Pseudo 2–Factor Isomorphic Regular Bipartite Graphs

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Abstract

A graph is pseudo 2–factor isomorphic if the numbers of circuits of length congruent to zero modulo four in each of its 2–factors, have the same parity. We prove that there exist no pseudo 2–factor isomorphic

k–regular bipartite graphs for $k \geq 4$. We also propose a characterization for 3-connected pseudo 2–factor isomorphic cubic bipartite graphs and obtain some partial results towards our conjecture.

1 Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted.

A graph with a 2–factor is said to be 2-*factor hamiltonian* if all its 2-factors are Hamilton circuits, and, more generally, 2-*factor isomorphic* if all its 2-factors are isomorphic. Examples of such graphs are K_4, K_5, $K_{3,3}$, the Heawood graph (which are all 2-factor hamiltonian) and the Petersen graph (which is 2-factor isomorphic).

Several recent papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties. It is shown in [1, 7] that k–regular 2–factor isomorphic bipartite graphs exist only when $k \in \{2, 3\}$ and an infinite family of 3–regular 2–factor hamiltonian bipartite graphs, based on $K_{3,3}$ and the Heawood graph, is constructed in [7]. It is conjectured in [7] that every 3–regular 2–factor hamiltonian bipartite graph belongs to this family, and, in [1], that every connected 3–regular 2–factor isomorphic bipartite graph is 2–factor hamiltonian. (We shall see in Section 3.2.4 of this paper that the latter conjecture is false.) Faudree, Gould and Jacobsen [6] determine the maximum number of edges in both 2–factor hamiltonian graphs and 2–factor hamiltonian bipartite graphs. In addition, Diwan [5] has shown that K_4 is the only 3–regular 2–factor hamiltonian planar graph.

In this paper, we extend the above mentioned results to the more general family of pseudo 2–factor isomorphic graphs i.e. graphs G with the property that the numbers of circuits of length congruent to zero modulo four in each 2–factor of G, have the same parity. We prove that pseudo 2–factor isomorphic k–regular bipartite graphs exist only when $k \in \{2, 3\}$. We then propose a conjectured characterization of 3–connected pseudo 2–factor isomorphic cubic bipartite graphs, and obtain some partial results towards our conjecture. We show in particular that there are no planar pseudo 2–factor isomorphic cubic bipartite graphs.

2 Preliminaries

An *r–factor* of a graph G is an r–regular spanning subgraph of G. A *1–factorization* of G is a partition of the edge set of G into 1–factors.

Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$, and A be its bipartite adjacency matrix. In general $0 \leq |det(A)| \leq per(A)$. We say that G is *det–extremal* if G has a 1-factor and $|det(A)| = per(A)$. Let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. For F a 1–factor of G, define the *sign* of F, $sgn(F)$, to be the sign of the permutation of $\{1, 2, \ldots, n\}$ corresponding to F. Then G is det–extremal if and only if G has a 1-factor and all its 1–factors have the same sign.

We shall need the following results. The first is elementary (and is a special case of [8, Lemma 8.3.1]).

Lemma 2.1 *Let F_1, F_2 be 1-factors in a bipartite graph G and t be the number of circuits in $F_1 \cup F_2$ of length congruent to zero modulo four. Then $sgn(F_1)sgn(F_2) = (-1)^t$.*

A *k–circuit* is a circuit of length k. A *central circuit* of a graph G is a circuit C such that $G - V(C)$ has a 1-factor. Lemma 2.1 easily implies:

Lemma 2.2 *Let G be a bipartite graph. Then G is det-extremal if and only if G has a* 1*-factor and every central circuit of G has length congruent to two modulo four.*

The next result follows from a more general theorem of Thomassen [11].

Theorem 2.3 *Let G be a det–extremal bipartite graph. If each edge of G is contained in a 1-factor then G has a vertex of degree at most three.*

We next describe a result of Asratian and Mirumyan [3], see also [2], concerning transformations between 1-factorizations of a regular bipartite graph. Let G be a t-regular bipartite graph, $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ be a 1-factorization of G, and C be a circuit of G.

Suppose $E(C) \subseteq F_i \cup F_j$ for some $1 \leq i < j \leq t$. Then we may obtain a new 1-factorization $\mathcal{F}'$ of G by putting $F'_i = F_i \triangle E(C)$, $F'_j = F_j \triangle E(C)$ and $\mathcal{F}' = (\mathcal{F} - \{F_i, F_j\}) \cup \{F'_i, F'_j\}$, where $\triangle$ denotes symmetric difference. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a 2-*transformation.*

Suppose $E(C) \subseteq F_i \cup F_j \cup F_k$ for some $1 \leq i < j < k \leq t$, and that $F_i \cap E(C)$ is a 1-factor of C. Let $X = (F_j \cup F_k) \triangle E(C)$. Since the edges of C alternate with respect to $F_j \cup F_k$, X is a 2-factor of G. Let $\{F'_j, F'_k\}$ be a 1-factorization of X. We may obtain a new 1-factorization $\mathcal{F}'$ of G by putting $F'_i = F_i \triangle E(C)$, and $\mathcal{F}' = (\mathcal{F} - \{F_i, F_j, F_k\}) \cup \{F'_i, F'_j, F'_k\}$. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a 3-*transformation.*

Theorem 2.4 *[2, 3] Let G be a t-regular bipartite graph. Then every* 1*-factorization of G can be obtained from a given* 1*-factorization by a sequence of* 2*- and* 3*-transformations.*

3 Pseudo 2-factor isomorphic regular bipartite graphs

Let G be a bipartite graph. For each 2–factor F of G let $t^*(F)$ be the number of circuits of F of length congruent to 0 modulo 4, and let

$$t(F) = \begin{cases} 0 & \text{if } t^*(F) \text{ is even} \\ 1 & \text{if } t^*(F) \text{ is odd} \end{cases}$$

We say that a bipartite graph G is *pseudo 2-factor isomorphic* if G has at least one 2-factor, and t has the same value on all 2-factors of G. In this case, we denote this constant value of t by $t(G)$.

3.1 Regular graphs of degree at least four

We show that there are no pseudo 2–factor isomorphic k–regular bipartite graphs for $k \geq 4$. Our proof uses the results of Thomassen, and Asratian and Mirumyan described in Section 2. We also use the fact that there is a close relationship between pseudo 2-factor isomorphic bipartite graphs and det–extremal bipartite graphs. This is illustrated by the following proposition.

Proposition 3.1 *Suppose G is a pseudo 2-factor isomorphic bipartite graph.*
(a) $G - F$ is det-extremal for all 1-factors F of G.
(b) If G is k-regular and $k \geq 3$ then $t^*(X) = 0$ for all 2-factors X of G. In particular, $t(G) = 0$.

Proof. (a) Let F be a 1–factor of G and $H = G - F$. Let F' be a 1–factor in H. Then $F \cup F'$ is a 2–factor of G, and hence has $t(G)$ circuits of length congruent to 0 modulo 4. By Lemma 2.1, $sign(F)sign(F') = (-1)^{t(G)}$. Since the choice of F' is arbitrary, all 1–factors of H have the same sign. Thus H is det–extremal.

(b) Let X be a 2-factor of G and F be a 1-factor of $G - X$. By (a), $H = G - F$ is det-extremal. Since every circuit of X is a central circuit of H, Lemma 2.2 implies that $t^*(X) = 0$. □

Theorem 3.2 *Let G be a pseudo 2–factor isomorphic k–regular bipartite graph. Then $k \in \{2, 3\}$.*

Proof. Suppose the theorem is false. Let G be a pseudo 2–factor isomorphic k–regular bipartite graph with $k \geq 4$. By Proposition 3.1(a), all 1-factors in any 1-factorization of G have the same sign. By Theorem 2.3, G contains two 1–factors with different signs. Since every 1–factor is contained in a 1–factorization of G, there are two 1–factorizations $\mathcal{F}_0, \mathcal{F}_1$ of G such that all

1–factors in $\mathcal{F}_0$ have positive sign and all 1–factors in $\mathcal{F}_1$ have negative sign. However, by Theorem 2.4, $\mathcal{F}_1$ can be obtained from $\mathcal{F}_0$ by a sequence of 2– and 3–transformations. Since $k \geq 4$, at least one 1–factor is preserved in every transformation, and hence the signs of all 1–factors in the resulting 1–factorization must be the same as those of the 1–factors in the original 1–factorization. This gives a contradiction. □

Theorem 3.2 generalises the analogous results for 2-factor hamiltonian graphs [7] and 2-factor isomorphic graphs [1]. Its proof is substantially simpler than the proofs given for the latter two results.

3.2 Cubic graphs

It is straightforward to show that $K_{3,3}$ and the Heawood graph H_0, shown in Figure 1(a), are 2–factor hamiltonian and hence pseudo 2–factor isomorphic, see [7]. We first show that the Pappus graph P_0, shown in Figure 1(b), is pseudo 2–factor isomorphic but not 2–factor isomorphic.

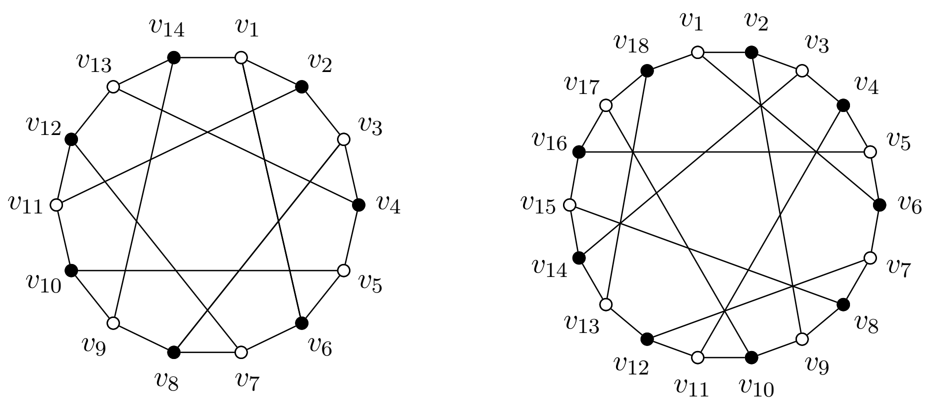

Figure 1: (a) Heawood H_0 (b) Pappus P_0

Proposition 3.3 *The Pappus graph P_0 is pseudo 2–factor isomorphic but not 2–factor isomorphic.*

Proof. We adopt the labelling of the Pappus graph P_0 given in Figure 1(b). Let F be a 2–factor of P_0 and C be a shortest circuit in F. Since P_0 is 3–arc–transitive, see [4], we may assume that the path $P = v_1v_2v_3v_4$ is contained in C. Since P_0 is bipartite, has 18 vertices, and has girth six, we have $|C| \in \{6, 8, 18\}$.

Suppose $|C| = 6$. By inspection, P is contained in exactly one 6–circuit $v_1v_2v_3v_4v_5v_6v_1$. This implies that edges $v_{18}v_1$, v_6v_7, v_2v_9, v_3v_{14}, v_4v_{11} do not belong to F, which in turn implies that F contains the 6–circuits $v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{13}$, and $v_7v_8v_9v_{10}v_{11}v_{12}v_7$. Thus F consists of exactly three 6–circuits.

Now, suppose that $|C| = 8$. Then, by inspection, C is either: $v_1v_2v_3v_4v_5v_{16}v_{17}v_{18}v_1$, $v_1v_2v_3v_4v_{11}v_{10}v_{17}v_{18}v_1$, $v_1v_2v_3v_4v_{11}v_{12}v_{13}v_{18}v_1$, or $v_1v_2v_3v_4v_{11}v_{12}v_7v_6v_1$. These in turn, respectively, imply that v_6, v_9, v_{14}, v_5 have degree 1 in F which is impossible. Thus we cannot have $|C| = 8$.

The remaining case, when $|C| = 18$, occurs when C is a hamiltonian circuit of P_0, which clearly can occur.

In both the cases $|C| = 6$ and $|C| = 18$, we have $t(F) = 0$. Thus P_0 is pseudo 2–factor isomorphic. It is not 2–factor isomorphic since, by the above, it has two non-isomorphic 2–factors. □

3.2.1 Star products

We show that $K_{3,3}$, H_0 and P_0 can be used to construct an infinite family of 3-connected pseudo 2–factor isomorphic cubic bipartite graphs.

Let G, G_1, G_2 be graphs such that $G_1 \cap G_2 = \emptyset$. Let $y \in V(G_1)$ and $x \in V(G_2)$ such that $d_{G_1}(y) = 3 = d_{G_2}(x)$. Let x_1, x_2, x_3 be the neighbours of y in G_1 and y_1, y_2, y_3 be the neighbours of x in G_2. If $G = (G_1 - y) \cup (G_2 - x) \cup \{x_1y_1, x_2y_2, x_3y_3\}$, then we say that G is a *star product* of G_1 and G_2 and write $G = (G_1, y) * (G_2, x)$, or more simply as $G = G_1 * G_2$ when we are not concerned which vertices are used in the star product. The set $\{x_1y_1, x_2y_2, x_3y_3\}$ is a 3–edge cut of G and we shall also say that G_1 and G_2 are 3*–cut reductions* of G.

We next show that star products preserve the property of being pseudo 2–factor isomorphic in the family of cubic bipartite graphs.

Lemma 3.4 *Let G be a star product of two pseudo 2–factor isomorphic cubic bipartite graphs G_1 and G_2. Then G is also pseudo 2–factor isomorphic.*

Proof. Suppose $G = (G_1, y) * (G_2, x)$ with x_1, x_2, x_3 the neighbours of y in G_1 and y_1, y_2, y_3 the neighbours of x in G_2. Suppose further that G is not pseudo 2–factor isomorphic. Then G has a 2–factor F with $t(F) = 1$. Since G is bipartite F contains exactly two edges of the 3–edge–cut $S = \{x_1y_1, x_2y_2, x_3y_3\}$. Let C be the circuit of F which intersects S and C_i be the circuit of G_i corresponding to C, $i = 1, 2$. Let F_i be the 2–factor of G_i consisting of the circuits of F which are contained in G_i together with C_i. Since $|C| = |C_1| + |C_2| - 2$, we have $1 = t(F) \equiv t(F_1) + t(F_2) \mod 2$. Hence $t(F_i) = 1$ for some $i \in \{1, 2\}$. Applying Proposition 3.1, we contradict the hypothesis that G_i is pseudo 2–factor isomorphic. □

Given a set $\{G_1, G_2, \ldots, G_k\}$ of 3-edge-connected cubic bipartite graphs let $\mathcal{SP}(G_1, G_2, \ldots, G_k)$ be the set of cubic bipartite graphs which can be obtained from $G_1, G_2, \ldots, G_k$ by repeated star products. Lemma 3.4 implies that all graphs in $\mathcal{SP}(K_{3,3}, H_0, P_0)$ are pseudo 2–factor isomorphic. We conjecture that these are the only 3-connected pseudo 2–factor isomorphic cubic bipartite graphs.

Conjecture 3.5 *Let G be a 3-connected cubic bipartite graph. Then G is pseudo 2–factor isomorphic if and only if G belongs to $\mathcal{SP}(K_{3,3}, H_0, P_0)$.*

Note that McCuaig [9] has shown that a 3-connected cubic bipartite graph G is det-extremal if and only if $G \in \mathcal{SP}(H_0)$.

Let G be a graph and E_1 be an edge-cut of G. We say that E_1 is a *non-trivial edge-cut* if all components of $G - E_1$ have at least two vertices. The graph G is *essentially 4-edge-connected* if G is 3-edge-connected and has no non-trivial 3-edge-cuts. It is easy to see that Conjecture 3.5 holds if and only if Conjectures 3.6 and 3.7 below are both valid.

Conjecture 3.6 *Let G be an essentially 4-edge-connected pseudo 2–factor isomorphic cubic bipartite graph. Then $G \in \{K_{3,3}, H_0, P_0\}$.*

Conjecture 3.7 *Let G be a 3-edge-connected pseudo 2–factor isomorphic cubic bipartite graph and suppose that $G = G_1 * G_2$. Then G_1 and G_2 are both pseudo 2–factor isomorphic.*

We will obtain partial results on Conjectures 3.6 and 3.7 in the following two subsections.

3.2.2 Essentially 4-edge-connected cubic bipartite graphs

We show that if G is an essentially 4-edge-connected pseudo 2–factor isomorphic cubic bipartite graph and G has a 4-circuit then $G = K_{3,3}$. We need the following result of Plummer [10].

Proposition 3.8 *[10] Let G be an essentailly 4-edge-connected cubic bipartite graph and e, f be independent edges of G. Then $\{e, f\}$ is contained in a 1-factor of G.*

□

Proposition 3.9 *Let G be an essentially 4-edge-connected cubic bipartite graph distinct from $K_{3,3}$, and C be a 4-circuit in G. Then C is contained in a 2-factor of G.*

Proof. Suppose the theorem is false and let G be a counterexample. Let $C = x_1y_2x_3y_4x_1$ and let y_1, x_2, y_3, x_4 be the neighbours in $V(G) - V(C)$ of x_1, y_2, x_3, y_4 respectively. If y_1, x_2, y_3, x_4 were not distinct then the essential 4-edge-connectivity of G would imply that $G = K_{3,3}$. Thus y_1, x_2, y_3, x_4 are distinct. By proposition 3.8, G has a 1-factor F with $\{x_1y_1, x_3y_3\} \subseteq F$. This implies that we must also have $\{x_2y_2, x_4y_4\} \subseteq F$. Thus $G - F$ is a 2-factor of G containing C. □

Propositions 3.1(b) and 3.9 immediately imply:

Theorem 3.10 *Let G be an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph. Suppose G contains a 4-circuit. Then $G = K_{3,3}$.*

3.2.3 Cubic bipartite graphs of edge-connectivity three

We present a partial converse of Lemma 3.4. We need the following definition.

Let G be a connected cubic bipartite graph. We say that G is *badly behaved* if there is an edge f of G with the property that, for every 2–factor F of G:

(i) $t(F) = 1$ if and only if $f \in F$;

(ii) if $t(F) = 0$ then each circuit of F has length congruent to two modulo four;

(iii) if $t(F) = 1$ then F has exactly one circuit C of length congruent to zero modulo 4 and $f \in E(C)$.

In this case f is said to be a *bad edge of* G. Note that a badly behaved graph cannot be pseudo 2–factor isomorphic by (i).

We next introduce some additional notation for working with 2–factors. Given a 2–factor F of a graph G containing a vertex x and and edge e, we use C_x and C_e to denote the circuits of F to which x and e belong. Let $G = (G_1, y) * (G_2, x)$ be a cubic bipartite graph with bipartition (X, Y). Let F_i be a 2–factor of G_i, $i = 1, 2$. We say that F_1 and F_2 are *compatible* 2–*factors* if for each $j \in \{1, 2, 3\}$, $yx_j \in C_y$ if and only if $xy_j \in C_x$. In this case we define a circuit $C_x * C_y$ in G by setting $C_x * C_y = (C_y - y) \cup (C_x - x) \cup \{x_jy_j \ : \ yx_j \in C_y, j = 1, 2, 3\}$, and a 2–factor $F_1 * F_2$ of G by setting $F_1 * F_2 = (F_1 - C_y) \cup (F_2 - C_x) \cup \{C_x * C_y\}$. The 2–factor $F_1 * F_2$ is said to be the *join* 2*–factor of* F_1 *and* F_2. Note that the circuit C has length $|C| = |C_x| + |C_y| - 2$. Using this notation we have the following lemma.

Lemma 3.11 *Let F_i be a 2-factor of G_i, $i = 1, 2$, such that F_1, F_2 are compatible. Then $t(F_1 * F_2) = 1$ if and only if $t(F_1) \neq t(F_2)$.*

Proof. It follows from the above definition that $|C_x * C_y| = |C_x| + |C_y| - 2$. Thus, $t^*(F_1 * F_2) \equiv t^*(F_1) + t^*(F_2) \mod 2$. Hence, $t(F_1 * F_2) = 1$ if and only if $t(F_1) \neq t(F_2)$. □

Theorem 3.12 *Let $G = (G_1, y) * (G_2, x)$ be a cubic bipartite graph with x_1, x_2, x_3 the neighbours of y in G_1 and y_1, y_2, y_3 the neighbours of x in G_2. Then G is pseudo 2-factor isomorphic if and only if either:*
(a) G_1, G_2 are both pseudo 2-factor isomorphic, or
(b) G_1, G_2 are both badly behaved and, for some $i \in \{1, 2, 3\}$, yx_i is a bad edge of G_1 and xy_i is a bad edge of G_2.

Proof. We first assume that (a) or (b) holds. If (a) holds, G is pseudo 2-factor isomorphic by Lemma 3.4. Hence we may suppose that (b) holds and, relabelling if necessary, that yx_3 and xy_3 are bad edges of G_1 and G_2, respectively. Let F be a 2–factor of G. Then $F = F_1 * F_2$ for 2-factors F_1 of G_1 and F_2 of G_2. If $x_3y_3 \notin F$ then $x_3y \notin F_1$ and $xy_3 \notin F_2$. This implies that $t(F_1) = 0 = t(F_2)$. Otherwise, if $x_3y_3 \in F$ then $x_3y \in F_1$ and $xy_3 \in F_2$. This implies that $t(F_1) = 1 = t(F_2)$. In both cases $t(F) = 0$ by Lemma 3.11. Since the choice of F was arbitrary, G is pseudo 2-factor isomorphic.

We next assume that G is pseudo 2-factor isomorphic. Choose $j \in \{1, 2, 3\}$ and let F_j, respectively F'_j, be a 2–factor of G_1, respectively G_2, avoiding x_jy, respectively y_jx. Then F_j and F'_j are compatible 2–factors and $F = F_j * F'_j$ is a 2–factor of G avoiding x_jy_j. Since G is pseudo 2-factor isomorphic, Proposition 3.1(b) and Lemma 3.11 imply that $t(F_j) = t(F'_j) = t_j$, say. It follows that every 2-factor X_j of G_1 which avoids yx_j satisfies $t(X_j) = t_j$ and every 2-factor X'_j of G_2 which avoids xy_j satisfies $t(X'_j) = t_j$. If $t_1 = t_2 = t_3$ then G_1 and G_2 are both pseudo 2-factor isomorphic and (a) holds. Hence we suppose without loss of generality that $1 = t_1 \geq t_2 \geq t_3 = 0$.

Suppose $t_2 = 0$. Let L_1, L_2, L_3 be a 1-factorization of G_1, labelled so that $yx_j \in L_j$ for all $1 \leq j \leq 3$. By Lemma 2.1, $sign(L_1)sign(L_2) = (-1)^{t_3} = 1$, $sign(L_1)sign(L_3) = (-1)^{t_2} = 1$, and $sign(L_2)sign(L_3) = (-1)^{t_1} = -1$. Clearly this is impossible. Hence $t_2 = 1$, and thus $t_3 = 0$.

Let F_j, respectively F'_j, be a 2–factor of G_1, respectively G_2, avoiding x_jy, respectively y_jx, for $1 \leq j \leq 3$. Let C_y, respectively C_x, be the circuit of F_j, respectively F'_j, containing y, respectively x. Then $F = F_j * F'_j$ is a 2-factor of G. Since G is pseudo 2-factor isomorphic, Proposition 3.1(b) implies that all circuits of F have length conguent to two modulo four. This implies that all circuits of $F_j \cup F'_j$ other than C_y, C_x have length congruent to two modulo four. Furthermore, the facts that $|C_y * C_x| = |C_y| + |C_x| - 2$ has length congruent to two modulo four, $t_1 = 1 = t_2$ and $t_3 = 0$, imply that $|C_x| \equiv |C_y| \equiv 0 \mod 4$ if $j \in \{1, 2\}$ and $|C_y| \equiv |C_x| \equiv 2 \mod 4$ if $j = 3$.

Thus G_1 and G_2 are both badly behaved, yx_3 is a bad edge of G_1 and xy_3 is a bad edge of G_2. $\square$

Theorem 3.12 implies that Conjecture 3.7 is equivalent to the statement that there are no 3-edge-connected badly behaved cubic bipartite graphs. We will see in the next subsection that 2-edge-connected badly behaved cubic bipartite graphs can exist. We close this subsection by showing that a 3-edge-connected badly behaved cubic bipartite graph can have at most one bad edge. This will follow easily from the following result.

Lemma 3.13 *Let G be a 3-edge-connected cubic bipartite graph and $e, f \in E(G)$. Then G has a 1-factor containing e and avoiding f.*

Proof. We proceed by contradiction. Suppose that G, e, f is a counterexample with as few vertices as possible. Choose an edge h of G incident with f but not incident with e. If G had a 1-factor F with $\{e, h\} \subseteq F$ then we would have $f \notin F$ and F would be the required 1-factor of G. Hence no such 1-factor exists and, by Proposition 3.8, G has a non-trivial 3-edge-cut $K = \{e_1, e_2, e_3\}$. Let H_1, H_2 be the components of $G - K$ and let G_i be obtained from G by contracting $E(H_i)$ for $i = 1, 2$. Without loss of generality, $e \in E(G_1)$. By induction, G_1 has a 1-factor F_1 containing e, and avoiding f if $f \in E(G_1)$. Relabelling e_1, e_2, e_3 if necessary we may suppose that $e_1 \in F_1$. By induction G_2 has a 1-factor F_2 containing e_1, and avoiding f if $f \in E(G_2)$. Then $F = F_1 \cup F_2$ is a 1-factor of G containing e and avoiding f. $\square$

Corollary 3.14 *Suppose that G is a badly behaved 3–connected cubic bipartite graph. Then G contains exactly one bad edge.*

Proof. Suppose f and f^* are distinct bad edges of G. By Lemma 3.13, G has a 1-factor F containing f and avoiding f^*. Let $X = G - F$. Since $f^* \in X$ we must have $t(X) = 1$ and since $f \notin X$ we must have $t(X) = 0$, a contradiction. $\square$

3-cut reductions

Let G be a cubic bipartite graph with bipartition (X, Y) and K be a non-trivial 3-edge-cut of G. Let H_1, H_2 be the components of $G - K$. We have seen that G can be expressed as a star product $G = (G_1, y_K) * (G_2, x_K)$ where $G_1 - y_K = H_1$ and $G_2 - x_K = H_2$. We say that y_K, repectively x_K, is the *marker vertex* of G_1, repectively G_2, *corresponding to the cut* K. Each non-trivial 3-edge-cut of G distinct from K is a non-trivial 3-edge-cut of G_1 or G_2, and vice versa. If G_i is not essentially 4-edge-connected for $i = 1, 2$, then we may reduce G_i along another non-trivial 3-edge-cut. We can continue this process until all the graphs we obtain are essentially 4-edge-connected. We call these resulting graphs the *constituents* of G. It is easy to see that the constituents of G are unique i.e. they are independent of the order we choose to reduce the non-trivial 3-edge-cuts of G. Furthermore, each vertex of G and each marker vertex belong to a unique constituent of G. Let $T(G)$ be the graph whose vertex set is the set of constituents of G, in which two vertices are adjacent if the corresponding constituents contain two marker vertices x_K, y_K corresponding to the same non-trivial 3-edge-cut K. It is straightforward to check that $T(G)$ is a tree, which we will call the 3-*cut reduction tree* of G. Conjecture 3.5 is equivalent to the statement that if G is a 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graph then every constituent of G is isomorphic to $K_{3,3}$, H_0 or P_0.

We can use Theorem 3.10 to deduce some evidence in favour of this statement.

Theorem 3.15 *Let G be a 3-edge-connected pseudo 2-factor isomorphic bipartite graph. Suppose G contains a 4-cycle C. Then C is contained in a constituent of G which is isomorphic to $K_{3,3}$.*

Proof. It is easy to see that no edge of C can be obtained in a non-trivial 3-edge-cut of G. Thus C is contained in a unique constituent G_1 of G and no vertex of C is a marker vertex of G_1. Suppose $G_1 \neq K_{3,3}$. By Theorem 3.10, C is contained in a 2-factor F_1 of G_1. It is straightforward to show, as in the proof of Theorem 3.12, that F_1 can be extended to a 2-factor F of G with $C \subseteq F$. This contradicts Proposition 3.1(b). □

3.2.4 Cubic bipartite graphs of edge-connectivity two

We shall construct infinite families of 2-edge-connected badly behaved cubic bipartite graphs and 2-edge-connected non-hamiltonian 2-factor isomorphic cubic bipartite graphs.

Let G, G_1, G_2 be graphs such that $G_1 \cap G_2 = \emptyset$. Let $e_i = u_i v_i \in V(G_i)$ for $i = 1, 2$. If $G = (G_1 - e_1) \cup (G_2 - e_2) \cup \{u_1u_2, v_1v_2\}$, then we say that G is a *2-join* of G_1 and G_2 and write $G = (G_1, e_1) \circ (G_2, e_2)$, or more simply $G = G_1 \circ G_2$ when we are not concerned which edges are used in the 2-join. The set $\{u_1u_2, v_1v_2\}$ is a 2–edge cut of G and we shall also say that G_1 and G_2 are 2–*cut reductions* of G.

Lemma 3.16 *Let G_i be a pseudo 2-factor isomorphic cubic bipartite graph and $e_i = u_i v_i \in E(G_i)$ for $i = 1, 2$. Let $G = (G_1, e_1) \circ (G_2, e_2)$. Then G is badly behaved and both u_1u_2 and v_1v_2 are bad edges of G.*

Proof. The lemma can be proved in a similar way to Lemma 3.4. □

Lemma 3.16 can be used to construct an infinite family of badly behaved cubic bipartite graphs of edge-connectivity two, by choosing any $G_1, G_2 \in \mathcal{SP}(K_{3,3}, H_0, P_0)$. The badly behaved graphs G constructed in this way will all have the property that their bad edges belong to 2-edge-cuts. We can modify the construction to obtain badly behaved graphs without this property. Let G_1, G_2 be graphs and $e_i = x_i y_i \in E(G_i)$ for $i = 1, 2$. Define $(G_1, e_1) \diamond (G_2, e_2)$ to be the graph consisting of the disjoint union of $G_1 - e_1$ and $G_2 - e_2$ and two new adjacent vertices u, v together with the new edges uv, x_1u, y_1v, x_2u, y_2v. It is straightforward to show that if G_1, G_2 are pseudo 2-factor isomorphic cubic bipartite graphs then $(G_1, e_1) \diamond (G_2, e_2)$ is badly behaved with uv as its bad edge.

We next state a similar result to Proposition 3.12 for 2-edge-cuts, which we will use in the following subsection to show that there are no planar pseudo 2-factor isomorphic cubic bipartite graphs.

Lemma 3.17 *Let G_i be a cubic bipartite graph and $e_i = u_i v_i \in E(G_i)$ for $i = 1, 2$. Let $G = (G_1, e_1) \circ (G_2, e_2)$ and suppose that G is pseudo 2-factor isomorphic. Then for some $\{i, j\} = \{1, 2\}$, G_i is pseudo 2-factor isomorphic and G_j is badly behaved with $u_j v_j$ as a bad edge.*

Proof. The lemma can be proved in a similar way to Lemma 3.12. □

We close this subsection by constructing an infinite family of non-hamiltonian connected 2-factor isomorphic cubic bipartite graphs.

Proposition 3.18 *Let G_i be a 2-factor hamiltonian cubic bipartite graph with k vertices and $e_i = u_i v_i \in E(G_i)$ for $i = 1, 2, 3$. Let G be the graph obtained from the disjoint union of the graphs $G_i - e_i$ by adding two new vertices w and z and new edges wu_i and zv_i for $i = 1, 2, 3$. Then G is a non-hamiltonian connected 2-factor isomorphic cubic bipartite graph of edge-connectivity two.*

Proof. The assertion that G has edge-connectivity two follows from the fact that connected cubic bipartite graphs are 2-edge-connected. The assertion that G is non-hamiltonian holds since $G - \{w, z\}$ has three components.

Let F be a 2–factor of G. By symmetry we may assume that $F = F' \cup F_3$, where F_3 is a 2–factor of G_3 avoiding $u_3 v_3$ and $F' = (F_1 - e_1) \cup (F_2 - e_2) \cup \{wu_1, wu_2, zv_1, zv_2\}$ is a 2–factor of $G - G_3$, with F_i a 2–factor of G_i containing $u_i v_i$ for $i = 1, 2$. Since G_i is 2-factor hamiltonian, F_i is a k-circuit for $i = 1, 2, 3$. Thus F has exactly two circuits, one of which has length k and the other length $2k + 2$. Hence G is 2-factor isomorphic. □

It was shown in [7] that all graphs in $\mathcal{SP}(K_{3,3}, H_0)$ are 2-factor hamiltonian. Thus we may apply Proposition 3.18 by taking $G_1 = G_2 = G_3$ to be any graph in $\mathcal{SP}(K_{3,3}, H_0)$ to obtain an infinite family of 2-edge-connected non-hamiltonian 2-factor isomorphic graphs. This family gives counterexamples to the conjecture [1, Conjecture 1.2] that all connected 2-factor isomorphic graphs are 2-factor hamiltonian. Note, however, that Conjecture 3.5 would imply the truth of the modified conjecture that all 3-edge-connected 2-factor isomorphic graphs are 2-factor hamiltonian.

3.2.5 Planar cubic bipartite graphs

We show that there are no planar pseudo 2-factor-isomorphic cubic bipartite graphs.

Theorem 3.19 *Let G be a pseudo 2-factor-isomorphic cubic bipartite graph. Then G is non-planar.*

Proof. Suppose the theorem is false and let G be a counterexample with as few edges as possible. Clearly G is connected, and hence 2-edge-connected. Since G is a planar cubic bipartite graph Euler's formula implies that G has a face of size four. Thus G contains a 4-circuit. If G were 3-edge-connected then Theorem 3.15 would imply that some constituent of G is isomorphic to $K_{3,3}$. This would contradict the planarity of G since each constituent of G can be obtained by edge-contractions (which preserve planarity). Hence G has edge-connectivity two. Lemma 3.17 now implies that some 2-cut reduction of G is a pseudo 2-factor-isomorphic planar cubic bipartite graph. This contradicts the minimality of G. □

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