrivial result since the iterates of the logistic map at $a = 2$ are not independent but exhibit complicated higher-order correlations described by forests of binary trees [22]. Gaussian limit behavior is also numerically observed for other typical parameter values in the chaotic region of the logistic map [18]. Indeed, whenever the Lyapunov exponent of the one-dimensional map is positive, one expects the CLT to be valid [21].

Now we are ready to discuss the behavior of the logistic map at the edge of chaos for which a standard CLT is not valid due to the lack of strong chaoticity. In order to calculate the average in Eq. (2), it is necessary to take the average over a large number of $N$ iterates as well as a large number $n_{ini}$ of randomly chosen initial values $x_1^{(j)}$, namely,

$$\langle x \rangle = \frac{1}{n_{ini}} \frac{1}{N} \sum_{j=1}^{n_{ini}} \sum_{i=1}^{N} x_i^{(j)} \ . \tag{3}$$

These conditions are important due to non-ergodicity.

In principle, at the edge of chaos, taking $N \to \infty$ is not the only ingredient for the system to attain its limit distribution. It is also necessary to localize the critical point (the chaos threshold) with infinite precision. In other words, theoretically, for a full description of the shape of the distribution function of the attractor, one needs to take the $a_c$ value with infinite precision as well as taking $N \to \infty$. On the other hand, in numerical experiments, neither the precision of $a_c$ nor the $N$ values can approach infinity. In fact, numerically one can see the situation as a kind of interplay between the precision of $a_c$ and the number $N$ of iterates. For a given finite precision of $a_c$ (slightly above the exact critical value), if we use a very large $N$, then the system quickly feels that it is not exactly at the chaos threshold, and the central part of its probability distribution function becomes a Gaussian (with only small deviations in the tails). On the other hand, for the same $a_c$, if we take $N$ too small, then the summation given by Eq. (2) starts to be inadequate to approach the edge-of-chaos limiting distribution, and the central part of the distribution around zero exhibits a sort of divergence. This is indeed a direct consequence of the fact that the attractor of the system at the edge of chaos is a fractal that only occupies a tiny part of the full phase space (see [23] for details). This is the reason why the central parts of the distributions shown in [18] do not present the typical smooth shape of $q$-Gaussians. Indeed, the values of $N$ chosen in [18] ($2^{14}$ and $2^{15}$) are too small for the precision of $a_c$ (1.401155189092) to obtain a complete construction of the entire distribution including both central parts and tails. On the other extreme, for this precision of $a_c$, one can think about numerical experiments with $N$ values at the level of, say, $2^{30}$ or more, for which the central part would approach a Gaussian since the system starts to realize that it is not exactly at the edge of

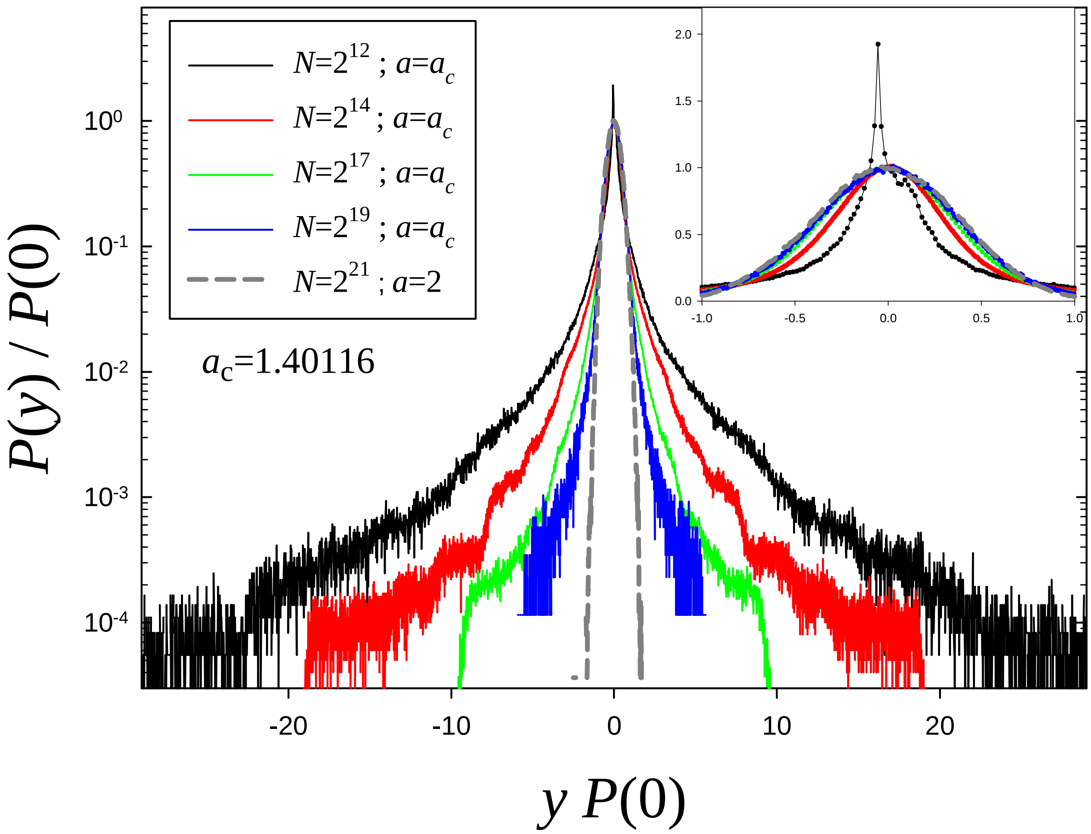


FIG. 1: Probability density function of the quantity $y$ rescaled by $P(0)$. The map is at the edge of chaos with 5 digits precision ($a_c = 1.40116$). The tendency to approach the Gaussian is evident as $N$ increases and $a_c$ is kept fixed.

chaos. We observe that between these two extremes, for a given precision of $a_c$, there exists a range of values of $N$ for which the probability density of the system happens to converge to a $q$-Gaussian in the entire region. Unfortunately, this kind of large $N$ values which are necessary to fully verify this observation if we take $a_c$ with say 12 digits precision (as we did in [18]) cannot normally be reached in numerical experiments. However, we can check this scenario using less precision for $a_c$. This will in turn make the appropriate $N$ value to become small enough so that we are able to handle the numerics with standard computers.

In the present illustration, we focus on $a_c$ values with three successive precisions, namely $a_c = 1.4012$, $a_c = 1.40116$ and $a_c = 1.401156$ (i.e., with 5, 6 and 7 digits precision respectively). For all these cases, we numerically verify the above-mentioned scenario. In our simulations, after omitting a transient of the first $2^{12}$ iterates, we calculate the quantity $y$ in Eq. (2) for various $N$ values and construct its probability distribution. A representative example is given in Fig. 1. It is worth mentioning that the same picture emerges for any precision of $a_c$, but of course with different $N$ values. It is clearly seen from the figure that, for this $a_c$ precision, $N = 2^{19}$ is so large that the density approaches a Gaussian in the central region with small deviations in the tails (further increase of $N$ values would make the whole curve become a Gaussian), whereas for $N = 2^{12}$ the curve has heavy tails and a peaked central part (this curve can be fitted neither by a Gaussian nor by a $q$-Gaussian in the entire region). On the other hand, between these two extreme cases, there is an appropriate range of $N$ values (around $2^{17}$ for this example of precision of $a_c$) for which the distribution appears to be a $q$-Gaussian for the entire region. In order to better illustrate this point, in Fig. 2 we plot the probability densities corresponding to the three $a_c$ cases with appropriate $N$ values. Three important aspects of the curves are evident: (i) the curves obtained for these three different cases exhibit a very clear data collapse, (ii) the curves can be well fitted *everywhere* (i.e. both in the tails and in the central region) by a $q$-Gaussian

$$P(y) \sim e_q^{-\beta y^2} := \frac{1}{(1+\beta(q-1)y^2)^{\frac{1}{q-1}}} \,, \tag{4}$$

where $q$ and $\beta$ are suitable parameters, and finally (iii) as the precision of $a_c$ and consequently the value of the appropriate $N$ increases, the region consistent with the $q$-Gaussian grows in size. We also determine and show in

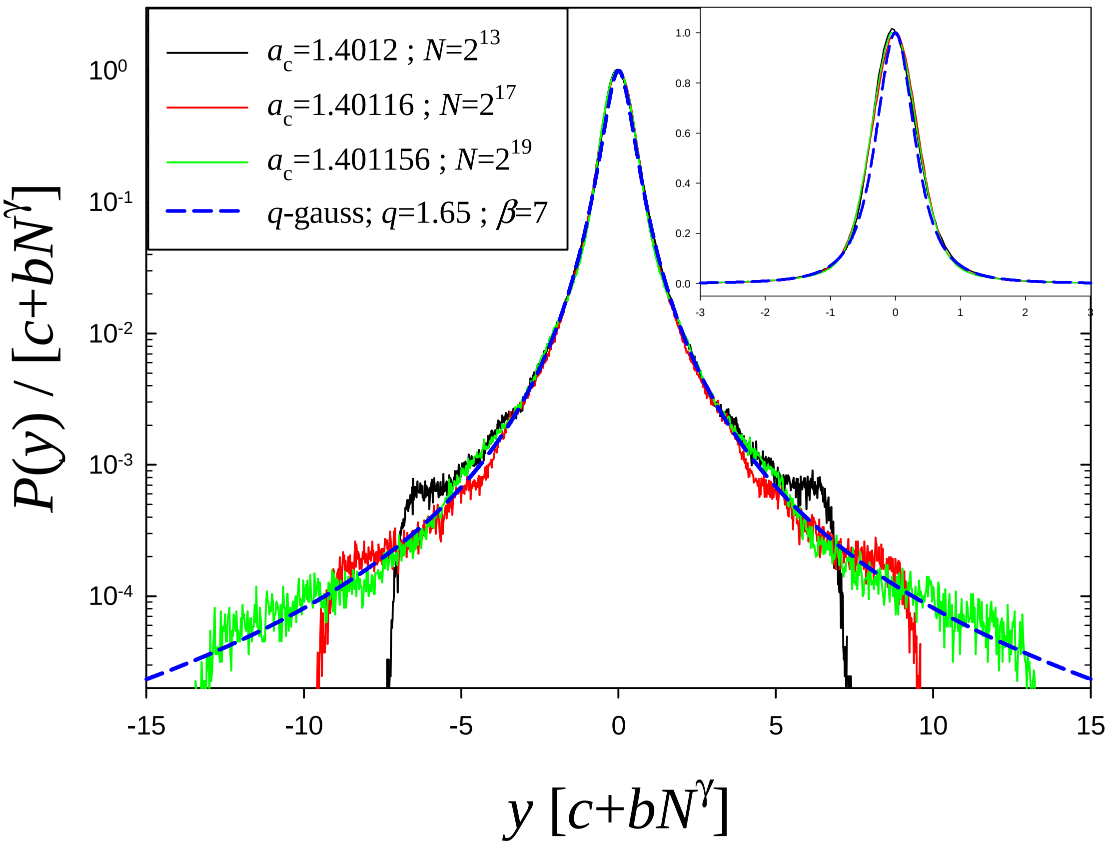


FIG. 2: Data collapse of probability density functions for the three different cases obtained by using 4,5 and 6 digits precision for $a_c$. A reasonably good fit using a $q$-Gaussian with $q = 1.65$ and $\beta = 7$ is obtained for the whole range of the histogram. Inset: The linear-linear plot of the data for a better visualization of the central part.

Fig. 3 how $P(0)$ evolves with $N$ for the three cases used in Fig. 2, i.e. the rescaling factor that has been used in Figs. 2 and 4.

In Fig. 4 we plot the $q$-logarithm (defined to be the inverse function of the $q$-exponential given in Eq. (4), namely $ln_q(x) = (x^{1-q} - 1)/(1 - q)$) of the same data. This provides a further visualisation of the aspect mentioned in (iii), namely that as the precision of $a_c$ and the value of $N$ increase, the region consistent with a $q$-Gaussian extends in size. This constitutes quite a clear numerical indication that the limit distribution will be adequately approached when both the precision of $a_c$ and the value of $N$ go to infinity.

Let us now provide a theoretical argument what the optimum value of $N$ could be to achieve best convergence to a $q$-Gaussian. Finite precision of $a_c$ means that the parameter $a$ of the system is at some distance $|a - a_c|$ from the exact critical point $a_c = 1.401155189092...$. Suppose we are slightly above the critical point ($a > a_c$), by an amount

$$|a - a_c| \sim \frac{1}{\delta^n}, \tag{5}$$

where $\delta = 4.6692011...$ is the Feigenbaum constant. Then there exist $2^n$ chaotic bands of the attractor with a selfsimilar structure, which approach the Feigenbaum attractor for $n \to \infty$ by the band splitting procedure (see e.g. [24], p.10, for more details). Suppose we perform $2^n$ iterations of the map for a given initial value with a parameter $a$ as given by Eq. (5). Then after $2^n$ iterations we are basically back to the starting value, because we fall into the same band of the band splitting structure. This means the sum of the iterates $\sum_{i=1}^{2^n} x_i$ will essentially approach a fixed value $w = 2^n\langle x\rangle$ plus a small correction $\Delta w_1$ which describes the small fluctuations of the position of the $2^n$th iterate within the chaotic band. Hence

$$y_1 = \sum_{i=1}^{2^n} (x_i - \langle x\rangle) = \Delta w_1. \tag{6}$$

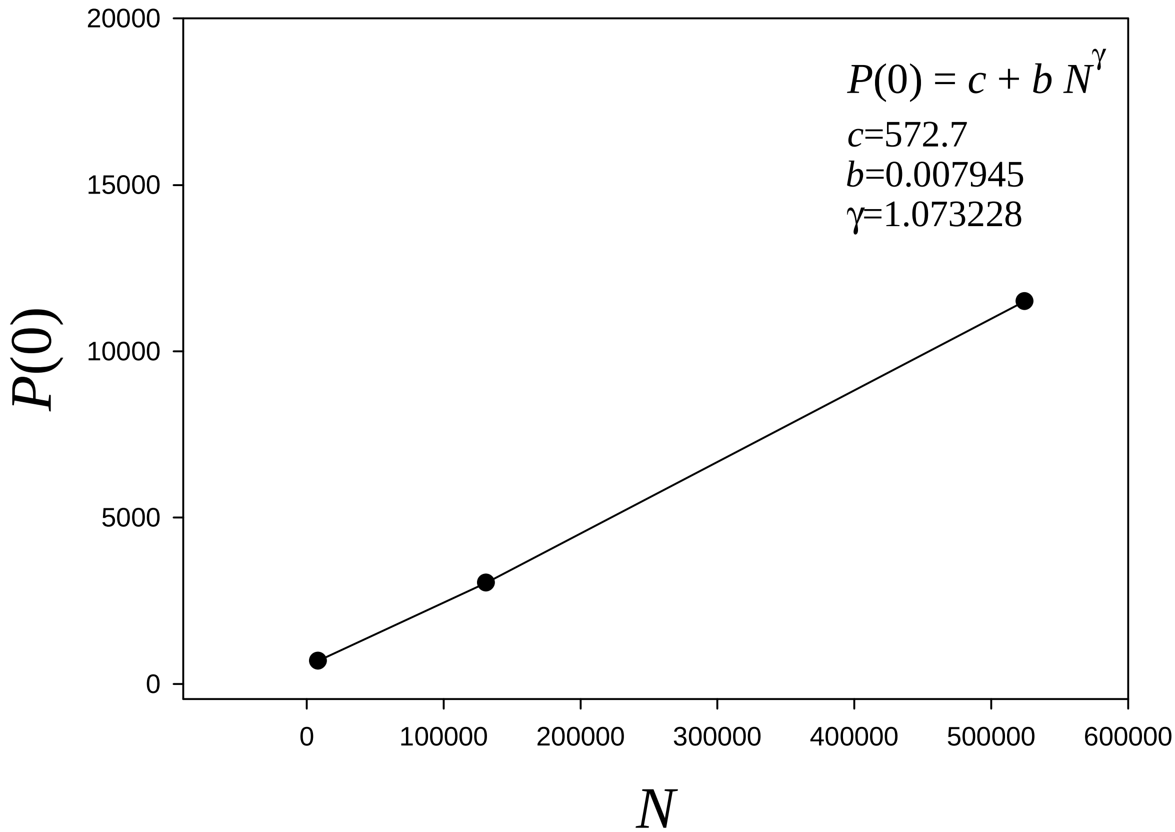


FIG. 3: Dependence of the rescaling factor $P(0)$ on $N$ in order to achieve data collapse as displayed in Fig. 2.

If we continue to iterate for another $2^n$ times, we obtain

$$y_2 = \sum_{i=2^n+1}^{2^{n+1}} (x_i - \langle x \rangle) = \Delta w_2. \tag{7}$$

The new fluctuation $\Delta w_2$ is not expected to be independent from the old one $w_1$, since correlations of iterates decay very slowly if we are close to the critical point. Continuing, we finally obtain

$$y_{2^n} = \sum_{i=4^n-2^n+1}^{4^n} (x_i - \langle x \rangle) = \Delta w_{2^n} \tag{8}$$

if we iterate the map $4^n$ times in total. The total sum of iterates

$$y = \sum_{i=1}^{4^n} (x_i - \langle x \rangle) = \sum_{j=1}^{2^n} \Delta w_j \tag{9}$$

can thus be regarded as a sum of $2^n$ strongly correlated random variables $\Delta w_j$, each being influenced by the structure of the $2^n$ chaotic bands at distance $a - a_c \sim \delta^{-n}$ from the Feigenbaum attractor. There is a 1-1 correspondence between these $2^n$ random variables $\Delta w_j, j = 1, \ldots, 2^n$ and the $2^n$ chaotic bands of the attractor, which remains preserved if $n$ is further increased. It is now most reasonable to assume that the above system of a sum of $2^n$ correlated random variables $\Delta w_j$ exhibits data collapse (and hence convergence to a well-defined limit distribution) under successive renormalization transformations $n \to n+1 \to n+2 \to n+3 \cdots$. The limit distribution may indeed be a $q$-Gaussian, as indicated by our numerical experiments. The above scaling argument implies that the optimum iteration time $N^*$ to observe convergence to a $q$-Gaussian limit distribution is given by

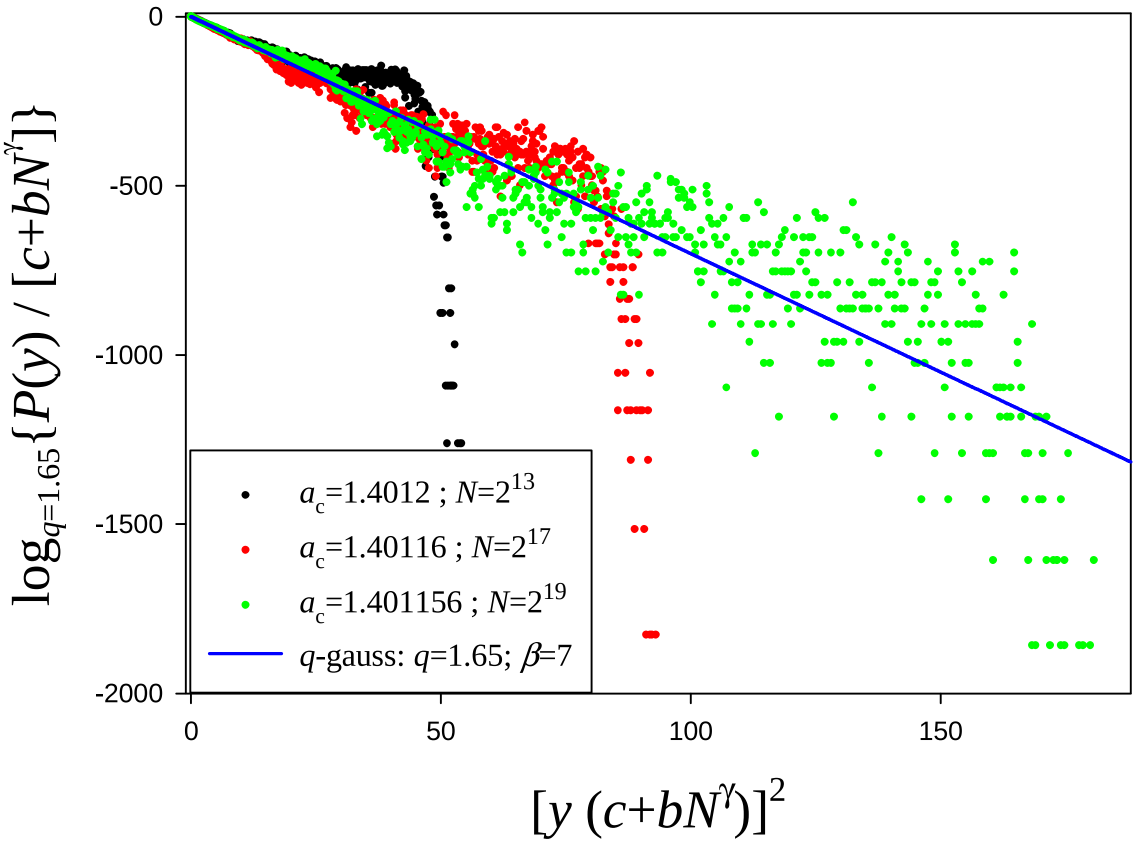


FIG. 4: $q$-logaritmic plot of the same data as in Fig. 2. As the precision of $a_c$ and the value of $N$ increase, it is clearly seen that the region consistent with $q$-Gaussian behavior is widening.

$$N^* \sim 4^n \tag{10}$$

where, at a given distance $a - a_c$, the integer $n$ is given by

$$n \approx -\frac{\log|a - a_c|}{\log \delta}. \tag{11}$$

Our numerical experiments decribed above were done at the distances $|a - a_c| = 4.48 \cdot 10^{-5}, 4.81 \cdot 10^{-6}, 8.11 \cdot 10^{-8}$, respectively, for which formula (10) and (11) yield the prediction $N^* \sim 2^{13}, 2^{16}, 2^{18}$, respectively, in good agreement with our numerical observation of $q$-Gaussians in Fig. 2 and 4 for $N \sim 2^{13}, 2^{17}, 2^{19}$.

Summarizing, we have presented numerical evidence that sums of iterates of the logistic map in a close vicinity of the edge of chaos approach a $q$-Gaussian probability distribution with $q = 1.65 \pm 0.05$, provided the typical number of iterations scale in line with Eqs.(10) and (11). This illustrates the *nonergodic* and *correlated* nature of this paradigmatic nonlinear dynamical system (which models a great variety of more complex physical situations, as it is well-known in the literature). The $q$-Gaussians are precisely the limit distributions of an important class of strongly correlated random systems in the realm of the recently proved $q$-generalized Central Limit Theorem. This feature, together with the fact that they optimize within appropriate constraints the nonadditive entropy $S_q$, constitutes a further manifestation of the mathematical grounds on which nonextensive statistical mechanics is based. Needless to say that the analytical proof of the results numerically obtained here would be most welcome. As open questions we may mention (i) the careful study of other dissipative and conservative maps, starting with the one-dimensional $z$-logistic one, and (ii) the investigation of the precise convergence radius to the $q$-Gaussian limit form as a function of $N$ and $a - a_c$. A full numerical study of these points certainly requires high computational power.

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