Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT

MAS 329 Topology

10.00am, Tuesday 25th May 2004

Duration: 2 hours

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

1. (a) Explain what is meant by the statement that the function $d : X \times X \longrightarrow \mathbb{R}$ is a *metric* on the set X.

(b) Give three metrics on $\mathbb{R}^n$ which define the same topology on $\mathbb{R}^n$.

(c) Let (X, d) be a metric space and r a real number greater than 0. Show that, for all $x \in X$, the set $E(x, r)$ defined by $E(x, r) = \{y \in X \mid d(x, y) \leq r\}$ is a closed subset of X.

(d) Give an example to show that the set $E(x, r)$ in (c) is not necessarily the closure of the open ball $B(x, r)$.

(e) Give an example to show that $E(x, r)$ is not necessarily compact.

2. (a) Explain what is meant by a *topology* on a set X.

(b) Suppose $\mathbb{R}$ has the usual topology and $X = (0, 2] \cup [3, 4]$ has the relative (subspace) topology. Say whether or not the following subsets are open in X (Justification is not required.)

$$\text{(i) } (0,2], \qquad \text{(ii) } [1,2], \qquad \text{(iii) } [3,4], \qquad \text{(iv) } \{1\}.$$

(c) Explain what is meant by the *cofinite topology* on a set X and verify that it is a topology on X.

(d) Define the *closure* of a subset U of a topological space X. If the topology on X is the cofinite topology and U is an infinite subset of X, prove that the closure of U is X.

3. (a) What is meant by a *Hausdorff* topological space?

(b) If (X, d) is a metric space, prove that X, with the topology induced by d, is Hausdorff.

(c) If X and Y are Hausdorff topological spaces, show that the product space $X \times Y$ is Hausdorff.

(d) Let X be a set with at least two elements and let $x_0 \in X$. Let $\mathcal{T}$ be the topology consisting of all subsets U of X such that $x_0 \notin U$, together with X. Is X Hausdorff? Justify your answer.

(e) Give an example of a continuous map $f : X \longrightarrow Y$ between topological spaces where X is Hausdorff but $f(X)$ is not Hausdorff.

4. Let X be a topological space.

(a) Explain the statement that X is *connected*.

(b) Show that X is disconnected if and only if there is a continuous surjective map $f : X \longrightarrow D$, where $D = \{0, 1\}$ with the discrete topology.

(c) If $(X_i \mid i \in I)$ is a family of connected subspaces of X, such that, for all $i, j \in I$, $X_i \cap X_j$ is non-empty, show that $\bigcup_{i \in I} X_i$ is connected.

(d) Let X be an infinite set with the cofinite topology. Show that X is connected.

(e) Let X be a subspace of $\mathbb{R}$ with $X \subseteq \mathbb{Q}$ and having at least two elements. Show that X is disconnected. (You may assume that, between any two real numbers, there is an irrational number.)

5. (a) Explain what is meant by the statement that a topological space X is compact.

(b) If $f : X \longrightarrow Y$ is a continuous mapping of topological spaces and X is compact, show that $f(X)$ is a compact subspace of Y.

(c) Explain what is meant by a bounded set in a metric space (X, d). Show that, if a subset E of X is compact in the topology induced by d, then E is bounded.

(d) Give an example of a bounded subset E in a metric space which is not compact in the topology induced by the metric.

(e) Let X be the set of positive integers. Let $\mathcal{T}$ be the topology consisting of the subsets U such that either $2 \notin U$ or $\{2, 3, 4\} \subseteq U$. Is X with this topology compact? (Justify your answer.)

[Next question overleaf]

6. (a) If $\sim$ is an equivalence relation on a topological space X, briefly explain the construction of the quotient space $X/\sim$, verifying that it is a topological space.

(b) Define the notion of a quotient map of topological spaces, and show that if $f : X \longrightarrow Y$ is a quotient map, Z is a topological space and $g : Y \longrightarrow Z$ is a mapping (of sets), then g is continuous if and only if the composite map $g \circ f$ is continuous.

(c) If $f : X \longrightarrow Y$ is a continuous surjective mapping of topological spaces, and f is a closed mapping, show that f is a quotient map.

7. (a) Let α, β be (standard) paths in a topological space X. Define the idea of a path homotopy from α to β.

(b) Explain why the relation $\sim$ defined by $\alpha \sim \beta$ if there is a path homotopy from α to β is an equivalence relation on the set of all paths in a topological space X. (Only brief reasons need be given.)

(c) Let X and Y be spaces and let $f : X \longrightarrow Y$ be a continuous function. Given $x \in X$, explain how f gives rise to a homomorphism $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$. (You need not prove that f_* is a homomorphism.)

(d) Prove Brouwer's fixed point theorem: if $f : E^2 \longrightarrow E^2$ is a continuous mapping then there exists $x \in E^2$ such that $f(x) = x$.

[Here $E^2 = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 1\}$. You may assume $\pi_1(E^2, x)$ is trivial, for any $x \in E^2$.]

[End of examination paper]