MOCK EXAM PAPER: MODEL SOLUTIONS

Question 1.

(a) $d^{(4)} = 0.07$ (*nominal* rate of discount convertible quarterly).

$$d = 1 - \left[1 - \frac{d^{(4)}}{4}\right]^4 = 1 - \left[1 - \frac{0.07}{4}\right]^4 = 0.0682 \qquad (\textit{effective annual} \text{ rate of discount})$$

Answer: 6.82% (to 2 decimal places).

(b) The amount to be paid is

$$\exp\left(\int_0^{\frac{1}{12}} \delta \, dt\right) - 1 = e^{\frac{\delta}{12}} - 1 = e^{0.01} - 1 = 0.01$$

Answer: 0.01 (to 2 decimal places).

(c) The following problem replaces part (c) of Question 1 on the original sample examination paper:

> A Sony playstation with a cash price of £189.99 can be purchased for 26 equal monthly repayments with the first repayment being due at the moment of purchase. Find the amount of each monthly repayment if the APR charged is 26.7%.

Set: (unit of time) = (one month); the interest rate per unit time is then $i^* = \frac{i^{(12)}}{12}$. Correspondingly,

$$v^* = \frac{1}{1+i^*} = \frac{1}{1+\frac{i^{(12)}}{12}} = \frac{1}{(1+i)^{\frac{1}{12}}}, \qquad i \text{ being the APR } (i{=}0.267)$$

Let P be the monthly repayment. The loan of £189.99 is to be repaid by an annuity-due of £P payable at unit time intervals over 26 units of time.

Hence, the equation of value:

$$P\,\ddot{a}_{\overline{26|}@v=v^*} = 189.99$$

From this,

$$P = \frac{189.99}{\ddot{a}_{\overline{26|}@v=v^*}} = \frac{189.99}{\frac{1-v^{26}}{1-v}\,\big|_{@v=v^*}} = 9.25$$

Answer: £9.25 (to 2 decimal places).

(d) Using the formulae appended to the end of the examination paper:

$$\begin{aligned}\frac{\ddot{a}^{(p)}_{\overline{n}|}}{\ddot{a}_{\overline{n}|}} &= \frac{\frac{1}{p}\left[\frac{1-v^n}{1-v^{\frac{1}{p}}}\right]}{\frac{v(1-v^n)}{1-v}} = \left[\frac{1}{p(1-v^{\frac{1}{p}})}\right]\left[\frac{1-v}{v}\right] \\ &= \frac{i}{d^{(p)}} \qquad \left(\text{as } \; p(1-v^{\frac{1}{p}}) = d^{(p)} \quad \text{and} \quad \frac{1-v}{v} = i\right)\end{aligned}$$

(e) Set: (unit of time)=(one quarter); the interest rate per unit time is then $i^* = \frac{i^{(4)}}{4} = \frac{0.12}{4} = 0.03$. Correspondingly, $v^* = \frac{1}{1+i^*} = \frac{1}{1.03}$.

Let P be the initial amount of the quarterly repayment. The loan is to be repaid by a combination of an immediate annuity and a deferred annuity. The immediate annuity pays £P at unit time intervals for 40 time units, the first payment being due at time $t = 1$. The deferred annuity pays £100 at unit time intervals for 20 time units, the first payment being due at time $t = 21$. Hence the equation of value reads

$$Pa_{\overline{40}|@v=v^*} + (v^*)^{20} \times (100a_{\overline{20}|@v=v^*}) = 10000$$

Therefore

$$P = \frac{10000 - (v^*)^{20}100\frac{v^*[1-(v^*)^{20}]}{1-v^*}}{\frac{v^*[1-(v^*)^{40}]}{1-v^*}} = 396.99$$

Answer: <u>£396.99</u> (to 2 decimal places).

Question 2.

(a) $\mu(x) = -\frac{1}{s(x)}\frac{d}{dx}s(x)$, where $s(x)$ is the survival function.

The force of mortality can attain values greater than 1, for example $\mu(x) = 2$ when $s(x) = e^{-2x}$.

(b) ${}_tq_x$ is the probability (conditional) that a life aged x at the present time will die within next t years.

${}_tq_x = \frac{s(x)-s(x+t)}{s(x)}$

$\mu(x) = \frac{1}{\omega - x}$ for De Moivre's Law of Mortality.

(c) $s(x) = e^{-\mu x}$ if X has an exponential distribution. Hence, ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{e^{-\mu(x+t)}}{e^{-\mu x}} = e^{-\mu t}$.

(d)

$$\begin{aligned} P(K(x)=k) &= P(k \le T(x) < k+1) = P(k < T(x) \le k+1) \\ &= P(T(x) \ge k) - P(T(x) \ge k+1) \\ &= {}_k p_x - {}_{k+1} p_x . \end{aligned}$$

The curtate expectation of life is defined as $\mathsf{e}_x = E(K(x))$.

$$\begin{aligned} \mathsf{e}_x &= \sum_{k=0}^{\infty} k \; P(K(x)=k) = \sum_{k=0}^{\infty} k \; {}_k p_x - \sum_{k=0}^{\infty} k \; {}_{k+1} p_x = \sum_{k=1}^{\infty} k \; {}_k p_x - \sum_{k=1}^{\infty} (k-1) \; {}_k p_x \\ &= \sum_{k=1}^{\infty} \; {}_k p_x . \end{aligned}$$

(e) Follows from

$${}_k p_x = \frac{s(x+k)}{s(x)} = \left[\frac{s(x+1)}{s(x)}\right]\left[\frac{s(x+1+k-1)}{s(x+1)}\right] = (p_x)({}_{k-1}p_{x+1}), \quad k \ge 1.$$

Namely,

$$\frac{\mathsf{e}_x}{1+\mathsf{e}_{x+1}} = \frac{\sum_{k=1}^{\infty} \; {}_k p_x}{1+\mathsf{e}_{x+1}} = \frac{p_x \sum_{k=1}^{\infty} \; {}_{k-1} p_x}{1+\mathsf{e}_{x+1}} = \frac{p_x \sum_{k=0}^{\infty} \; {}_k p_{x+1}}{1+\mathsf{e}_{x+1}} = p_x$$

(f) The p.d.f. of the random variable $X|(x_1 < X < x_2)$ is $\frac{f_X(x)}{P(x_1<X<x_2)}$ when $x_1 < x < x_2$ and zero otherwise, $f_X(x)$ being the p.d.f of X. Therefore

$$\begin{aligned} E(X|(x_1 < X < x_2)) &= \int_{x_1}^{x_2} \frac{x f_X(x)}{P(x_1 < X < x_2)} \; dx = \int_{x_1}^{x_2} \frac{x(-\frac{d}{dx} s(x))}{s(x_1) - s(x_2)} \; dx \\ &= \int_{x_1}^{x_2} \frac{x s(x) \mu(x)}{s(x_1) - s(x_2)} \; dx, \end{aligned}$$

where we have used

$$f_X(x) = -\frac{d}{dx} s(x) = s(x)\mu(x) \quad \text{and} \quad P(x_1 < X < x_2) = s(x_1) - s(x_2).$$

Question 3.

(a) 1) ${}_{10}p_{50} = \frac{l_{60}}{l_{50}} = \frac{78924}{90085} = \underline{0.876}$ (to 3 s.d.).

2) ${}_{5|5}q_{60} = \frac{l_{65}-l_{70}}{l_{60}} = \frac{68490-54806}{78924} = \underline{0.173}$ (to 3 s.d.).

3) $1000\frac{d_{60}+d_{61}}{l_0} = \frac{1805+1947}{1000} = \underline{38}$ (to the nearest integer).

4) Age of selection is 50 years, hence [50]. The duration of the select period is 1 year, hence $l_{[50]+1+k} = l_{51+k}$ for all $k \geq 0$.
$p_{[50]} = \frac{p_{50}}{2} = \frac{0.99272}{2}$; $l_{[50]} = \frac{l_{51}}{p_{[50]}} = \frac{2\times 89429}{0.99272} = 180169.6349$.

$$\begin{aligned}
\mathring{e}_{[50]} \simeq \frac{1}{2} + \mathsf{e}_{[50]} &= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{l_{[50]+k}}{l_{[50]}} = \frac{1}{2} + \frac{1}{l_{[50]}}(l_{51} + l_{52} + \ldots) \\
&= \frac{1}{2} + \frac{l_{50}}{l_{[50]}} \frac{1}{l_{50}}(l_{51} + l_{52} + \ldots) = \frac{1}{2} + \frac{l_{50}}{l_{[50]}} \mathsf{e}_{50} \\
&\simeq \frac{1}{2} + \frac{l_{50}}{l_{[50]}}\left(\mathring{e}_{50} - \frac{1}{2}\right) \\
&= 0.5 + \frac{90085}{180169.6349}\Big(22.68 - 0.5\Big) = \underline{11.59}
\end{aligned}$$

(b) Fix an integer x and consider ${}_tp_x$ as a function of t in the interval $0 \leq t \leq 1$. Then

$$ {}_tp_x\mu(x+t) = -\frac{s(x+t)}{s(x)} \frac{1}{s(x+t)} \frac{d}{d(x+t)} s(x+t) = -\frac{1}{s(x)} \frac{d}{dt} s(x+t).$$

The assumption of uniform distribution of death is equivalent to linear interpolation of $s(x+t)$ for fractional t. Hence, under this assumption,

$$\frac{d}{dt}s(x+t) = \frac{d}{dt}[(1-t)s(x) + ts(x+1)] = s(x+1) - s(x)$$

and

$$ {}_tp_x\mu(x+t) = \frac{s(x) - s(x+1)}{s(x)} = q_x.$$

(c) $z_1 = v^{T(x)}$, where $v = \frac{1}{1+i}$, i being the interest rate.

(d) $z_2 = v^{K(x)+1}$.

(e)

$$\begin{aligned}
\bar{A}_x = E(v^{T(x)}) &= \int_0^{\infty} v^t f_{T(x)}(t)\ dt = \int_0^{\infty} v^t\ {}_tp_x\mu(x+t)\ dt \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} v^t\ {}_tp_x\mu(x+t)\ dt \\
&= \sum_{k=0}^{\infty} \int_0^{1} v^{k+t}\ {}_{t+k}p_x\mu(x+k+t)\ dt
\end{aligned}$$

Now apply ${}_{t+k}p_x = {}_kp_x \, {}_tp_{x+k}$ and (b) to obtain that

$$\begin{aligned}
\bar{A}_x &= \sum_{k=0}^{\infty} v^k \, {}_kp_x \int_0^1 v^t \, {}_tp_{x+k}\mu(x+k+t) \; dt = \sum_{k=0}^{\infty} v^k \, {}_kp_x \, q_{x+k} \int_0^1 v^t \; dt \\
&= \sum_{k=0}^{\infty} v^k \, {}_kp_x \, (1 - \, p_{x+k}) \frac{1-v}{\delta} = \frac{1-v}{v\delta} \sum_{k=0}^{\infty} v^{k+1} (\, {}_kp_x - \, {}_kp_x \, p_{x+k}) \\
&= \frac{i}{\delta} \sum_{k=0}^{\infty} v^{k+1} (\, {}_kp_x - \, {}_{k+1}p_x) = \frac{i}{\delta} E(v^{K(x)+1}) \\
&= \frac{i}{\delta} A_x
\end{aligned}$$

Question 4.

(a) $\ddot{a}_{[40]} = \frac{N_{[40]}}{D_{[40]}} = \frac{131995.19}{6981.5977} = \underline{18.91}$

(b) The expected present value of the benefit payment is £100000 $A_{[40]}$. By the conversion relationship, $A_{[40]} = 1 - d\ddot{a}_{[40]}$. Hence

$$100000 A_{[40]} = 100000(1 - \frac{0.04}{1.04} 18.91) = \underline{27269.23}$$

(c) The annual premium:

$$P = \frac{100000 A_{[40]}}{\ddot{a}_{[40]}} = \frac{27269.23}{18.91} = \underline{1442.05}$$

(d) Consider unit death benefits. Then the minimum amount is such a value of h that $P(Z \le h) = 0.95$, where Z is the total present value of death benefits, $Z = \sum_{j=1}^{100} z_j$, $z_j = v^{T_j}$, being the present value of the death benefit for life j.

Assuming a constant force of mortality μ and a constant force of interest δ,

$$E(z_j) = E(v_j^T) = \int_0^{\infty} v^t \, {}_tp_x \mu \; dt = \int_0^{\infty} v^t \, e^{-\mu t} \mu \; dt = \frac{\mu}{\delta + \mu}$$

$$\operatorname{var}(z_j) = E(v^{2T}) - [E(z_j)]^2 = \frac{\mu}{\mu + 2\delta} - \left[\frac{\mu}{\mu + \delta} \right]^2 .$$

Therefore $E(Z) = 100 \frac{\mu}{\mu+\delta}$ and $\operatorname{var}(Z) = 100 \left\{ \frac{\mu}{\mu+2\delta} - \left[\frac{\mu}{\mu+\delta} \right]^2 \right\}$.

Approximating the distribution of Z by a normal one,

$$\frac{h - E(Z)}{\sqrt{\mathrm{var}(Z)}} = 1.645$$

and therefore

$$h = E(Z) + 1.645\sqrt{\mathrm{var}(Z)} = 100\frac{\mu}{\mu+\delta} + 1.645 \times 10\sqrt{\frac{\mu}{\mu+2\delta} - \left[\frac{\mu}{\mu+\delta}\right]^2}$$

Substituting $\delta = \ln(1+i) = \ln(1+0.08)$ and $\mu = 0.1$, we obtain

$$h = 60.99922658$$

As the sum assured is £50000 the minimum amount to be invested is

$$£50000 \times h = \underline{£3049961.33}.$$

(e) $z = e^{-n\delta} \times \mathrm{Bernoulli}(e^{-n\mu})$.

z is a discrete-type random variable, its p.m.f. is

values	$z = 0$	$z = e^{-n\delta}$
probs.	$1 - e^{-n\mu}$	$e^{-n\mu}$

Question 5.

(a) The probability that a newborn will survive to its third year of life is $P(X > 2) = P(X > 2|X > 1))P(X > 1) = p_0 p_1$. From the Leslie matrix $p_0 = \frac{1}{2}$ and $p_1 = \frac{1}{3}$. Hence this probability is $\underline{\frac{1}{6}}$

$$\underline{n}(1) = M\underline{n}(0) = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 500 \\ 333\frac{1}{3} \end{bmatrix}$$

$$\underline{n}(2) = M^2\underline{n}(0) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \\ 166\frac{2}{3} \end{bmatrix}$$

$$\underline{n}(3) = M^3\underline{n}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

As M^3 is the identity matrix, so is $M^{15} = (M^3)^5$. Therefore, $\underline{n}(15) = \underline{n}(0)$.

$\underline{n}(t)$ is oscillating with a period of 3 years.

(b) Identical to the derivation given in lectures (simple birth-death continuous time population model).