# Compatible Circuit Decompositions of 4-Regular Graphs

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**Abstract**

A transition system $T$ of an Eulerian graph $G$ is a family of partitions of the edges incident to each vertex of $G$ into transitions i.e. subsets of size two. A circuit decomposition $\mathcal{C}$ of $G$ is compatible with $T$ if no pair of adjacent edges of $G$ is both a transition of $T$ and consecutive in a circuit of $\mathcal{C}$. We give a conjectured characterization of when a 4-regular graph has a transition system which admits no compatible circuit decomposition. We show that our conjecture is equivalent to the statement that the complete graph on five vertices and the graph with one vertex and two loops are the only essentially 6-edge-connected 4-regular graphs which have a transition system which admits no compatible circuit decomposition. In addition, we show that our conjecture would imply the Circuit Double Cover Conjecture.

## 1 Introduction

The circuit double cover conjecture asserts that the edges of every bridgeless graph $G$ can be covered with circuits in such a way each edge of $G$ appears in exactly two circuits. An obvious, but fallacious, proof would be to double each edge of $G$ and then choose a circuit decomposition of the resulting Eulerian graph $2G$. We can try to repair the hole in this proof by putting restrictions on which edges can be consecutive in the circuit decomposition

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of $2G$ by means of a transition system. This gives rise to the notion of a compatible circuit decomposition of a transitioned graph. We present a conjectured characterisation of when a 4-regular transitioned graph has a compatible circuit decomposition and show that our conjecture is equivalent to the statement that $K_5$ and the graph with one vertex and two loops are the only essentially 6-edge-connected 4-regular graphs which admit transition systems with no compatible circuit decomposition. In addition, we use a construction due to F. Jaeger to show that our conjecture would imply the Circuit Double Cover Conjecture.

Let $G = (V, E)$ be an Eulerian graph. A *tour*, respectively *circuit*, of $G$ is a closed walk in $G$ which does not repeat edges, respectively edges and vertices. A *tour decomposition* of $G$ is a set of tours whose edge-sets partition $E$. *Circuit decompositions* are defined analogously. For $e = uv \in E(G)$ we consider $e$ to be made up of two half-edges $e_u$ incident to $u$ and $e_v$ incident to $v$. (We distinguish between the two half-edges of a loop.) For $v \in V(G)$ let $E_v$ denote the set of half-edges of $G$ incident with $v$. A *partial transition system* for $G$ is a function $T$ defined on $V(G)$ such that $T(v)$ is either the empty set or else is a partition of $E_v$ into subsets of size two for each $v \in V(G)$. We refer to the vertices of $G$ for which $T(v) \neq \emptyset$ as the *transition vertices* of $(G, T)$ and to the elements of $T(v)$ as *transitions* at $v$. We also refer to the pair $(G, T)$ as a *transitioned graph*. We say that two partial transition systems for $G$, $T_1$ and $T_2$, are *compatible* if for each vertex $v$ of $G$, $T_1(v) \cap T_2(v) = \emptyset$.

A partial transition system for $G$ in which each vertex of $G$ is a transition vertex is said to be a *transition system* for $G$. There is a natural bijection $f$ between transition systems for $G$ and tour decompositions for $G$: two half-edges are consecutive at $v$ in some tour of $f(T)$ if and only if they form a transition in $T(v)$. We use this bijection to extend the concept of compatibility to tour decompositions: a tour decomposition $X$ for an Eulerian graph $G$ is said to be *compatible* with a partial transition system $T$ (or another tour decomposition $Y$) if and only if $f^{-1}(X)$ and $T$ (or $f^{-1}(X)$ and $f^{-1}(Y)$) are compatible partial transition systems. Given a tour decomposition $X$ for $G$, we refer to the transitions of $f^{-1}(X)$ as *transitions* of $X$.

The main concern of this paper is the following open problem.

**Problem 1.1** *Characterise when a transitioned graph $(G, T)$ has a circuit decomposition which is compatible with $T$.*

Note that the analogous problem for compatible Euler tours was solved by Kotzig [7], who showed that a transitioned graph $(G, T)$ has an Euler tour which is compatible with $T$ if and only if each transition vertex of $G$ has degree at least four.

There is a natural link between Problem 1.1 and the Circuit Double Cover Conjecture. Let $G$ be a graph, $2G$ be the graph obtained by replacing each edge $e$ of $G$ by two parallel edges $e'$ and $e''$, and $T$ be the transition system for $2G$ corresponding to the circuit decomposition of $2G$ into the pairs of parallel edges $\{e', e''\}$. Then $G$ has a circuit double cover if and only if $(2G, T)$ has a compatible circuit decomposition. Thus one may expect that a solution to Problem 1.1 would lead to a solution of the Circuit Double Cover Conjecture.

An obvious necessary condition for the existence of a compatible circuit decomposition is that $(G, T)$ contains no *separating transitions*, that is to say pairs $\{e_v, f_v\} \in T(v)$, $v \in V$, such that $G - \{e, f\}$ has more components than $G$. The first named author [2] showed that this condition is also sufficient when $G$ is planar. This result was extended to graphs with no $K_5$-minor by Fan and Zhang [1]. We shall discuss examples in Section 2 which show that it is not sufficient in general. The following attractive conjecture made by G. Sabidussi in 1975, see [2], would give a sufficient condition for the existence of a compatible circuit decomposition in a transitioned graph $(G, T)$, namely that the transition system $T$ should correspond to an Euler tour of $G$.

**Conjecture 1.2** *Let $G$ be an Eulerian graph of minimum degree at least four and $T$ be an Euler tour of $G$. Then $G$ has a circuit decomposition which is compatible with $T$.*

The purpose of this paper is to formulate a conjectured solution to Problem 1.1 for the case when $G$ is 4-regular, and to show that our conjecture is equivalent to the statement that, if $(G, T)$ is an essentially 6-edge-connected 4-regular transitioned graph, then $(G, T)$ has a compatible circuit decomposition unless $G = K_5$ and $T$ is a transition system for $K_5$ corresponding to a circuit decomposition into two circuits of length five, or $G$ is the graph with one vertex and two loops $T$ is a transition system corresponding to the circuit decomposition into two loops. Our motivation for restricting to 4-regular transitioned graphs is a result of Jaeger [6] that the Circuit Double Cover Conjecture can be restated in terms of compatible circuit decompositions of a special family of 4-regular transitioned graphs (line graphs of cubic graphs). Jaeger's construction will be used later in this paper to show that our conjecture would imply the Circuit Double Cover Conjecture. We refer the reader to [5] for a survey on compatible Euler tours and circuit decompositions, and to [6] for a survey on the Circuit Double Cover Conjecture.

## 2 A recursive construction

Two examples of 4-regular transitioned graphs with no compatible circuit decompositions are:

- *the bad double loop*: this is the graph $G$ with one vertex incident with two loops together with the transition system induced by the decomposition of $G$ into two loops.

- *the bad* $K_5$: this is the complete graph on five vertices $K_5$ together with the transition system induced by the decomposition of $E(K_5)$ into two circuits of length five.

Other examples can be obtained by using the following two operations for combining two transitioned graphs $(G_1, T_1)$ and $(G_2, T_2)$.

- 1-*sum*: $(G, T) = (G_1, T_1) + (G_2, T_2)$. The graph $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by deleting an edge $e_i = u_i v_i$ from $G_i$, $i \in \{1, 2\}$ and then adding two new edges $f = u_1 u_2$ and $h = v_1 v_2$. The partial transition system $T$ is obtained from $T_1 \cup T_2$ by replacing the half-edges $(e_i)_{u_i}$ and $(e_i)_{v_i}$ by the half-edges $f_{u_i}$ and $h_{v_i}$, respectively, in any transition of $T_i$ which contains them, $i \in \{1, 2\}$.

- *star-product*: $(G, T) = (G_1, T_1) * (G_2, T_2)$. Choose a transition vertex $v_1$ of degree 4 of $(G_1, T_1)$ and two edges $e_2 = w_2 x_2$ and $f_2 = y_2 z_2$ of $G_2$. Let $e_1 = v_1 w_1, f_1 = v_1 x_1, g_1 = v_1 y_1, h_1 = v_1 z_1$ be the edges incident with $v_1$ where $T_1(v_1) = \{\{(e_1)_{v_1}, (f_1)_{v_1}\}, \{(g_1)_{v_1}, (h_1)_{v_1}\}\}$. The graph $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by deleting $v_1, e_2, f_2$ and then adding four new edges $e = w_1 w_2, f = x_1 x_2, g = y_1 y_2, h = z_1 z_2$. The partial transition system $T$ is obtained from $(T_1 - T_1(v_1)) \cup T_2$ by replacing the deleted half-edges by the new half-edges in any transition which contains them, in an obvious way.

Let $\mathcal{R}$ be the family of 4-regular transitioned graphs $(G, T)$ which can be obtained recursively from the bad double loop and the bad $K_5$ as follows.

- If $(G_1, T_1) \in \mathcal{R}$ and $(G_2, T_2)$ is any connected 4-regular transitioned graph then $(G, T) = (G_1, T_1) + (G_2, T_2) \in \mathcal{R}$.

- If $(G_1, T_1), (G_2, T_2) \in \mathcal{R}$ then $(G, T) = (G_1, T_2) * (G_2, T_2) \in \mathcal{R}$.

The following lemmas will imply that no transitioned graph in $\mathcal{R}$ can have a compatible circuit decomposition.

**Lemma 2.1** *Suppose $(G,T),(G_1,T_1),(G_2,T_2)$ are 4-regular transitioned graphs and $(G,T) = (G_1,T_1) + (G_2,T_2)$. Then $(G,T)$ has a compatible circuit decomposition if and only if $(G_1,T_1)$ and $(G_2,T_2)$ both have compatible circuit decompositions.*

*Proof:* Straightforward. •

Given a bipartition $(X,Y)$ of the vertex set of a graph $G$, let $E(X,Y)$ denote the set of all edges of $G$ from $X$ to $Y$. We refer to $E(X,Y)$ as a *$k$-edge-cut* of $G$ if $|E(X,Y)| = k$.

**Lemma 2.2** *Suppose $(G,T),(G_1,T_1),(G_2,T_2)$ are 4-regular transitioned graphs and $(G,T) = (G_1,T_1) * (G_2,T_2)$. If $(G,T)$ has a compatible circuit decomposition then either $(G_1,T_1)$ or $(G_2,T_2)$ has a compatible circuit decomposition.*

*Proof:* Let $\mathcal{C}$ be a compatible circuit decomposition of $(G,T)$ and let $A = \{e,f,g,h\}$ be the 4-edge-cut of $G$ separating $V(G_1) - v_1$ from $V(G_2)$. We adopt the labelling of edges given in the definition of the star product. Suppose that the edges in $A$ belong to the same circuit $C_1$ of $\mathcal{C}$. In that case the edges in $E(C_1) \cap E(G_2)$ correspond to two vertex disjoint paths, which determine one or two circuits of $G_2$ when completed with $e_2$ and $f_2$. Adding the circuits of $\mathcal{C}$ wholly contained in $G_2$, we obtain a compatible circuit decomposition of $(G_2,T_2)$. Hence, we can suppose that $A$ intersects two circuits of $\mathcal{C}$, say $C_1$ and $C_2$. If the circuit of $\mathcal{C}$ containing $e$ also contains $f$, then the paths determined by $E(C_1) \cap E(G_2)$ and $E(C_2) \cap E(G_2)$ can be extended as before to circuits of $G_2$ and then to a compatible circuit decomposition of $(G_2,T_2)$. If not, say $E(C_1) \cap A = \{e,g\}$ and $E(C_2) \cap A = \{f,h\}$, then $(E(C_1) \cap E(G_1)) \cup \{e_1,g_1\}$ and $(E(C_2) \cap E(G_1)) \cup \{f_1,h_1\}$ together with the circuits of $\mathcal{C}$ wholly contained in $G_1$ form a compatible circuit decomposition of $(G_1,T_1)$. •

**Corollary 2.3** *If $(G,T) \in \mathcal{R}$ then $(G,T)$ has no compatible circuit decomposition.*

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We conjecture that the family $\mathcal{R}$ characterizes the 4-regular transitioned graphs which have no compatible circuit decomposition.

**Conjecture 2.4** *Let $(G,T)$ be a connected 4-regular transitioned graph. Then $(G,T)$ has no compatible circuit decomposition if and only if $(G,T) \in \mathcal{R}$.*

Note that Conjecture 2.4 would follow if we could show that every 4-regular transitioned graph $(G,T) \notin \mathcal{R}$ has a compatible circuit $C$ such that $(G - E(C), T|_{V(G)-V(C)}) \notin \mathcal{R}$. A conjecture equivalent to Conjecture 2.4 was formulated by the second author in [3]. He showed that his conjecture would imply Conjecture 1.2 in [4].

We say that an edge-cut of a graph $G$ is *trivial* if it consists of all the edges incident with $v$ for some vertex $v \in V(G)$. A graph is *essentially 6-edge-connected* if it is connected and all edge-cuts of size less than six are trivial. It is conjectured in [5, Conjecture 4.3] that every 6-edge-connected transitioned graph has a compatible circuit decomposition and stated in the ensuing discussion that this may even be true under the weaker hypothesis that the graph has minimum degree four and is essentially 6-edge connected (and is not the bad double loop or the bad $K_5$). The following conjecture is the special case of this statement when $G$ is 4-regular.

**Conjecture 2.5** *Let $G$ be an essentially 6-edge-connected 4-regular graph and $T$ be a transition system for $G$. Then $(G,T)$ has no compatible circuit decomposition if and only if $(G,T)$ is the bad double loop or the bad $K_5$.*

Conjecture 2.5 would follow from Conjecture 2.4, since the bad double loop and the bad $K_5$ are the only essentially 6-edge-connected transitioned graphs in $\mathcal{R}$. We shall show in Section 4 that the two conjectures are, in fact, equivalent. We first need to establish some preliminary results on the family $\mathcal{R}$, on compatible circuit decompositions and on the family of 4-edge-connected 4-regular graphs.

**Lemma 2.6** *Suppose $(G,T) \in \mathcal{R}$ and $v \in V(G)$ with $T(v) = \emptyset$. Let $T'(v)$ be a partition of $E_v$ into subsets of size two, and put $T' = T \cup T'(v)$. Then $(G,T') \in \mathcal{R}$.*

*Proof:* We use induction on the number of vertices of $G$. Since $T(v) = \emptyset$, $(G,T)$ is not the bad double loop or the bad $K_5$. Hence we have one of the following two cases.

**Case 1** $(G,T) = (G_1,T_1) + (G_2,T_2)$ where $(G_1,T_1) \in \mathcal{R}$ and $(G_2,T_2)$ is arbitrary. For $i \in \{1,2\}$ let $T_i' = T_i \cup T'(v)$ if $v \in V(G_i)$ and $T_i' = T_i$ otherwise. Then $(G,T') = (G_1,T_1') + (G_2,T_2')$. Futhermore, $(G_1,T_1') \in \mathcal{R}$,

by induction if $v \in V(G_1)$, and trivially otherwise. Hence $(G, T') \in \mathcal{R}$.
**Case 2** $(G, T) = (G_1, T_1) * (G_2, T_2)$ where $(G_i, T_i) \in \mathcal{R}$ for $i \in \{1, 2\}$. We proceed as in Case 1. $\bullet$

**Lemma 2.7** *Let $(G, T)$ be a 4-regular transitioned graph and $v \in V(G)$ with $T(v) = \emptyset$. Suppose $(G, T)$ has a compatible circuit decomposition $\mathcal{C}$. Then $(G, T)$ has a compatible circuit decomposition $\mathcal{C}'$ such that either:*
*(a) The circuits of $\mathcal{C}'$ which contain $v$ intersect only at $v$, or*
*(b) The transition systems for $G$ induced by $\mathcal{C}$ and $\mathcal{C}'$ are compatible at $v$.*

*Proof:* Let $X$ and $Y$ be the circuits of $\mathcal{C}$ which contain $v$. Let $G'$ be the graph obtained from $X \cup Y$ by suppressing all vertices of degree two, and $X'$ and $Y'$ be the circuits of $G'$ induced by $X$ and $Y$, respectively. Let $T'$ be a transition system for $G'$ which agrees with $T$ at each vertex of $G'$ which is a transition vertex of $(G, T)$ and uses different transitions to those induced by $X'$ and $Y'$ at each vertex of $G'$ which is not a transition vertex of $(G, T)$. Let $H$ be the cubic graph obtained by separating each vertex $u$ of $G'$ into two vertices $u_1, u_2$ of degree two along the transitions in $T'$, and then adding a 1-factor $F_1 = \{u_1 u_2 \,:\, u \in V(G')\}$. Let $F_2$ and $F_3$ be the sets of edges of $H$ corresponding to the edges in $X'$ and $Y'$, respectively. Then $\{F_1, F_2, F_3\}$ is a proper 3-edge-colouring of $H$.

Let $C$ be the circuit of $F_2 \cup F_3$ which contains $v_1$. Let $Z$ be the path or circuit of $H$ defined by putting $Z = C$ if $v_2 \notin V(C)$ and otherwise letting $Z$ be the $v_1 v_2$-path in $C$ which starts with an edge of $F_2$. Let $\{F_1, F_2', F_3'\}$ be the new 3-edge-colouring of $H$ obtained by interchanging the colours $F_2, F_3$ along $Z$. Note that all vertices of $H$ other than perhaps $v_1, v_2$ are incident with one edge of each colour. Furthermore, if $Z$ is a circuit then $v_1, v_2$ are also incident with an edge of each colour; if $Z$ is a $v_1 v_2$-path which enters $v_2$ along an edge of $F_2$ then $v_1, v_2$ are each incident with two edges in $F_3'$ and one edge in $F_1$; if $Z$ is a $v_1 v_2$-path which enters $v_2$ along an edge of $F_3$ then $v_1$ is incident with one edge of $F_1$ and two edges in $F_3'$, and $v_2$ is incident with one edge of $F_1$ and two edges in $F_2'$. In all cases we may obtain a 2-edge-colouring $\{F_2', F_3'\}$ of $G'$ by contracting $F_1$.

Let $\mathcal{D}$ be a tour decomposition of $G'$ in which the edge set of each tour in $\mathcal{D}$ is a monochromatic component in this 2-edge-colouring. By construction, $\mathcal{D}$ is compatible with $T'$ at all vertices of $G'$ other than perhaps $v$. Furthermore, all tours in $\mathcal{D}$ are circuits with the possible exception that there may be a tour $W$ which contains $v$ as a vertex of degree four and has all other vertices of degree two. In the former case the circuits of $\mathcal{D}$ containing $v$ use different transitions at $v$ than $X'$ and $Y'$. In the latter case we may decompose $W$ into two circuits $W_1, W_2$ with $V(W_1) \cap V(W_2) = \{v\}$. Let $\mathcal{C}_1 = \mathcal{D}$ if $\mathcal{D}$ is a circuit decomposition of $G'$, and otherwise put $\mathcal{C}_1 = (\mathcal{D} - \{W\}) \cup \{W_1, W_2\}$. Finally let $\mathcal{C}'$ be the compatible circuit decomposition of $(G, T)$ obtained from $\mathcal{C}$ by replacing $X$ and $Y$ by the circuits of $G$ induced by $\mathcal{C}_1$. If $\mathcal{D}$ is a circuit decomposition of $G'$ then $\mathcal{C}$ and $\mathcal{C}'$ are compatible at $v$. Otherwise the circuits in $\mathcal{C}'$ which contain $v$ are induced by $W_1$ and $W_2$ and hence intersect only at $v$. •

Let $G$ be a 4-edge-connected 4-regular graph. Suppose that $E(U_1, U_2) = \{w_1w_2, x_1x_2, y_1y_2, z_1z_2\}$ is a non-trivial 4-edge-cut of $G$ with $\{w_1, x_1, y_1, z_1\} \subseteq U_1$. The *cleavage graphs* of $G$ along $E(U_1, U_2)$ are the graphs $G_i$, $1 \le i \le 2$ obtained by adding a new vertex $v$ and new edges $vw_i, vx_i, vy_i, vz_i$ to $G[U_i]$. It can easily be seen that both cleavage graphs are 4-edge-connected. We call the new vertex $v$ the *marker vertex* of $G_i$. A partial transition system $T$ for $G$ induces a partial transition system $T_i$ for $G_i$ in an obvious way, taking $T_i(v) = \emptyset$. We say that $(G_1, T_1)$ and $(G_2, T_2)$ are obtained by *cleaving* $(G, T)$ along $E(U_1, U_2)$.

**Lemma 2.8** *Suppose that $(G, T)$ is a 4-edge-connected 4-regular transitioned graph, and that $(G_1, T_1)$ and $(G_2, T_2)$ are obtained by cleaving $(G, T)$ along a 4-edge-cut $E(U_1, U_2)$.*
*(a) For $j \in \{1, 2, 3\}$, let $T_i^j$ be the three distinct transition systems for $G_i$ which agree with $T$ on $U_i$ and for which the marker vertex $v$ is a transition vertex. If $(G_i, T_i^j)$ has a compatible circuit decomposition $\mathcal{C}_i^j$ for all $j \in \{1, 2, 3\}$ and $i \in \{1, 2\}$, then $(G, T)$ has a compatible circuit decomposition.*
*(b) If $(G_1, T_1)$ has a compatible circuit decomposition $\mathcal{C}_1$ and $U_2$ contains no transition vertices of $T$ then $(G, T)$ has a compatible circuit decomposition.*

*Proof:* We adopt the labelling of edges and vertices used in the definition of cleavage graphs.
(a) We can relabel the $\mathcal{C}_i^j$ such that $\mathcal{C}_1^1$ and $\mathcal{C}_2^1$ 'agree' at $v$. More precisely, we can relabel so that $vw_i, vx_i \in E(C_i)$ and $vy_i, vz_i \in E(D_i)$ for circuits $C_i, D_i \in \mathcal{C}_i^1$ for each $i \in \{1, 2\}$. Putting $C = (C_1 - v_1) \cup (C_2 - v_2)$ and $D = (D_1 - v_1) \cup (D_2 - v_2)$, we have that $\mathcal{C} = (\mathcal{C}_1^1 - \{C_1, D_1\}) \cup (\mathcal{C}_2^1 - \{C_2, D_2\}) \cup \{C, D\})$ is a compatible circuit decomposition of $G$.
(b) This follows in a similar way to (a) since the 4-edge connectedness of $G_2$ implies that we may choose a circuit decomposition $\mathcal{C}_2$ of $G_2$ such that $\mathcal{C}_1$

and $\mathcal{C}_2$ 'agree' at $v$. $\bullet$

## 3 Essentially 6-edge-connected 4-regular graphs

Let $G = (V, E)$ be a graph. For $A, B$ disjoint subsets of $V$ let $d_G(A, B)$ denote the number of edges in $G$ between $A$ and $B$, and put $d_G(A) = d_G(A, V - A)$. We will suppress the subscript $G$ when it is obvious to which graph we are referring. We need the following well known identities.

**Lemma 3.1** *Let $G = (V, E)$ be a graph and $X, Y \subseteq V$. Then:*
*(a) $d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X - Y, Y - X)$;*
*(b) $d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y))$.*

It is easy to see that an essentially 6-edge-connected 4-regular graph $G$ of order at least four is simple. Let $v$ be a vertex of $G$ and $e = vw, f = vx, g = vy, h = vz$ be the edges incident to $v$. The graph $G_v^{e,f}$ obtained by *splitting $v$ along $e, f$* is obtained from $G - v$ by adding two new edges $wx$ and $yz$.

**Lemma 3.2** *Let $G = (V, E)$ be an essentially 6-edge-connected 4-regular graph with at least six vertices and $v \in V$. Let the edges incident to $v$ be $e_i = vx_i$, for $1 \leq i \leq 4$. Suppose that $G_v^{e_1,e_i}$ is not essentially 6-edge-connected for all $i \in \{2, 3, 4\}$. Then, relabelling $x_1, x_2, x_3, x_4$ if necessary, we have $\{x_1x_2, x_1x_3, x_1x_4\} \in E$ and $G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ is essentially 6-edge-connected.*

*Proof:* Since $G_i := G_v^{e_1,e_i}$ is not essentially 6-edge-connected, there exist sets $X_i \subset V - v$ such that $d_{G_i}(X_i) = 4$, $2 \leq |X_i| \leq |V| - 3$, and $x_1, x_i \in X_i$ for all $i \in \{2, 3, 4\}$. Let $Y_i = (V - v) - X_i$. Since $G$ is essentially 6-edge-connected we must have $\{x_2, x_3, x_4\} - \{x_i\} \subseteq Y_i$.

Consider the sets $X_2, X_3$. We have $x_1 \in X_2 \cap X_3$, $x_2 \in X_2 - X_3$, $x_3 \in X_3 - X_2$, and $x_4 \in (V - v) - (X_2 \cup X_3) = Y_2 \cap Y_3$. Applying Lemma 3.1(a) to $G - v$ we have

$$\begin{aligned} 4 + 4 &= d_{G-v}(X_2) + d_{G-v}(X_3) \\ &= d_{G-v}(X_2 \cap X_3) + d_{G-v}(X_2 \cup X_3) + \\ &\quad 2d_{G-v}(X_2 - X_3, X_3 - X_2). \end{aligned} \tag{1}$$

If $|X_2 \cap X_3| \geq 2$ and $|Y_2 \cap Y_3| \geq 2$ then the essential 6-edge-connectivity of $G$ will imply that $d_{G-v}(X_2 \cap X_3) \geq 5$ and $d_{G-v}(X_2 \cup X_3) = d_{G-v}(Y_2 \cap Y_3) \geq 5$. This would contradict (1) and hence either $|X_2 \cap X_3| = 1$ or $|Y_2 \cap Y_3| = 1$.

We may use Lemma 3.1(b) to deduce similarly that either $|X_2 - X_3| = 1$ or $|X_3 - X_2| = 1$. Thus, at least one of the sets $X_2, Y_2, X_3, Y_3$ has size two. Similarly, at least one of the sets $X_2, Y_2, X_4, Y_4$ has size two, and at least one of the sets $X_3, Y_3, X_4, Y_4$ has size two. Since $|V| \geq 6$, we cannot have $|X_i| = 2 = |Y_i|$ for all $i \in \{2, 3, 4\}$. It follows that we may relabel $e_1, e_2, e_3, e_4$ if necessary, such that $X_2 = \{x_1, x_2\}$ and $X_3 = \{x_1, x_3\}$. Since $d_{G-v}(X_2) = 4 = d_{G-v}(X_3)$, we have $x_1x_2, x_1x_3 \in E$.

By considering the sets $X_3, X_4$, we may use Lemma 3.1(b) and the fact that $x_1x_2 \in E(G - v)$ to deduce that $d_{G-v}(X_4 - X_3) = 3$. The essential 6-edge-connectivity of $G$ now implies that $|X_4 - X_3| = 1$. Thus $X_4 = \{x_1, x_4\}$ and $x_1x_4 \in E$. Hence $\{x_1x_2, x_1x_3, x_1x_4\} \in E$.

To complete the proof of the lemma, we now suppose that $H = G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ has an edge-cut $E(X, Y)$ of size at most four, with $|X| \geq 2$ and $|Y| \geq 2$. If $\{x_2, x_3, x_4\} \subseteq X$ then $E(X \cup \{v, x_1\}, Y)$ would be a non-trivial edge-cut of $G$ of size at most four. This would contradict the essential 6-edge-connectedness of $G$. Thus, we may assume by symmetry that $x_2 \in X$ and $x_3, x_4 \in Y$. Then $E(X, Y \cup \{v, x_1\})$ is a non-trivial edge-cut of $G$ of size at most four. This again contradicts the essential 6-edge-connectedness of $G$. •

## 4 Equivalence of the two conjectures

**Theorem 4.1** *Let $(G, T)$ be a counterexample to Conjecture 2.4, chosen such that the number of transition vertices of $G$ is as small as possible and, subject to this condition, such that the number of vertices of $G$ is as small as possible. Then $G$ is essentially 6-edge-connected and $T$ is a transition system for $G$.*

*Proof:* Since $(G, T)$ is a counterexample to Conjecture 2.4, $(G, T)$ is a connected 4-regular transition graph which has no compatible circuit decomposition and $(G, T) \notin \mathcal{R}$.

**Claim 1** *$G$ is 4-edge connected.*

*Proof:* Suppose $G$ is not 4-edge-connected. Since $G$ is Eulerian all edge-cuts of $G$ contain an even number of edges and hence $G$ has a 2-edge-cut. Hence we can express $(G, T)$ as a 1-sum $(G, T) = (G_1, T_1) + (G_2, T_2)$. If both $(G_1, T_1)$ and $(G_2, T_2)$ have compatible circuit decompositions then so would $(G, T)$. Thus we may assume that $(G_1, T_1)$ has no compatible circuit decomposition. Since $(G_1, T_1)$ has at most as many transition vertices as $(G, T)$ and $G_1$ has fewer vertices than $G$, it follows that $(G_1, T_1) \in \mathcal{R}$. Hence $(G, T) \in \mathcal{R}$. This contradicts the fact that $G$ is a counterexample to Conjecture 2.4. Thus $G$ is 4-edge-connected. •

**Claim 2** *$G$ does not have a non-trivial 4-edge-cut which separates two transition vertices of $(G, T)$.*

*Proof:* We proceed by contradiction. Let $E(U_1, U_2) = \{e, f, g, h\}$ be a non-trivial 4-edge-cut of $G$ such that both $U_1$ and $U_2$ contain transition vertices of $G$. Suppose $e = w_1w_2, f = x_1x_2, g = y_1y_2, h = z_1z_2$ where $w_1, x_1, y_1, z_1 \in U_1$. For $i \in \{1, 2\}$, let $(G_i, T_i)$ be obtained by cleaving $G$ along $E(U_1, U_2)$ and let $v$ be the corresponding marker vertex of $G_i$. Using Lemma 2.8(a), we may assume without loss of generality that $(G_1, T_1^1)$ has no compatible circuit decomposition, where $T_1^1$ agrees with $T$ on $U_1$ and $T_1^1(v) = \{\{vw_1, vx_1\}, \{vy_1, vz_1\}\}$. Since $(G_1, T_1^1)$ has at most as many transition vertices as $(G, T)$ and $G_1$ has fewer vertices than $G$, it follows that $(G_1, T_1^1) \in \mathcal{R}$.

Let $T'$ be the partial transition system for $G$ obtained by putting $T'(u) = T(u)$ for all $u \in U_1$, and $T'(u) = \emptyset$ for all $u \in U_2$. Then $(G, T')$ has fewer transition vertices than $G$. If $(G, T')$ has no compatible circuit decomposition, then the choice of $G$ will imply that $(G, T') \in \mathcal{R}$. This would imply that $(G, T) \in \mathcal{R}$ by Lemma 2.6.

Thus $(G, T')$ has a compatible circuit decomposition, $\mathcal{C}'$. Let $\mathcal{X}_1'$ be the tour decomposition of $G_1$ obtained from $\mathcal{C}'$ by contracting all edges which join vertices of $U_2$. Then $\mathcal{X}_1'$ has at most one element which is not a circuit and, in any such element, $v$ is the unique vertex of degree other than two. Let $\mathcal{C}_1$ be the circuit decomposition of $G_1$ which is either equal to $\mathcal{X}_1'$ or is obtained from $\mathcal{X}_1'$ by decomposing the unique tour of $\mathcal{X}_1'$ which is not a circuit, into two edge-disjoint circuits. Then $\mathcal{C}_1$ is a compatible circuit decomposition of $(G_1, T_1)$. Since $(G_1, T_1^1)$ has no compatible circuit decomposition, there are circuits $X_1, Y_1 \in \mathcal{C}_1$ such that $vw_1, vx_1 \in E(X_1)$ and $vy_1, vz_1 \in E(Y_1)$. Using Lemma 2.7 and the fact that every compatible circuit decomposition of $(G_1, T_1)$ must induce the transitions $\{vw_1, vx_1\}$ and $\{vy_1, vz_1\}$ at $v$, we may modify $\mathcal{C}_1$ if necessary to ensure that $V(X_1) \cap V(Y_1) = \{v\}$.

Let $G_2'$ be the graph obtained from $G_2$ by splitting $v$ along $vw_2, vx_2$. Let $T_2'$ be the partial transition system for $G_2'$ defined by $T_2'(u) = T_2(u)$ for all $u \in U_2$. If $(G_2', T_2')$ has a compatible circuit decomposition $\mathcal{C}_2$, then we can combine $\mathcal{C}_1$ and $\mathcal{C}_2$ to obtain a compatible circuit decomposition of $(G, T)$. Thus $(G_2', T_2')$ has no compatible circuit decomposition.

Since $(G'_2, T'_2)$ has fewer transition vertices than $G$, $(G'_2, T'_2) \in \mathcal{R}$. Since $(G,T) = (G_1, T_1^1) * (G'_2, T'_2)$ we have $(G,T) \in \mathcal{R}$. This contradicts the choice of $G$. •

**Claim 3** *$G$ is essentially 6-edge-connected.*

*Proof:* Suppose $G$ is not essentially 6-edge-connected. Let $W$ be the set of transition vertices of $(G,T)$. We may define an equivalence relation on $V(G)$ by saying that two vertices $u, v$ of $G$ are related if every 4-edge-cut which separates $u$ and $v$ in $G$ is trivial. Claim 2 implies that $W$ is contained in an equivalence class $U_0$ of this relation. Choose a non-trivial 4-edge cut $E(U_1, U'_1)$ of $G$. Relabelling if necessary, we have $U_0 \subseteq U_1$. Let $(G_1, T_1)$ and $(G'_1, T'_1)$ be obtained by cleaving $(G,T)$ along $E(U_1, U'_1)$, where $U'_1 = V - U_1$. Then $W$ is the set of transition vertices of $(G_1, T_1)$, and $G_1$ does not have a non-trivial 4-edge cut which separates two vertices of $W$. We may continue this process to obtain a sequence $(G_0, T_0), (G_1, T_1), \ldots, (G_m, T_m)$ where $(G_0, T_0) = (G,T)$, $W$ is the set of transition vertices of $(G_i, T_i)$ for all $1 \leq i \leq m$, and $G_m$ is essentially 6-edge-connected. Since $(G_m, T_m)$ contains at least one marker vertex $v$ for which $T_m(v) = \emptyset$, $(G_m, T_m)$ is not the bad double loop or the bad $K_5$. Thus $(G_m, T_m) \notin \mathcal{R}$. The choice of $(G,T)$ now implies that $(G_m, T_m)$ has a compatible circuit decomposition $\mathcal{C}_m$. We can now apply Lemma 2.8(b) recursively to deduce that $(G,T)$ has a compatible circuit decomposition and hence contradict the choice of $G$. •

**Claim 4** *$T$ is a transition system for $G$.*

*Proof:* Suppose some vertex $v$ is not a transition vertex of $(G,T)$. Let the edges incident to $v$ be $e_i = vx_i$, for $1 \leq i \leq 4$.

We first consider the case when $|V| \leq 5$. Let $K_n^{(m)}$ be the graph with $n$ vertices in which each pair of vertices is joined by $m$ parallel edges. Since $G$ is 4-regular and essentially 6-edge-connected, $G$ is either the double loop, $K_2^{(4)}$, $K_3^{(2)}$, or $K_5$. It can easily be seen that none of these graphs has a partial transition system $T$ with $T(v) = \emptyset$ for some vertex $v$ and for which there is no compatible circuit decomposition. Thus $|V| \geq 6$.

Let $G' = G_v^{e_1, e_i}$, for some $i \in \{2, 3, 4\}$, and $T'$ be the partial transition system for $G'$ induced by $T$. If $(G', T')$ had a compatible circuit decomposition $\mathcal{C}'$, then $\mathcal{C}'$ would readily give rise to a compatible circuit decomposition for $(G,T)$. Thus $(G', T')$ has no compatible circuit decomposition. The choice of $(G,T)$ now implies that $(G', T') \in \mathcal{R}$.

Suppose that $G'$ is essentially 6-edge-connected. Since $(G', T') \in \mathcal{R}$, and $|V(G)| \geq 6$, $(G', T')$ is the bad $K_5$. We now show directly that $(G, T)$ has a compatible circuit decomposition. Let $f(T')$ be the circuit decomposition of $K_5$ corresponding to $T'$. The edges of $G'$ created by the splitting of $v$ must be independent because $G$ is essentially 6-edge-connected. Depending on whether these edges are part of the same circuit of length 5 in $f(T')$ or not, there are two possible configurations for $(G, T)$ and they both admit compatible circuit decompositions as seen in Figures 1 and 2.

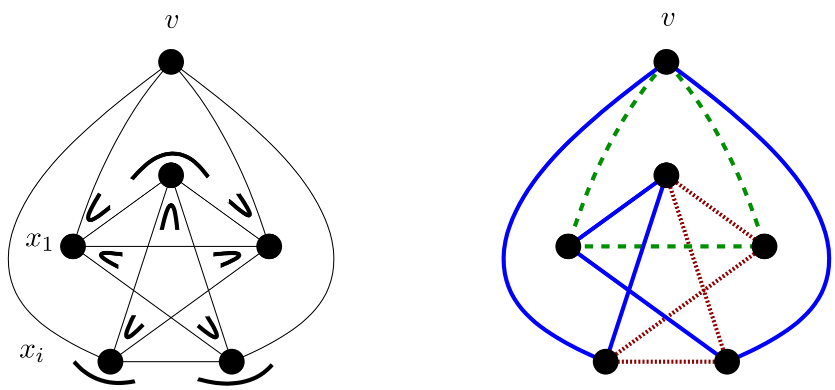


Figure 1: A compatible circuit decomposition for $(G, T)$ in the case when splitting $v$ along $e_1, e_i$ creates two edges which belong to the same 5-circuit in $f(T')$.

Thus we may suppose that $G_v^{e_1,e_i}$ is not essentially 6-edge-connected for all $2 \leq i \leq 4$. By Lemma 3.2, there is a relabelling of $x_1, x_2, x_3, x_4$ such that $\{x_1x_2, x_1x_3, x_1x_4\} \in E$ and $G'' = G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ is essentially 6-edge-connected. Let $(G_1, T_1)$ and $(G_2, T_2)$ be obtained by cleaving $(G', T')$ along the 4-edge-cut $E(\{x_1, x_i\}, V(G') - \{x_1, x_i\})$, where $x_1, x_i \in V(G_1)$. Then $G_1 = K_3^{(2)}$. Since $(G', T')$ has no compatible circuit decomposition, and every transition system for $K_3^{(2)}$ admits a compatible circuit decomposition, Lemma 2.8(a) implies that $(G_2, T_2^j)$ does not have a compatible circuit decomposition for some transition system $T_2^j$ which agrees with $T_2$ on $V(G_2) - z$, where $z$ is the marker vertex of $G_2$. The choice of $(G, T)$ now implies that $(G_2, T_2^j) \in \mathcal{R}$. Since $G_2$ is isomorphic to $G''$, $G_2$ is essentially 6-edge-connected, and hence $(G_2, T_2^j)$ must be the bad $K_5$. Thus $G'$ is the graph shown on the left of Figure 3 (where we disregard for the moment the indicated transitions of $T'$). Hence $G'$ is 4-edge-connected

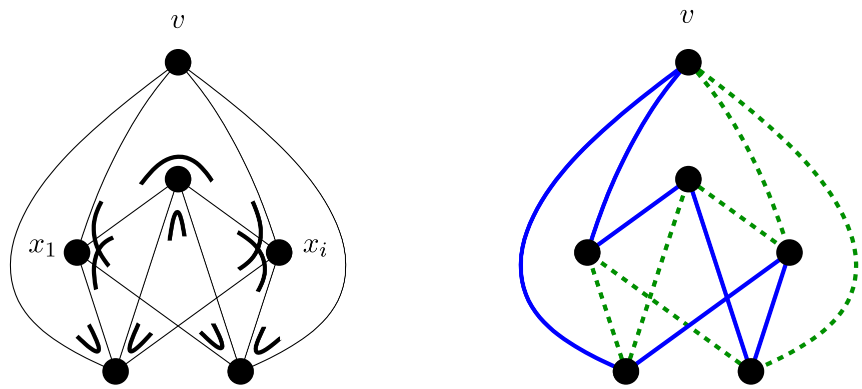


Figure 2: A compatible circuit decomposition for $(G,T)$ in the case when splitting $v$ along $e_1, e_i$ creates two edges which do not belong to the same 5-circuit in $f(T')$.

and has exactly one non-trivial 4-edge-cut. Since $(G',T') \in \mathcal{R}$, we have $(G',T') = (G_3,T_3) * (G_4,T_4)$ where: $(G_3,T_3)$ is the bad $K_5$; $G_4$ is the 4-regular graph with two vertices, two loops, and two parallel edges i.e. the 1-sum of two double loops, at least one of which is the bad double loop; the star product deletes a vertex from $G_3$ and the two loops from $G_4$. Thus the transitions $T'(x)$ for $x \in V(G') - \{x_1, x_i\}$ are as shown on the left of Figure 3. Hence $(G,T)$ is as shown in the centre of Figure 3. Now let $G^* = G_v^{e_1,e_k}$ for some $k \in \{2,3,4\} - \{i\}$, and $T^*$ be the partial transition system for $G^*$ induced by $T$ (see the third graph of Figure 3). By the same reasoning as before, cleaving $(G^*,T^*)$ along its unique non-trivial 4-edge-cut should produce a cleavage graph with a partial transition system that can be extended at the marker vertex to give a bad $K_5$. Since this is not possible, we get a contradiction. •

The equivalence of Conjectures 2.4 and 2.5 follows immediately from Theorem 4.1.

We close by using a construction due to Jaeger [6, Section 2.3] to show that the Circuit Double Cover Conjecture would follow from Conjecture 2.5. It is well known that the Circuit Double Cover Conjecture can be reduced to the special case of 3-connected cubic graphs. Let $G$ be such a graph and $H$ be the line graph of $G$. Then $H$ is 4-regular and essentially 6-edge-connected. There is a natural decomposition of $E(H)$ into triangles $\Delta_v$, $v \in V(G)$, the vertex set of $\Delta_v$ being the set of edges of $G$ incident to $v$. Let $T$ be the

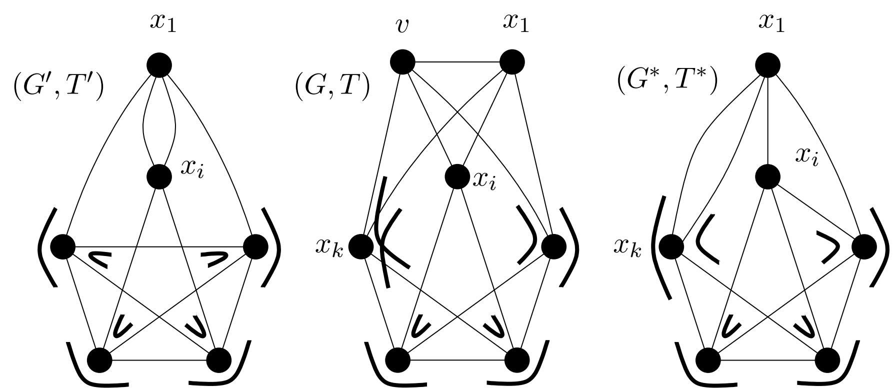


Figure 3: The graphs $(G',T'),(G,T),(G^*,T^*)$. The transitions at $x_1, x_i$, if any, are not shown.

transition system of $H$ corresponding to this triangle decomposition. Then $(H,T)$ is not the bad double loop or the bad $K_5$ so, by Conjecture 2.5, $(H,T)$ has a compatible circuit decomposition $\mathcal{C}$. It is straightforward to check that the vertex sets of the circuits in $\mathcal{C}$ are the edge sets of circuits in a circuit double cover of $G$.

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