Infinite dimensional holomorphic homogeneous regular domains

Cho-Ho Chu · Kang-Tae Kim · Sejun Kim

Abstract. We extend the concept of a finite dimensional *holomorphic homogeneous regular* (HHR) domain and some of its properties to the infinite dimensional setting. In particular, we show that infinite dimensional HHR domains are domains of holomorphy and determine completely the class of infinite dimensional bounded symmetric domains which are HHR. We compute the greatest lower bound of the squeezing function of all HHR bounded symmetric domains, including the two exceptional domains. We also show that uniformly elliptic domains in Hilbert spaces are HHR.

1. Introduction

The concept of a *holomorphic homogeneous regular* (*HHR*) complex manifold M of finite dimension has been introduced by Liu, Sun and Yau [20] in connection with the estimation of several invariant metrics on the moduli and Teichmüller spaces of Riemann surfaces. It can be described by saying that a particular function $\sigma : M \to (0,1]$, called the *squeezing function*, has strictly positive lower bound (cf. [7]). These manifolds possess many important geometric properties (e.g. all classical metrics on them are equivalent) [20, 21] and have also been studied by several authors (see, for example, [7, 8, 9, 15, 25]) in the case of complex domains. In particular, it has been shown in [25] that a holomorphic homogeneous regular bounded domain D in $\mathbb{C}^n$ must be pseudoconvex and all strongly convex domains in $\mathbb{C}^n$ are holomorphic homogeneous regular. Recently, it has been shown in [15] that all bounded convex domains in $\mathbb{C}^n$ are holomorphic homogeneous regular. The squeezing function on a bounded homogeneous domain in $\mathbb{C}^n$ is constant, by its holomorphic invariance, and has been computed explicitly for the four classical series of Cartan domains in [18]. In view of these interesting works, it is natural to ask if they can be extended to the setting of infinite dimensional domains.

The object of this paper is to begin a study of infinite dimensional holomorphic homogeneous regular domains. We extend the concept of a holomorphic homogeneous regular

C-H. Chu
School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK
e-mail: c.chu@qmul.ac.uk.
K.-T. Kim
Center for Geometry and its applications and Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, South Korea
e-mail: kimkt@postech.ac.kr.
S. Kim
Center for Geometry and its applications and Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, South Korea
e-mail: rlatpwns21@postech.ac.kr.

domain and generalise the aforementioned results to the infinite dimensional setting. In addition, we also obtain new results in finite dimensions, in particular, the squeezing functions are explicitly computed for all bounded symmetric domains, including the two exceptional domains, which were left untreated in [18].

The concept of the squeezing function for domains in $\mathbb{C}^n$ involves comparing a given domain with various Euclidean balls via embeddings. For infinite dimensional domains, we consider their holomorphic embeddings in *Hilbert balls*, that is, open unit balls of complex Hilbert spaces.

Throughout, all Banach spaces V are over the complex field $\mathbb{C}$ and the dual of V is denoted by V^*. Let D be a bounded domain in a (complex) Banach space V. We will call a map $f : D_1 \to D_2$ between two domains a *holomorphic embedding* of D_1 in D_2 if $f(D_1)$ is a domain in D_2 and f is biholomorphic onto $f(D_1)$.

Let $B_H = \{x \in H : \|x\| < 1\}$ be the open unit ball of a Hilbert space H and denote by $H(D, B_H)$ the set of all holomorphic embeddings of D into B_H, which may be an empty set. For instance, if D is the open unit ball of the Banach space ℓ^∞ of bounded sequences, then $H(D, B_H) = \emptyset$ for any Hilbert ball B_H.

In fact, $H(D, B_H) \neq \emptyset$ if and only if the ambient Banach space V of D is linearly homeomorphic to H. Indeed, if there is a holomorphic embedding $f : D \to B_H$, then V, as the tangent space at a point p in D, must be linearly homeomorphic to H, which is the tangent space of $f(D)$ at $f(p)$. Conversely, if $\varphi : V \to H$ is a linear homeomorphism, then we have $\varphi(D) \subset RB_H$ for some $R > 0$, and for each $p \in D$, the map $f : z \in D \mapsto \varphi(z-p)/2R \in B_H$ is a biholomorphic map onto the domain $f(D)$ in B_H, with $f(p) = 0$ and $rB_H \subset f(D) \subset B_H$ for some $r > 0$.

Given $H(D, B_H) \neq \emptyset$, then for each $p \in D$, the set

$$\mathcal{F}(p, D) = \{f \in H(D, B_H) : f(p) = 0\}$$

is nonempty, as noted previously. Hence we can define the *squeezing function* $\sigma_D : D \to (0, 1]$ by

$$\sigma_D(p) = \sup_{f \in \mathcal{F}(p,D)} \{r > 0 : rB_H \subset f(D)\}.$$

The *squeezing constant* $\hat{\sigma}_D$ for D is defined by

$$\hat{\sigma}_D = \inf_{p \in D} \sigma_D(p).$$

Both the squeezing function and squeezing constant are biholomorphic invariants.

Remark 1.1. We note that, if $H(D, B_H) \neq \emptyset$, then the definition of the squeezing function for a domain $D \subset V$ does not depend on the chosen Hilbert ball B_H. Indeed, if there is a holomorphic embedding of D into another Hilbert ball B_K of a Hilbert space K, then the previous remarks imply that there is a continuous linear isomorphism $T : H \to K$. Let $\alpha : H^* \to H$ and $\beta : K \to K^*$ be the canonical isometries. Then the linear isomorphism $\alpha T^* \beta T : H \to H$ satisfies

$$\langle \alpha T^* \beta T x, y\rangle_H = \langle Tx, Ty\rangle_K \qquad (x, y \in H)$$

and the linear isomorphism $T(\alpha T^* \beta T)^{-1/2} : H \to K$ is an isometry. It follows that the squeezing functions σ_D defined in terms B_H and B_K respectively are identical.

We now extend the concept of a finite dimensional HHR manifold introduced in [20, 21] to infinite dimensional complex domains. A finite dimensional HHR domain is also called a domain with *uniform squeezing property* in [25].

Definition 1.2. A bounded domain D in a complex Banach space V is called *holomorphic homogeneous regular* (HHR) if D admits a holomorphic embedding into some Hilbert ball B_H and its squeezing function $\sigma_D : D \to (0,1]$ has a strictly positive lower bound, that is, $\hat{\sigma}_D > 0$.

Remark 1.3. If D is an HHR domain in a Banach space V, then as noted previously, V must be linearly homeomorphic to a Hilbert space. We call V an *isomorph of a Hilbert space*. The class of of these Banach spaces has been characterised by many authors, for instance, it has been shown in [19] that a Banach space is an isomorph of a Hilbert space if and only if it is of type 2 and cotype 2. We refer to [23, Chapter IV] for more details.

For infinite dimensional bounded *symmetric* domains, we shall see that only those of finite rank can be embedded holomorphically in a Hilbert ball. We prove the following main results.

Theorem 2.5. *An HHR domain is a domain of holomorphy.*

This result extends the finite dimensional result in [25, Lemma 2] since a domain of holomorphy in a Banach space is pseudoconvex (cf. [22, 11.4, 37.7]). We note that a domain of holomorphy need not be HHR even in finite dimensions, as shown in [9, Theorem 1].

The following result reveals the connection between the rank of a symmetric domain and the extent to which a Hilbert ball can be squeezed inside it.

Theorem 4.5. *Let D be a bounded symmetric domain in a complex Banach space V. Then D is HHR if and only if D is of finite rank. In this case, D is biholomorphic to a finite product*

$$D_1 \times \cdots \times D_k$$

of irreducible bounded symmetric domains and we have

$$\hat{\sigma}_D = \left(\frac{1}{\hat{\sigma}^2_{D_1}} + \cdots + \frac{1}{\hat{\sigma}^2_{D_k}} \right)^{-1/2}.$$

If $\dim D_j < \infty$, then D_j is a classical Cartan domain or an exceptional domain, and $\hat{\sigma}_{D_j} = 1/\sqrt{p_j}$ where p_j is the rank of D_j.

If $\dim D_j = \infty$, then D_j is either a Lie ball or a type I domain of finite rank p_j. For a Lie ball D_j, we have $\hat{\sigma}_{D_j} = 1/\sqrt{2}$. For a rank p_j type I domain D_j, we have $\hat{\sigma}_{D_j} = 1/\sqrt{p_j}$.

Theorem 5.3. *Let Ω be a uniformly elliptic domain in a Hilbert space H. Then Ω is HHR.*

We introduce the concept of a uniformly elliptic domain in Section 5, which generalises the notion of strong convexity. This theorem generalises the finite dimensional result in [25, Proposition 1].

2. Holomorphic homogeneous regular domains

We begin our discussion of infinite dimensional HHR domains in this section by showing some properties of the squeezing function and conclude with a proof of pseudoconvexity for these domains.

Given two (nonempty) sets A and B in a Banach space V, we write

$$d(A,B) = \inf\{\|x-y\| : x \in A, y \in B\}$$

and for $p \in V$, write $d(p,B)$ for $d(\{p\},B)$ which is the distance from p to B. Let D be a bounded domain in V, a closed subset K of D is said to be *strictly contained* in D if $d(K, V\backslash D) > 0$. Let

$$B_V(p,r) := \{z \in V : \|z-p\| < r\}$$

denote the norm-open ball centred at p with radius $r > 0$. The open unit ball $B_V(0,1)$ is often written simply B_V. We will make use of the Carathéodory distance C_D on D, which is equivalent to the norm-distance on any closed ball (for the norm) strictly contained in D (see [10, Theorem IV.2.2]). For each $v \in B_V$, we have $C_{B_V}(v,0) = \tanh^{-1}\|v\|$, by [10, Theorem IV.1.8].

In what follows, the boundary of a topological subspace E of V will be denoted by ∂E. The complement of E in V will be denoted by E^c and as usual, $\overline{E}$ denotes the closure of E.

We first show that the squeezing function is continuous. Our proof follows the arguments in [7, Theorem 3.1]. It is included for completeness.

Proposition 2.1. *Let D be a bounded domain in a Banach space V linearly homeomorphic to a Hilbert space H. Then the squeezing function $\sigma_D : D \to (0,1]$ is continuous.*

Proof. Let (z_k) be a sequence converging to $a \in D$. We show

$$\lim_{k\to\infty} \inf \sigma_D(z_k) \geq \sigma_D(a) \geq \lim_{k\to\infty} \sup \sigma_D(z_k).$$

Let $0 < 2\varepsilon < \sigma_D(a)$ and pick $\sigma_D(a) \geq \rho > \sigma_D(a) - \varepsilon$ such that there is a holomorphic embedding $f : D \to B_H$ satisfying $f(a) = 0$ and $\rho B_H \subset f(D)$. By continuity, we have

$$\|f(z_k)\| < \varepsilon$$

for $k > K$, for some $K > 0$. Consider the holomorphic embedding $f_k : D \to B_H$ given by

$$f_k(\omega) = \frac{f(\omega) - f(z_k)}{1+\varepsilon} \qquad (\omega \in D)$$

which satisfies $f_k(z_k) = 0$ and

$$\frac{\rho - \varepsilon}{1+\varepsilon} B_H \subset f_k(D).$$

This gives

$$\sigma_D(z_k) \geq \frac{\rho-\varepsilon}{1+\varepsilon} > \frac{\sigma_D(a) - 2\varepsilon}{1+\varepsilon}$$

for $k > K$ and hence $\lim_{k\to\infty} \inf \sigma_D(z_k) \geq \sigma_D(a)$ since $\varepsilon > 0$ was arbitrary.

For the upper limit, let $0 < 2\varepsilon < \lim_k \inf \sigma_D(z_k)$ and let $f_k : D \to B_H$ be a holomorphic embedding satisfying $f_k(z_k) = 0$ and $\rho_k B_H \subset f_k(D)$ for some $\sigma_D(z_k) \geq \rho_k > \sigma_D(z_k) - \varepsilon$. Since $C_D(z_k, a) \to 0$ as $k \to \infty$, we have

$$\tanh^{-1} \|f_k(a)\| = C_{B_H}(0, f_k(a)) \leq C_D(0, a) \to 0$$

and hence there exists some $M > 0$ such that $\|f_k(a)\| < \varepsilon$ for $k > M$. By analogous arguments as before, one obtains

$$\sigma_D(a) \geq \frac{\rho_k - \varepsilon}{1+\varepsilon} > \frac{\sigma_D(z_k) - 2\varepsilon}{1+\varepsilon}$$

for $k > M$, which gives $\sigma_D(a) \geq \lim_{k\to\infty} \sup \sigma_D(z_k)$. □

Although the continuity of the squeezing function implies readily that if there is a sequence (p_k) in a *finite dimensional* bounded domain D with $\lim_k \sigma_D(p_k) = 0$, then the sequence admits a subsequence (p_j) converging to a boundary point $p \in \partial D$, this is not immediately clear for infinite dimensional domains. Nevertheless, one can still show, in infinite dimension, (p_k) has a subsequence (p_j) for which the distance $d(p_j, \partial D)$ to the boundary tends to 0. We prove a lemma first.

Lemma 2.2. *Let Ω be a bounded domain in an isomorph V of a Hilbert space H and $\varphi : V \to H$ a linear homeomorphism. Then there is a constant $m > 0$ such that for each $q \in \Omega$ satisfying $B_V(q, s) \subset \Omega$ for some $s > 0$, we have*

$$\sigma_\Omega(q) \geq \frac{s}{m^2 \|\varphi\| \|\varphi^{-1}\|}.$$

Proof. By a translation, we may assume $q = 0$. Since Ω is bounded, we have $\Omega \subset B_V(0, m)$ for some $m > 0$ and

$$\frac{1}{\|\varphi^{-1}\|} B_H(0, m) \subset \varphi(B_V(0, m)) \subset B_H(0, m\|\varphi\|) = m\|\varphi\| B_H. \tag{2.1}$$

The restriction of φ to Ω, still denoted by φ, is a holomorphic embedding of Ω into $m\|\varphi\|B_H$ satisfying $\varphi(q) = 0$. It follows from (2.1) that

$$\frac{s}{m\|\varphi^{-1}\|} B_H(0, m) \subset \varphi(B_V(0, s)) \subset \varphi(\Omega) \subset \varphi(B_V(0, m)) \subset m\|\varphi\| B_H.$$

Hence we have

$$\sigma_\Omega(q) \geq \frac{s}{m^2 \|\varphi\| \|\varphi^{-1}\|}.$$

□

Lemma 2.3. *Let (p_k) be a sequence in a bounded convex domain Ω in an isomorph V of a Hilbert space such that $\lim_{k\to\infty} \sigma_\Omega(p_k) = 0$. Then there is a subsequence (p_j) of (p_k) such that*

$$\lim_{j\to\infty} d(p_j, \partial\Omega) = 0.$$

Further, there is a sequence (p'_j) in Ω such that $\lim_{j\to\infty} \sigma_\Omega(p'_j) = 0$, and for each j, there exists a boundary point $q_j \in \partial\Omega$ with $\|p'_j - q_j\| = d(p'_j, \partial\Omega)$.

Proof. Let (p_k) be the given sequence satisfying

$$\lim_{k\to\infty} \sigma_\Omega(p_k) = 0. \tag{2.2}$$

Since the bounded domain Ω is relatively weakly compact in V, there is a subsequence (p_j) in Ω converging weakly to some point $p \in \overline{\Omega}$. We do not know if the squeezing function σ_Ω is weakly continuous on Ω.

Let $r_j = d(p_j, \partial\Omega)$ denote the distance from p_j to the boundary $\partial\Omega$. We first show that $\lim_{j\to\infty} r_j = 0$. Otherwise, we may assume (by choosing a subsequence)

$$r_j \geq s, \quad \text{for some} \quad s > 0$$

for all j. For all $z \in \partial\Omega$, we have $\|z - p_j\| \geq r_j$. Observe that $B_V(p_j, r_j) \subset \Omega$, for if there exists some $\omega \in B_V(p_j, r_j)\backslash\Omega$, then we must have $\omega \notin \overline{\Omega}$. Therefore the (real) line joining p_j and ω must intersect $\partial\Omega$ at a point z_0 say, which gives a contradiction that

$$r_j \leq \|z_0 - p_j\| \leq \|\omega - p_j\| < r_j.$$

By Lemma 2.2, there exists $m > 0$ such that

$$\sigma_\Omega(p_j) \geq \frac{r_j}{m^2\|\varphi\|\|\varphi^{-1}\|} \geq \frac{s}{m^2\|\varphi\|\|\varphi^{-1}\|} > 0,$$

contradicting $\lim_j \sigma_\Omega(p_j) = 0$. Therefore we have established

$$r_j = d(p_j, \partial\Omega) \to 0 \quad \text{as} \quad j \to \infty.$$

To show the second assertion, we make use of a result in [3, Theorem 3.2], which states that in a reflexive Banach space V, if the complement $V\backslash C$ of a non-empty closed set C in V is convex, then C is almost proximinal, in other words, there is a dense G_δ set A in $V\backslash C$ such that for each $x \in A$, there is a point $z \in C$ satisfying

$$\|x - z\| = d(x, C).$$

The given Banach space V is reflexive. We apply the above result to the set $C = V\backslash\Omega$, which is almost proximinal. By continuity of the squeezing function σ_Ω, there is an open neighbourhood N_j of p_j such that $\sigma_\Omega(x) < 2\sigma_\Omega(p_j)$ for all $x \in N_j$ and for each j. By density of A, one can find $p_j' \in A \cap N_j$ for which there exists $q_j \in C$ satisfying

$$\|p_j' - q_j\| = d(p_j', C) = d(p_j', V\backslash\Omega) \leq d(p_j', \partial\Omega)$$

where the last inequality holds because the boundary $\partial\Omega$ is contained in $V\backslash\Omega$.

If $q_j \notin \partial\Omega$, then $q_j \notin \overline{\Omega}$ since $q_j \notin \Omega$. Hence the line segment $\{p_j' + \alpha(q_j - p_j') : 0 \leq \alpha \leq 1\}$ joining p_j' and q_j must intersect the boundary $\partial\Omega$ at some point $\omega = p_j' + \beta(q_j - p_j') \in \partial\Omega$ with $0 < \beta < 1$. It follows that

$$\|p_j' - q_j\| \leq d(p_j', \partial\Omega) \leq \|p_j' - \omega\| = \beta\|p_j' - q_j\| < \|p_j' - q_j\|$$

which is impossible. Hence we have $q_j \in \partial\Omega$ and $\|p_j' - q_j\| = d(p_j', \partial\Omega)$.

Finally, $\sigma_\Omega(p_j') < 2\sigma_\Omega(p_j)$ for all j implies $\lim_j \sigma_\Omega(p_j') = 0$. $\square$

To show that an HHR domain D in a complex Banach space V is pseudoconvex, we show that D is a domain of holomorphy, as defined in [22]. In finite dimensions, a bounded domain D is a domain of holomorphy if (D, C_D) is complete [16, p.368]. We first extend this useful result to infinite dimension.

Lemma 2.4. *Let D be a bounded domain in a complex Banach space V. If D is complete with respect to the Carathéodory distance, then it is a domain of holomorphy.*

Proof. Suppose that D is not a domain of holomorphy. Then by definition, there are open subsets U, W of V satisfying the following conditions:

(i) U is connected.
(ii) $D \cap U \neq \emptyset$.
(iii) $U \not\subset D$.
(iv) $\emptyset \neq W \subset D \cap U$.
(v) For each holomorphic function $f\colon D \to \mathbb{C}$, there is a holomorphic function $\tilde{f}\colon U \to \mathbb{C}$ such that $\tilde{f}(z) = f(z)$ for each $z \in W$.

We deduce a contradiction. Without loss of generality, we may assume that U is bounded. Let W_0 be a connected component of $U \cap D$ with $W_0 \cap W \neq \emptyset$. Then we have

$$\partial W_0 \cap \partial D \cap U \neq \emptyset.$$

Indeed, if this is not the case, then for each $p \in U \backslash W_0 \neq \emptyset$, either $p \notin \partial W_0$ or $p \notin \partial D$. If $p \notin \partial W_0$, then there is a norm-open ball $B_p \subset U$ containing p such that either $B_p \cap W_0 = \emptyset$ or $B_p \cap W_0^c = \emptyset$. Since $p \notin W_0$, we must have $B_p \cap W_0 = \emptyset$. On the other hand, if $p \notin \partial D$, then there is an open ball B_p containing p such that $B_p \cap D = \emptyset$ or $B_p \cap D^c = \emptyset$. In either case, we have $B_p \cap W_0 = \emptyset$ since, if $B_p \subset D$, then the connected ball B_p resides in a connected component W_1 of $U \cap D$ and we must have $W_1 \neq W_0$ as $p \notin W_0$. Now the disconnection

$$U = W_0 \cup \left(\bigcup_{p \in U \backslash W_0} B_p \right)$$

contradicts the connectedness of U.

Pick a point $p \in \partial W_0 \cap \partial D \cap U$ and let (z_n) be a sequence in W_0 norm-converging to p. By omitting the first few terms of the sequence if necessary, we may assume that (z_n) and p are contained in a closed ball strictly contained in U. It follows that (z_n) also converges to p with respect to the Carathéodory distance C_U.

By condition (v) above, each holomorphic function $f\colon D \to \mathbb{C}$ with $|f(z)| < 1$ extends to a holomorphic function $\tilde{f} : U \to \mathbb{C}$, which coincides with f on the connected component W_0 by the identity principle. Moreover, if $|\tilde{f}(u)| > 1$ for some $u \in U$, then we deduce a contradiction by considering the extension to U of the function $\frac{1}{f - \tilde{f}(u)}$ on D. Hence we must have $|\tilde{f}(u)| \leq 1$ for all $u \in U$ and, by the maximum principle, $|\tilde{f}(u)| < 1$ for all $u \in U$. It follows that

$$C_D(z_n, z_m) \leq C_U(z_n, z_m)$$

for $n, m = 1, 2, \ldots$, where $C_U(z_n, z_m)$ converges to $C_U(p, p) = 0$ as $n, m \to \infty$. Hence (z_n) is a Cauchy sequence in D with respect to C_D. However, (z_n) does not converge in D, with respect to C_D. Indeed, if (z_n) C_D-converges to some point $z \in D$ say, then by [10, Lemma 2.1], there is a constant $\alpha > 0$ such that

$$\alpha \|z_n - z\| \leq C_D(z_n, z) \to 0 \quad \text{as} \quad n \to \infty$$

which is impossible since (z_n) does not converge in D with respect to the norm-distance. This shows that (D, C_D) fails to be complete, which is a contradiction. We therefore conclude that D is a domain of holomorphy. □

We now extend the result in [25, Lemma 2] to the following infinite dimensional setting.

Theorem 2.5. *Let D be an HHR domain in a complex Banach space V. Then D is a domain of holomorphy.*

Proof. In view of Lemma 2.4, we need only show that the Carathéodory distance in D is complete.

By the hypothesis, the squeezing constant $\hat{\sigma}_D$ takes the value, say, $r \in (0, 1]$. Let (x_n) be a C_D-Cauchy sequence in D. We show that (x_n) C_D-converges.

Let $\varepsilon = \tanh^{-1} \frac{r}{2}$. Then there is a number $N > 0$ such that $C_D(x_n, x_N) < \varepsilon$ for $n > N$.

Let $f \colon D \to B_H$ be a holomorphic embedding into a Hilbert ball B_H with $f(x_N) = 0$ and $B_H(0, \frac{3r}{4}) \subset f(D)$. Then the inverse holomorphic map $g := f^{-1} \colon f(D) \to D$ is well-defined on the ball $B_H(0, \frac{3r}{4})$.

We have, for $n > N$,

$$C_{B_H}(0, f(x_n)) = C_{B_H}(f(x_N), f(x_n)) \leq C_D(x_N, x_n) < \varepsilon = \tanh^{-1} \frac{r}{2}$$

as well as

$$\lim_{n,m\to\infty} C_{B_H}(f(x_m), f(x_n)) \leq \lim_{n,m\to\infty} C_D(x_m, x_n) = 0.$$

Since B_H is complete in the Carathéodory distance, there is a subsequence (x_{n_k}) of (x_n) such that $f(x_{n_k})$ converges to some $y_0 \in B_H$ with respect to C_{B_H}, and $C_{B_H}(0, y_0) \leq \varepsilon$. Hence, as noted previously, we have $y_0 \in \overline{B}_H(0, \frac{r}{2}) \subset B_H(0, \frac{3r}{4}) \subset f(D)$ and also,

$$\begin{aligned}
\lim_{k\to\infty} C_D(x_{n_k}, g(y_0)) &\leq \lim_{k\to\infty} C_D(g(y_{n_k}), g(y_0)) \\
&\leq \lim_{k\to\infty} \frac{4}{3r} C_{B_H}(f(g(y_{n_k})), f(g(y_0))) \\
&= \lim_{k\to\infty} \frac{4}{3r} C_{B_H}(y_{n_k}, y_0) = 0.
\end{aligned}$$

It follows that the sequence (x_n) converges to $g(y_0)$ in D with respect to C_D and the proof is complete. □

In the remaining sections, we will show that various infinite dimensional domains are HHR, including the finite-rank bounded symmetric domains and the class of strongly convex domains in Hilbert spaces.

3. Bounded symmetric domains

In this section, we discuss infinite dimensional bounded symmetric domains and some basic results which are needed later. We will make use of the underlying Jordan algebraic structures of a bounded symmetric domain to study the squeezing function.

Let D be a bounded symmetric domain in a complex Banach space V. Then V carries the structure of a *JB*-triple*, by Kaup's Riemann mapping theorem [12] (see also [4, Theorem 2.5.26]). More precisely, V is equipped with an *equivalent norm* $\|\cdot\|$ and a continuous *Jordan triple product*

$$\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$$

which is linear in the outer variables but conjugate linear in the middle one, and satisfies the following conditions:

(i) $\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\}$;
(ii) $z \,\square\, z : v \in V \mapsto \{z, z, v\} \in V$ is a hermitian operator on V, that is, $\|\exp it(z \,\square\, z)\| = 1$ for all $t \in \mathbb{R}$;
(iii) $z \,\square\, z$ has non-negative spectrum;
(iv) $\|z \,\square\, z\| = \|z\|^2$

for all $u, v, x, y, z \in V$. In this case, D is biholomorphic to the open unit ball $\{v \in V : \|v\| < 1\}$ of the JB*-triple $(V, \|\cdot\|)$ and, we say that D is realised as the open unit ball of the JB*-triple $(V, \|\cdot\|)$.

The *rank* of D can be defined in terms of the Jordan structures of V. A closed subspace E of a JB*-triple V is called a *subtriple* if $a, b, c \in E$ implies $\{a, b, c\} \in E$. For each $a \in V$, let $V(a)$ be the smallest subtriple of V containing a. For $V \neq \{0\}$, the *rank* of V is defined to be

$$r(V) = \sup\{\dim V(a) : a \in V\} \in \mathbb{N} \cup \{\infty\}.$$

The *rank* of D is defined to be $r(V)$. A (nonzero) JB*-triple V has *finite rank*, that is, $r(V) < \infty$ if, and only if, V is a reflexive Banach space (see [13, Proposition 3.2]). In particular, if there is a holomorphic embedding of D into a Hilbert ball, then V is linearly homeomorphic to a Hilbert space and hence D must be of finite rank.

A finite-rank JB*-triple can be *coordinatised* by elements called *tripotents*. An element e in a JB*-triple V is called a *tripotent* if $\{e, e, e\} = e$. A nonzero tripotent e is called *minimal* if $\{e, V, e\} = \mathbb{C}\, e$. The Banach subspace $K_0(V)$ of V generated by the minimal tripotents has been studied in [6]. Two elements $a, b \in V$ are said to be mutually (triple) *orthogonal* if $a \,\square\, b = b \,\square\, a = 0$, where $a \,\square\, b$ denotes the continuous linear operator

$$a \,\square\, b : x \in V \mapsto \{a, b, x\} \in V.$$

In fact, it can be shown that $a \,\square\, b = 0$ is equivalent to $b \,\square\, a = 0$ [4, Lemma 1.2.32]. For a finite-rank JB*-tiple V, its rank $r(V)$ is the (unique) cardinality of a maximal family of mutually orthogonal minimal tripotents in V, which is an ℓ^∞-sum of a finite number of finite-rank Cartan factors. There are six types of finite-rank Cartan factors, which can be infinite dimensional, listed below.

$$\begin{array}{rl}
\text{Type I} & L(\mathbb{C}^\ell, K) \quad (\ell = 1, 2, \ldots),\ \text{rank } = \ell \le \dim K, \\
\text{Type II} & \{z \in L(\mathbb{C}^\ell, \mathbb{C}^\ell) : z^t = -z\} \quad (\ell = 5, 6, \ldots),\ \text{rank } = \left[\dfrac{\ell}{2}\right] \\
\text{Type III} & \{z \in L(\mathbb{C}^\ell, \mathbb{C}^\ell) : z^t = z\} \quad (\ell = 2, 3, \ldots),\ \text{rank } = \ell \\
\text{Type IV} & \text{spin factor, rank } = 2 \\
\text{Type V} & M_{1,2}(\mathbb{O}) = \{1 \times 2 \text{ matrices over the Cayley algebra } \mathbb{O}\},\ \text{rank } = 2 \\
\text{Type VI} & H_3(\mathbb{O}) = \{3 \times 3 \text{ hermitian matrices over } \mathbb{O}\},\ \text{rank } = 3
\end{array}$$

where $L(\mathbb{C}^\ell, K)$ is the JB*-triple of linear operators from $\mathbb{C}^\ell$ to a Hilbert space K and z^t denotes the transpose of z in the JB*-triple $L(\mathbb{C}^\ell, \mathbb{C}^\ell)$ of $\ell \times \ell$ complex matrices. The Jordan triple product in the first three types is given by

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$$

where y^* denotes the adjoint of y.

A *spin factor* is a JB*-triple V equipped with a complete inner product $\langle \cdot, \cdot \rangle$ and a conjugation $* : V \to V$ satisfying

$$\langle x^*, y^* \rangle = \langle y, x \rangle \quad \text{and} \quad \{x, y, z\} = \frac{1}{2}\big(\langle x, y \rangle z + \langle z, y \rangle x - \langle x, z^* \rangle y^*\big).$$

The Cartan factor $H_3(\mathbb{O})$ is a Jordan algebra with product

$$x \cdot y = \frac{1}{2}(xy + yx)$$

where the product on the right-hand side is the usual matrix product. The Jordan triple product of $H_3(\mathbb{O})$ is given by

$$\{x, y, z\} = (x \cdot y) \cdot z + x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

The Cartan factor $M_{1,2}(\mathbb{O})$ can be identified as a subtriple of $H_3(\mathbb{O})$.

The only possible infinite dimensional finite-rank Cartan factors are the spin factors and $L(\mathbb{C}^\ell, K)$, with $\dim K = \infty > \ell$, where a spin factor has rank 2 and $L(\mathbb{C}^\ell, K)$ has rank ℓ. The open unit balls of the finite dimensional Cartan factors are exactly the six types of irreducible bounded symmetric domains in É. Cartan's classification. The last two types are the exceptional domains. This explains the etymology of *Cartan factor*. The open unit ball of a spin factor is known as a *Lie ball*.

For a finite-rank JB*-triple V with rank ℓ, each element $z \in V$ has a *spectral decomposition*

$$z = \alpha_1 e_1 + \cdots + \alpha_\ell e_\ell$$

where $e_1, \ldots, e_\ell$ are mutually (triple) orthogonal minimal tripotents and $\alpha_1 \ge \cdots \ge \alpha_\ell \ge 0$ with $\alpha_1 = \|z\|$, also called the *spectral norm* of z.

4. Squeezing functions of bounded symmetric domains

In finite dimensions, it is well-known that a bounded symmetric domain of rank ℓ contains a polydisc of dimension ℓ as a totally geodesic submanifold [17, p.41]. To see that this is also the case for infinite dimensional bounded symmetric domains of finite rank, we only need to consider the *irreducible* ones. As remarked previously, there are only two classes of such domains, namely, the Lie balls, which are of rank 2, and the type I domains of rank ℓ, which can be realised as the open unit ball of the Banach space $L(\mathbb{C}^\ell, K)$ of bounded linear operators between Hilbert spaces $\mathbb{C}^\ell$ and K, with $\ell \leq \dim K \leq \infty$ and $\ell < \infty$.

Given $\ell < \infty$, every operator $T \in L(\mathbb{C}^\ell, K)$ is a Hilbert-Schmidt operator in the Hilbert-Schmidt norm

$$\|T\|_2 = (\sum_{k=1}^{\ell} \|Te_k\|^2)^{1/2}$$

satisfying $\|T\| \leq \|T\|_2 \leq \sqrt{\ell}\|T\|$, where $\{e_1, \ldots, e_\ell\}$ is the standard orthonormal basis in $\mathbb{C}^\ell$.

Let $\overline{\mathbb{D}}$ be the closure of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{D}$ the closure of the open unit ball

$$D = \{T \in L(\mathbb{C}^\ell, K) : \|T\| < 1\}.$$

Fix orthonormal basis vectors $u_{\alpha_1}, \ldots, u_{\alpha_\ell}$ from an orthonormal basis $\{u_\alpha\}$ in K. Then the continuous map $\varphi : \overline{\mathbb{D}} \times \cdots \times \overline{\mathbb{D}} \to \overline{D}$, defined by

$$\varphi(z_1, \ldots z_\ell) = \sum_{k=1}^{\ell} z_k(e_k \otimes u_{\alpha_k}) \qquad (z_1, \ldots, z_\ell) \in \overline{\mathbb{D}}^\ell, \tag{4.1}$$

restricts to an injective holomorphic map

$$\varphi : \mathbb{D} \times \cdots \times \mathbb{D} \to D$$

with $\varphi(0, \ldots, 0) = 0$, where $e_k \otimes u_{\alpha_k} : \mathbb{C}^\ell \to K$ is the rank-one operator

$$e_k \otimes u_{\alpha_k}(h) = \langle h, e_k \rangle u_{\alpha_k} \qquad (h \in \mathbb{C}^\ell)$$

with $\|e_k \otimes u_{\alpha_k}\| = \|e_k \otimes u_{\alpha_k}\|_2 = 1$. This also implies that φ maps the boundary $\partial\mathbb{D}^\ell$ of $\mathbb{D}^\ell$ into the boundary $\partial D = \{T \in L(\mathbb{C}^\ell, K) : \|T\| = 1\}$.

Let D be the open unit ball of a spin factor V, which is of rank 2. Let e_1 and e_2 be two mutually (triple) orthogonal minimal tripotents in V. Then we have $\|\lambda e_1 + \mu e_2\| = \max\{|\lambda|, |\mu|\}$ for $\lambda, \mu \in \mathbb{C}$ [4, Corollary 3.1.21]. Hence one can define a continuous map

$$\varphi : (z_1, z_2) \in \overline{\mathbb{D}}^2 \mapsto z_1 e_1 + z_2 e_2 \in \overline{D} \tag{4.2}$$

which restricts to an injective holomorphic map from $\mathbb{D}^2$ to D satisfying $\varphi(0) = 0$ and $\varphi(\partial\mathbb{D}^2) \subset \partial D$.

Given a Hilbert space H, a holomorphic map $f : \mathbb{D}^n \to H$ admits a power series representation in terms of homogeneous polynomials from $\mathbb{C}^n$ to H (cf. [4, p.65]). A *homogeneous polynomial p of degree d* from $\mathbb{C}^n$ to H is given by

$$p(z_1, \ldots, z_n) = P(\,(z_1, \ldots, z_n), \ldots, (z_1, \ldots, z_n)\,) \in H, \qquad (z_1, \ldots, z_n) \in \mathbb{C}^n$$

where $P : \underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{d\text{-times}} \to H$ is a d-linear map. Let $\{e_\alpha\}$ be an orthonormal basis in H. We can write

$$p(z_1, \ldots, z_n) = \sum_\alpha \mathfrak{q}_\alpha(z_1, \ldots, z_n) e_\alpha$$

where $\mathfrak{q}_\alpha(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree d in n complex variables $z_1, \ldots, z_n$ and has the form

$$\mathfrak{q}_\alpha(z_1, \ldots, z_n) = \sum_{j_1+\cdots+j_n=d} c_{\alpha;\, j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n} \qquad (c_{\alpha;\, j_1,\ldots,j_n} \in \mathbb{C}).$$

A holomorphic map $f : \mathbb{D}^n \to H$ has a power series representation

$$f(z_1, \ldots, z_n) = f(0) + \sum_{d=1}^\infty p^d(z_1, \ldots, z_n), \qquad (z_1, \ldots, z_n) \in \mathbb{D}^n$$

where p^d is a homogeneous polynomial of degree d from $\mathbb{C}^n$ to H and has the from

$$p^d(z_1, \ldots, z_n) = \sum_\alpha \sum_{j_1+\cdots+j_n=d} c^d_{\alpha;\, j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n} e_\alpha \qquad (c^d_{\alpha;\, j_1,\ldots,j_n} \in \mathbb{C}). \tag{4.3}$$

Let $h : D \to D'$ be a biholomorphic map between two *open unit balls* D, D' of Banach spaces V and V' respectively. If $h(0) = 0$, then it follows from Cartan's uniqueness theorem that h is the restriction of the derivative $h'(0) : V \to V'$, which is a linear isometry (cf. [11, Corollary 6] and [14]). In particular, h extends to a continuous map $\bar{h} : \bar{D} \to \overline{D}'$ between the closures $\overline{D}$ and $\overline{D}'$, where $\bar{h} = h'(0)|_{\bar{D}}$. Moreover, $\bar{h}(\partial D) = \partial D'$.

Let D be a bounded symmetric domain, realised as the open unit ball of a JB*-triple V. Given a holomorphic embedding $f : D \to B_H$ of D into a Hilbert ball B_H, the image $f(D)$ is a bounded symmetric domain and hence there is an equivalent norm $\|\cdot\|_\infty$ on H such that $(H, \|\cdot\|_\infty)$ is a JB*-triple and $f(D)$ identifies (via a biholomorphic map) as the open unit ball of $(H, \|\cdot\|_\infty)$ (cf. [4, Theorem 2.5.26]). If $f(0) = 0$, then the previous remark implies that f extends to a continuous map $\bar{f}$, which maps ∂D onto the boundary $\partial f(D)$ of the domain $f(D)$.

The following lemma is a simple infinite dimensional extension of Alexander's result in [2, Proposition 1] (see also [18, Lemma 1]).

Lemma 4.1. *Let D be a bounded domain with boundary ∂D and B a Hilbert ball such that the following two continuous maps*

$$\overline{\mathbb{D}}^\ell \overset{\varphi}{\to} \overline{D} \overset{f}{\to} \overline{B}$$

on the closures restrict to holomorphic maps

$$\mathbb{D}^\ell \overset{\varphi}{\to} D \overset{f}{\to} B$$

with open image $f(D)$, satisfying $\varphi(\partial \mathbb{D}^\ell) \subset \partial D$ and $f(\partial D) \subset \partial f(D)$. If $\rho B \subset f(D)$ for some $\rho > 0$, then $\ell \rho^2 \leq 1$.

Proof. Let $\{e_\alpha\}$ be an orthonormal basis in the Hilbert space containing the ball B. By (4.3), the holomorphic map $f\circ\varphi$ on $\mathbb{D}^\ell$ has a power series representation

$$f\circ\varphi(z_1,\ldots,z_\ell)=\sum_{d=1}^{\infty}p^d(z_1,\ldots,z_\ell)$$

where $p^d(z_1,\ldots,z_\ell)$ is a d-homogeneous polynomial of the form

$$p^d(z_1,\ldots,z_\ell)=\sum_\alpha\sum_{j_1+\cdots+j_\ell=d}c^d_{\alpha;\,j_1,\ldots,j_\ell}z_1^{j_1}\cdots z_\ell^{j_\ell}e_\alpha \qquad (c^d_{\alpha;\,j_1,\ldots,j_\ell}\in\mathbb{C}).$$

Since $\rho B\subset f(D)$, we have $\|f(w)\|\geq\rho$ for each $w\in\partial D$. Noting that $f\circ\varphi(\partial\mathbb{D}^\ell)\subset\partial f(D)$, we deduce

$$\begin{aligned}
\rho^2 &\leq \frac{1}{2\pi}\int_0^{2\pi}\|f\circ\varphi(0,\ldots,e^{i\theta_j},0,\ldots,0)\|^2d\theta_j\\
&= \frac{1}{2\pi}\lim_{r\to1}\int_0^{2\pi}\|f\circ\varphi(0,\ldots,re^{i\theta_j},0,\ldots,0)\|^2d\theta_j\\
&= \frac{1}{2\pi}\lim_{r\to1}\int_0^{2\pi}\sum_\alpha\left|\sum_d c^d_{\alpha;\,0,\ldots,0,d,0,\ldots,0}r^de^{id\theta_j}\right|^2d\theta_j\\
&= \frac{1}{2\pi}\lim_{r\to1}\sum_\alpha\int_0^{2\pi}\left|\sum_d c^d_{\alpha;\,0,\ldots,0,d,0,\ldots,0}r^de^{id\theta_j}\right|^2d\theta_j\\
&= \lim_{r\to1}\sum_\alpha\sum_d\left|c^d_{\alpha;\,0,\ldots,0,d,0,\ldots,0}\right|^2r^{2d}\\
&= \sum_\alpha\sum_d\left|c^d_{\alpha;\,0,\ldots,0,d,0,\ldots,0}\right|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
1 &\geq \lim_{r\to1}\left(\frac{1}{2\pi}\right)^\ell\int_0^{2\pi}\cdots\int_0^{2\pi}\|f\circ\varphi(re^{i\theta_1},\ldots,re^{i\theta_\ell})\|^2d\theta_1\cdots d\theta_\ell\\
&= \lim_{r\to1}\sum_\alpha\sum_d\sum_{\nu_1+\cdots+\nu_\ell=d}\left|c^d_{\alpha;\,\nu_1,\ldots,\nu_\ell}\right|^2r^{2d}\\
&= \sum_\alpha\sum_d\sum_{\nu_1+\cdots+\nu_\ell=d}\left|c^d_{\alpha;\,\nu_1,\ldots,\nu_\ell}\right|^2\\
&\geq \sum_\alpha\sum_d\left|c^d_{\alpha;\,d,0,\ldots,0}\right|^2+\cdots+\sum_\alpha\sum_d\left|c^d_{\alpha;\,0,\ldots,0,d}\right|^2\geq\ell\rho^2.
\end{aligned}$$

□

In finite dimensions, the squeezing constant of the four series of classical Cartan domains has been computed by Kubota in [18]. We will now compute the squeezing constants of the remaining finite rank bounded symmetric domains of all dimensions.

We begin with the two exceptional domains which are realised as the open unit balls of the JB*-triples $M_{1,2}(\mathbb{O})$ and $H_3(\mathbb{O})$ respectively, where $\dim M_{1,2}(\mathbb{O})=16$ and $\dim H_3(\mathbb{O})=27$.

Both JB*-triples are equipped with the spectral norm, as noted previously. They also carry a Hilbert space structure, with inner product

$$\langle x, y\rangle = \frac{1}{18}\text{Trace}\, D(x,y) \qquad (x, y \in H_3(\mathbb{O})), \tag{4.4}$$

shown in [24, Corollary 2.14], where $D(x,y) = 2x \,\square\, y$. Given a minimal tripotent $e \in H_3(\mathbb{O})$, we have $\langle e, e\rangle = 1$ [24, Proposition 2.8]. If e and u are two mutually (triple) orthogonal tripotents in $H_3(\mathbb{O})$, then $\langle e, u\rangle = 0$ [24, Lemma 2.9].

The 27-dimensional domain $D_{27} \subset H_3(\mathbb{O})$ has rank 3 whereas the 16-dimensional domain $D_{16} \subset M_{1,2}(\mathbb{O})$ has rank 2. The following two propositions, together with Kubota's results in [18], give a complete list of squeezing constants of all finite dimensional irreducible bounded symmetric domains.

Proposition 4.2. *The squeezing constant of the exceptional domain D_{27} is given by $\hat{\sigma}_{D_{27}} = 1/\sqrt{3}$.*

Proof. We compute $\sigma_{D_{27}}(0) = \hat{\sigma}_{D_{27}}$. We have $D_{27} = \{z \in H_3(\mathbb{O}) : \|z\| < 1\}$, where $\|\cdot\|$ is the spectral norm. Given $z \in H_3(\mathbb{O})$ with spectral decomposition

$$z = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \qquad (\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0),$$

the spectral norm $\|z\|$ equals α_1, where the minimal tripotents e_1, e_2, e_3 are mutually orthogonal with respect to the inner product given in (4.4). The Hilbert space norm $\|z\|_2$ of z is given by

$$\|z\|_2^2 = \langle z, z\rangle = \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$

It follows that

$$\|z\| \leq \|z\|_2 \leq \sqrt{3}\|z\|$$

for all $z \in H_3(\mathbb{O})$. This implies

$$B_{27} \subset D_{27} \subset \sqrt{3}B_{27}$$

where $B_{27} = \{z \in H_3(\mathbb{O}) : \|z\|_2 < 1\}$ is the Hilbert ball in $H_3(\mathbb{O})$. Hence we have $\hat{\sigma}_{D_{27}} \geq 1/\sqrt{3}$. To show the reverse inequality, we define a continuous map $\varphi : \overline{\mathbb{D}}^3 \to \overline{D}_{27}$ by

$$\varphi(z_1, z_2, z_3) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} = z_1 e_{11} + z_2 e_{22} + z_3 e_{33}$$

where e_{jj} is the diagonal matrix in $H_3(\mathbb{O})$ with 1 in the jj-entry and 0 elsewhere. Since e_{11}, e_{22}, e_{33} are mutually (triple) orthogonal minimal tripotents in $H_3(\mathbb{O})$, we see that φ restricts to an injective holomorphic map from $\mathbb{D}^3$ into D_{27} with $\varphi(0) = 0$ and $\varphi(\partial\mathbb{D}^3) \subset \partial D_{27}$. By Lemma 4.1 and the remarks before it, for each holomorphic embedding $f : D_{27} \to B_{27}$ with $f(0) = 0$ and $\rho B_{27} \subset f(D_{27})$, we must have $3\rho^2 \leq 1$. This proves the reverse inequality. $\square$

Proposition 4.3. *The squeezing constant of the exceptional domain D_{16} is given by $\hat{\sigma}_{D_{16}} = 1/\sqrt{2}$.*

Proof. The arguments are similar to those in the proof of Lemma 4.2, we recapitulate for completeness. We consider $M_{1,2}(\mathbb{O})$ as a subtriple of $H_3(\mathbb{O})$. It suffices to show $\sigma_{D_{16}}(0) = 1/\sqrt{2}$. We have $D_{16} = \{z \in M_{1,2}(\mathbb{O}) : \|z\| < 1\}$, where $\|\cdot\|$ is the spectral norm. Given $z \in M_{1,2}(\mathbb{O})$ with spectral decomposition

$$z = \alpha_1 e_1 + \alpha_2 e_2 \qquad (\alpha_1 \geq \alpha_2 \geq 0),$$

the spectral norm $\|z\|$ equals α_1, where the minimal tripotents e_1, e_2 are mutually orthogonal with respect to the inner product given in (4.4). The Hilbert space norm $\|z\|_2$ of z is given by

$$\|z\|_2^2 = \langle z, z\rangle = \alpha_1^2 + \alpha_2^2$$

and

$$\|z\| \leq \|z\|_2 \leq \sqrt{2}\|z\|$$

for all $z \in M_{1,2}(\mathbb{O})$. This implies

$$B_{16} \subset D_{16} \subset \sqrt{2}B_{16},$$

where $B_{16} = \{z \in M_{1,2}(\mathbb{O}) : \|z\|_2 < 1\}$ is the Hilbert ball in $M_{1,2}(\mathbb{O})$. Hence $\hat{\sigma}_{D_{16}} \geq 1/\sqrt{2}$. For the reverse inequality, one defines a continuous map $\varphi : \overline{\mathbb{D}}^2 \to \overline{D}_{16}$ by

$$\varphi(z_1, z_2) = z_1 e_{11} + z_2 e_{22}$$

where $e_{11} = (1, 0)$ and $e_{22} = (0, 1)$ are mutually (triple) orthogonal minimal tripotents in $M_{1,2}(\mathbb{O})$, and φ restricts to an injective holomorphic map from $\mathbb{D}^2$ into D_{16} with $\varphi(0) = 0$ and $\varphi(\partial\mathbb{D}^2) \subset \partial D_{16}$. As before, for each holomorphic embedding $f : D_{16} \to B_{16}$ satisfying $f(0) = 0$ and $\rho B_{16} \subset f(D_{16})$, we must have $2\rho^2 \leq 1$. This proves the reverse inequality. $\square$

The following result extends Kubota's result [18] for the classical Cartan domains to all finite dimensional irreducible bounded symmetric domains.

Corollary 4.4. *Let D be a finite dimensional irreducible bounded symmetric domain of rank p. Then its squeezing constant is given by $\hat{\sigma}_D = 1/\sqrt{p}$.*

We are now ready to show that finite-rank bounded symmetric domains, which can be infinite dimensional, are HHR and compute their squeezing constants.

Theorem 4.5. *Let D be a bounded symmetric domain in a complex Banach space. Then D is HHR if and only if it is of finite rank. In this case, D is biholomorphic to a finite product*

$$D_1 \times \cdots \times D_k$$

of irreducible bounded symmetric domains and we have

$$\hat{\sigma}_D = \left(\frac{1}{\hat{\sigma}_{D_1}^2} + \cdots + \frac{1}{\hat{\sigma}_{D_k}^2}\right)^{-1/2}. \tag{4.5}$$

If $\dim D_j < \infty$, then D_j is a classical Cartan domain or an exceptional domain, and $\hat{\sigma}_{D_j} = 1/\sqrt{p_j}$ where p_j is the rank of D_j.

If $\dim D_j = \infty$, then D_j is either a Lie ball or a Type I domain of finite rank p_j. For a Lie ball D_j, we have $\hat{\sigma}_{D_j} = 1/\sqrt{2}$. For a rank p_j Type I domain D_j, we have $\hat{\sigma}_{D_j} = 1/\sqrt{p_j}$.

Proof. Let D be HHR, realised as the open unit ball of a JB*-triple V. Then V is linearly homeomorphic to some Hilbert space H. In particular, V is reflexive and hence D is of finite rank. Conversely, a finite-rank bounded symmetric domain D decomposes into a finite Cartesian product $D = D_1 \times \cdots \times D_k$ of irreducible bounded symmetric domains, where each D_j is of finite rank p_j and realised as the open unit ball of a Cartan factor V_j for $j = 1, \ldots, k$.

To complete the proof, we show that each domain D_j of rank p_j has squeezing constant $\hat{\sigma}_{D_j} = 1/\sqrt{p_j}$ and $\hat{\sigma}_D = (p_1 + \cdots + p_k)^{-1/2}$.

By Corollary 4.4, we have $\hat{\sigma}_{D_j} = 1/\sqrt{p}_j$ if $\dim V_j < \infty$. In fact, this is also the case even if V_j is infinite dimensional, in which case V_j is either a spin factor or the type I Cartan factor $L(\mathbb{C}^\ell, K)$ with $\dim K = \infty > \ell$. We now compute the squeezing constant in these two cases.

First, let D_j be a Lie ball, that is, the open unit ball of a spin factor $(V, \|\cdot\|)$, which has rank 2. In this case, V is a Hilbert space with norm $\|\cdot\|_h$ satisfying

$$\|\cdot\| \leq \|\cdot\|_h \leq \sqrt{2}\|\cdot\|$$

(cf. [5, Section 2]). This gives $\hat{\sigma}_{D_j} \geq 1/\sqrt{2}$. Making use of the map φ in (4.2) and analogous arguments in the proof of Lemma 4.3, one concludes that $\hat{\sigma}_{D_j} = \hat{\sigma}_{D_j}(0) = 1/\sqrt{2}$.

Next, let D_j be a Type I domain of rank p_j, realised as the open unit ball

$$D_j = \{T \in L(\mathbb{C}^{p_j}, K) : \|T\| < 1\}$$

of $L(\mathbb{C}^{p_j}, K)$ with $\dim K = \infty$. Equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$, the vector space $L(\mathbb{C}^{p_j}, K)$ is a Hilbert space. Let $B = \{T \in L(\mathbb{C}^{p_j}, K) : \|T\|_2 < 1\}$ be its open unit ball. Since $\|\cdot\| \leq \|\cdot\|_2 \leq \sqrt{p_j}\|\cdot\|$, we have $B \subset D_j \subset \sqrt{p_j}B$ and therefore $\hat{\sigma}_{D_j}(0) \geq 1/\sqrt{p_j}$. As before, using the map φ in (4.1) and similar arguments, we deduce $\hat{\sigma}_{D_j} = \hat{\sigma}_{D_j}(0) = 1/\sqrt{p_j}$.

It remains to establish (4.5). The domain $D = D_1 \times \cdots \times D_k$ is the open unit ball of the ℓ_∞-sum

$$V_1 \oplus \cdots \oplus V_k$$

of Cartan factors, where D_j is the open unit ball of V_j of rank p_j for $j = 1, \ldots, k$. We observe from the previous arguments that for each domain D_j, one can construct a continuous map $\varphi_j : \overline{\mathbb{D}}^{p_j} \to \overline{D}_j$ which restricts to a holomorphic map from $\mathbb{D}^{p_j}$ to D_j satisfying $\varphi_j(0) = 0$ and $\varphi_j(\partial\mathbb{D}^{p_j}) \subset \partial D_j$. Hence the product map

$$\varphi := \varphi_1 \times \cdots \times \varphi_k : \overline{\mathbb{D}}^{p_1} \times \cdots \times \overline{\mathbb{D}}^{p_k} \to \overline{D}_1 \times \cdots \times \overline{D}_k = \overline{D}$$

is continuous, which restricts to a holomorphic map from $\mathbb{D}^{p_1} \times \cdots \times \mathbb{D}^{p_k}$ to $D_1 \times \cdots \times D_k$ satisfying $\varphi(0, \ldots, 0) = (0, \ldots, 0)$ and maps the boundary of $\mathbb{D}^{p_1} \times \cdots \times \mathbb{D}^{p_k}$ into the boundary of $D_1 \times \cdots \times D_k = D$. Applying Lemma 4.1 again, we deduce that

$$\hat{\sigma}_D \leq \frac{1}{\sqrt{p_1 + \cdots + p_k}} = \frac{1}{\sqrt{\hat{\sigma}_{D_1}^{-2} + \cdots + \hat{\sigma}_{D_k}^{-2}}}.$$

For each $j = 1, \ldots, k$, the previous arguments reveal that there is a Hilbert space H_j with open unit ball B_j such that

$$B_j \subset D_j \subset \sqrt{p_j}B_j.$$

Let B be the open unit ball of the Hilbert space direct sum $H_1 \oplus_2 \cdots \oplus_2 H_k$. Then we have

$$B \subset D_1 \times \cdots \times D_k \subset \sqrt{p_1 + \cdots + p_k}\, B.$$

This implies that

$$\hat{\sigma}_D \geq \frac{1}{\sqrt{p_1 + \cdots + p_k}}$$

which completes the proof. $\square$

5. Uniformly elliptic domains

Bounded symmetric domains can be realised as convex domains in Banach spaces and those which are HHR have been completely determined previously. We conclude the paper in this section by introducing a large class of bounded convex domains, which include the strongly convex domains, and show that these domains are HHR in Hilbert spaces. The domains to be introduced are called *uniformly elliptic domains*.

We begin with a preamble. Recall that a finite dimensional bounded domain $D \subset \mathbb{C}^n$ with a C^2 boundary ∂D is called *strongly convex* if all normal curvatures of ∂D are positive (cf. [1, p.108]). Such a domain is a manifold with curvature pinched which entails the existence of two positive constants $R > r > 0$ such that for each $q \in \partial D$, there are two points q', q'' in $\mathbb{C}^n$ with the property that q is a common boundary point of the Euclidean balls $B_{\mathbb{C}^n}(q', r)$ and $B_{\mathbb{C}^n}(q'', R)$ satisfying $B_{\mathbb{C}^n}(q', r) \subset D \subset B_{\mathbb{C}^n}(q'', R)$. For fixed r and R, it can be seen that q' and q'' are unique and colinear with q. For instance, an ellipsoid is strongly convex and has this property.

In view of the fact that Hilbert balls are the only bounded symmetric domains with a C^2 boundary, we generalise the concept of strong convexity to infinite dimension without the assumption of a smooth boundary, to cover a wider class of domains, as follows.

Definition 5.1. A bounded convex domain Ω in a complex Banach space V is called *uniformly elliptic* if there exist universal constants r, R with $0 < r < R$ such that to each $q \in \partial\Omega$, there correspond two unique points $q', q'' \in V$, colinear to q, satisfying

(5.1.1) $B_V(q', r) \subset \Omega \subset B_V(q'', R)$;

(5.1.2) $q \in \partial B_V(q', r) \cap \partial B_V(q'', r)$, that is, q is a common boundary point of $B_V(q', r)$, $B_V(q'', R)$ and Ω.

Evidently, the definition of uniform ellipticity depends on the norm of the ambient Banach space. By the previous remarks, strongly convex domains are uniformly elliptic, but the converse is false. In fact, all open balls in Banach spaces are uniformly elliptic. Indeed, if say, $\Omega = B_V$ is the open unit ball of a Banach space V, then for each boundary point $q \in \partial\Omega$, we have $\|q\| = 1$ and

$$B_V(q/2, 1/2) \subset \Omega = B_V(0, 1)$$

and $q \in \partial B_V(\frac{q}{2}, \frac{1}{2}) \cap \partial\Omega \cap \partial B_V(0, 1)$. For $R = 1$ and $r = 1/2$, the points $q' = q/2$ and $q'' = 0$ are unique and colinear to q.

By definition, each point p in a uniformly elliptic domain Ω in a Banach space V lies in the ball $B_V(q'', R)$ for all $q \in \partial\Omega$, as in (5.1.1) above, although p need not be colinear with q and q''. We consider the question of colinearity below.

Lemma 5.2. *Let Ω be a uniformly elliptic domain in a Banach space V and for each $q \in \partial\Omega$, let*

$$B_V(q', r) \subset \Omega \subset B_V(q'', R), \quad q \in \partial B_V(q', r) \cap \partial B_V(q'', r)$$

be as in the definition of uniform ellipticity. Then for each $p \in \Omega$ and $q \in \partial\Omega$ with $\|p-q\| = d(p, \partial\Omega)$, either p is colinear with q and q'' or, there exists $q_1 \in \partial\Omega$ such that p is colinear with q_1 and $q_1'' = q''$ satisfying $\|p - q_1\| = \|p - q\|$.

Proof. Let $q \in \partial\Omega \cap \partial B_V(q'', R)$ satisfy $\|p - q\| = d(p, \partial\Omega)$. Suppose p is not colinear with q and q''. We show the existence of q_1 in the lemma.

Consider $p \in \Omega \subset B_V(q'', R)$. Extend the (real) line through q'' and p to a point $q_1 \in \partial B_V(q'', R)$. Then we have $\|p - q_1\| = d(p, \partial B_V(q'', R)) \leq \|p - q\|$. We show that $q_1 \in \partial\Omega$, which would imply $\|p - q_1\| \geq \|p - q\|$ and complete the proof by uniqueness of q_1' and q_1''.

If $q_1 \notin \partial\Omega$, we deduce a contradiction. Since $q_1 \notin \overline{\Omega}$ and $p \in \Omega$, the line joining p and q_1 must intersect $\partial\Omega$ at some point ω, say. Now we have the contradiction

$$\|p - q\| \geq \|p - q_1\| > \|p - \omega\| \geq d(p, \partial\Omega) = \|p - q\|.$$

□

We will discuss uniformly elliptic domains in greater detail in another work, but complete this section presently by showing that these domains are HHR in Hilbert spaces, which generalises the finite dimensional result for strongly convex domains in [25, Proposition 1].

Theorem 5.3. *Let Ω be a uniformly elliptic domain in a Hilbert space H. Then Ω is HHR.*

Proof. We need to show that the squeezing function σ_Ω of Ω has a strictly positive lower bound. Suppose, to the contrary, that there is a sequence (p_ν) in Ω such that

$$\lim_{\nu \to \infty} \sigma_\Omega(p_\nu) = 0. \tag{5.1}$$

We deduce a contradiction. By Lemma 2.3, we may assume, by choosing another sequence if necessary, that $d(p_\nu, \partial\Omega)$ converges to 0 as $\nu \to \infty$ and one can find a boundary point $q_\nu \in \partial\Omega$ such that

$$\|q_\nu - p_\nu\| = d(p_\nu, \partial\Omega) > 0.$$

Write $\lambda_\nu = d(p_\nu, \partial\Omega)$ and let

$$B_V(q_\nu', r) \subset \Omega \subset B_V(q_\nu'', R), \quad q_\nu \in \partial B_V(q_\nu', r) \cap \partial B_V(q_\nu'', R)$$

be as in the definition of uniformly ellipticity of Ω where, by Lemma 5.2, q_ν can be chosen so that p_ν lies on the line through q_ν and q_ν''.

We complete the proof by a contradiction that there is a subsequence $(p_{\nu'})$ of (p_ν) and a constant $\delta > 0$ satisfying

$$\sigma_\Omega(p_{\nu'}) > \delta \quad \text{for all } \nu'.$$

In fact, δ depends only on r and R.

For each ν, we define a holomorphic embedding

$$\Phi \circ L_\nu : \Omega \to H$$

as follows. Let $\mathbf{e^1}$ be the unit vector

$$\mathbf{e^1} := \frac{q_\nu'' - q_\nu}{\|q_\nu'' - q_\nu\|}.$$

We have

(S1) $q_\nu'' = R\mathbf{e}^1 + q_\nu,\ q_\nu' = r\mathbf{e}^1 + q_\nu,$

(S2) $p_\nu = \lambda_\nu \mathbf{e}^1 + q_\nu$ ($\lambda_\nu \to 0$ as $\nu \to \infty$).

Since $\sigma_\Omega(p_\nu) = \sigma_{\Omega - q_\nu}(p_\nu - q_\nu)$, taking a translation, we may assume $q_\nu = 0$. Then we have

(S1^0) $q''_\nu = R\mathbf{e}^1$, $\varphi q'_\nu = r\mathbf{e}^1$,

(S2^0) $p_\nu = \lambda_\nu \mathbf{e}^{\mathbf{1}}$.

We now have

$$p_\nu = \lambda_\nu \mathbf{e}^1 \in B_H(r\mathbf{e}^1, r) \subset \Omega \subset B_H(R\mathbf{e}^1, R). \tag{5.2}$$

where

$$q'_\nu = r\mathbf{e}^1, \quad q''_\nu = R\mathbf{e}^1.$$

Extend $\{\mathbf{e}^1\}$ to an orthonormal basis $\{\mathbf{e}^\gamma\}_{\gamma\in\Gamma}$ in H. For each $z \in H$, we will write

$$z = \sum_{\gamma\in\Gamma} z_\gamma \mathbf{e}^\gamma = z_1\mathbf{e}^1 + \sum_{\gamma\neq 1} z_\gamma \mathbf{e}^\gamma$$

with $z_\gamma \in \mathbb{C}$. We have

$$z \in B_H(r\mathbf{e}^1, r) \Leftrightarrow \|z - r\mathbf{e}^1\| < r$$

where

$$\|z - r\mathbf{e}^1\|^2 = |z_1 - r|^2 + \sum_{\gamma\neq 1} |z_\gamma|^2 = |z_1|^2 - 2r\mathrm{Re}\, z_1 + r^2 + \sum_{\gamma\neq 1}|z_\gamma|^2. \tag{5.3}$$

We definite a dilation $L_\nu : H \to H$ by

$$L_\nu(z) = \frac{z_1}{\lambda_\nu}\mathbf{e}^1 + \frac{1}{\sqrt{\lambda_\nu}}\sum_{\gamma\neq 1} z_\gamma \mathbf{e}^\gamma, \quad z = \sum_{\gamma\in\Gamma} z_\gamma\mathbf{e}^\gamma$$

which satisfies $L_\nu(p_\nu) = \mathbf{e}^1$. The map L_ν is a linear homeomorphism of H, with inverse

$$L_\nu^{-1}(z) = \lambda_\nu z_1 \mathbf{e}^1 + \sqrt{\lambda_\nu}\sum_{\gamma\neq 1} z_\gamma \mathbf{e}^\gamma.$$

Define a Cayley transform $\Phi : \{z \in H : \mathrm{Re}\, z_1 > 0\} \to H$ by

$$\Phi(z) := \frac{z_1 - 1}{z_1 + 1}\mathbf{e}^1 + \sum_{\gamma\neq 1}\frac{\sqrt{2}z_\gamma}{z_1+1}\mathbf{e}^\gamma, \quad z = \sum_\gamma z_\gamma \mathbf{e}^\gamma$$

and the holomorphic embedding

$$\Phi \circ L_\nu : \Omega \to H$$

where $\Phi(L_\nu(p_\nu)) = 0$. Although Φ depends on ν, we omit the subscript ν indicating this, to simplify notation, since confusion is unlikely.

We will show that

$$B_H\left(0, \sqrt{\frac{r}{2+2r}}\right) \subset \Phi(L_\nu(\Omega)) \subset B_H(0, \sqrt{1+R})$$

for sufficiently large ν.

Substituting R for r in (5.3), we see that

$$B_H(R\mathbf{e}^1, R) = \{z \in H : \|z - R\mathbf{e}^1\|^2 < R^2\} = \{z \in H : \sum_{\gamma\in\Gamma}|z_\gamma|^2 < 2R\ \mathrm{Re}\ z_1\}.$$

Given $\zeta = \sum_\gamma \zeta_\gamma \mathbf{e}^\gamma \in B_H$, we have

$$\Phi^{-1}(\zeta) = \frac{1+\zeta_1}{1-\zeta_1}\mathbf{e}^1 + \sum_{\gamma\neq 1} \frac{\sqrt{2}\zeta_\gamma}{1-\zeta_1}\mathbf{e}^\gamma.$$

Hence

$$\begin{aligned}
&\zeta \in \Phi L_\nu(B_H(R\mathbf{e}^1, R)) \Leftrightarrow L_\nu^{-1}\Phi^{-1}\zeta \in B_H(R\mathbf{e}^1, R)\\
\Leftrightarrow\ & \lambda_\nu \left(\frac{1+\zeta_1}{1-\zeta_1}\right)\mathbf{e}^1 + \sum_{\gamma\neq 1} \frac{\sqrt{2\lambda_\nu}}{1-\zeta_1}\,\mathbf{e}^\gamma \in B_H(R\mathbf{e}^1, R)\\
\Leftrightarrow\ & \alpha\lambda_\nu^2|1+\zeta_1|^2 + 2\lambda_\nu \sum_{\gamma\neq 1}|\zeta_\gamma|^2 < 2R\lambda_\nu(1-|\zeta_1|^2)\\
\Rightarrow\ & \sum_{\gamma\neq 1}|\zeta_\gamma|^2 < R(1-|\zeta_1|^2)\\
\Rightarrow\ & \|\zeta\|^2 = |\zeta_1|^2 + \sum_{\gamma\neq 1}|\zeta_\gamma|^2 < 1+R
\end{aligned}$$

and therefore we have, by (5.2),

$$\Phi L_\nu(\Omega) \subset \Phi L_\nu(B_H(R\mathbf{e}^1, R)) \subset B_H(0, \sqrt{1+R}). \tag{5.4}$$

We now show that $B_H(0, \sqrt{\frac{r}{2+2r}}\,) \subset \Phi L_\nu(\Omega)$ for sufficiently large ν. For this, we will make use of the inclusion $B_H(r\mathbf{u}, r) \subset \Omega$.

We have $L_\nu^{-1}\Phi^{-1}(\zeta) \in B_H(r\mathbf{u}, r)$ if and only if $\|L_\nu^{-1}\Phi^{-1}(\zeta) - r\mathbf{u}\| < r$, where

$$\begin{aligned}
&\|L_\nu^{-1}\Phi^{-1}(\zeta) - r\mathbf{u}\|^2 < r^2 \Leftrightarrow \left|\lambda_\nu\left(\frac{1+\zeta_1}{1-\zeta_1}\right) - r\right|^2 + \frac{2\lambda_\nu}{|1-\zeta_1|^2}\sum_{\gamma\neq 1}|\zeta_\gamma|^2 < r^2 \\
\Leftrightarrow\ & \lambda_\nu(\lambda_\nu|1+\zeta_1|^2 - 2r(1-|\zeta_1|^2) + 2\sum_{\gamma\neq 1}|\zeta_\gamma|^2) < 0.
\end{aligned} \tag{5.5}$$

For $\zeta \in B_H(0, \sqrt{\frac{r}{2+2r}}\,)$, we have $2r - (2r|\zeta_1|^2 + 2\|\zeta\|^2) > r$ and $|1+\zeta_1|^2 \le \left(1 + \sqrt{\dfrac{r}{2+2r}}\right)^2$. Since $\lambda_\nu \to 0$ as $\nu \to \infty$, there exists ν_0 such that $\nu \ge \nu_0$ implies

$$\lambda_\nu < \frac{r}{2\left(1+\sqrt{\frac{r}{2+2r}}\right)^2}$$

and hence

$$\begin{aligned}
&\lambda_\nu|1+\zeta_1|^2 - 2r(1-|\zeta_1|^2) + 2\sum_{\gamma\neq 1}|\zeta_\gamma|^2 \le \lambda_\nu|1+\zeta_1|^2 - 2r + 2r|\zeta_1|^2 + 2\|\zeta\|^2\\
&< \frac{r}{2\left(1+\sqrt{\frac{r}{2+2r}}\right)^2}|1+\zeta_1|^2 - r < -r/2
\end{aligned}$$

which gives $\lambda_\nu(\lambda_\nu|1+\zeta_1|^2 - 2r(1-|\zeta_1|^2) + 2\sum_{\gamma\neq 1}|\zeta_\gamma|^2) < -r\lambda_\nu/2 < 0$ and by (5.5),

$$\|L_\nu^{-1}\Phi^{-1}(\zeta) - r\mathbf{u}\|^2 < r^2.$$

We have therefore shown that, for $\nu \geq \nu_0$, the inclusions

$$B_H(0, \sqrt{r/(2+2r}\,) \subset \Phi L_\nu(B_H(r\mathbf{e}^1, r)) \subset \Phi L_\nu(\Omega)$$

are satisfied.

Now it follows from this and (5.4) that

$$\sigma_\Omega(p_\nu) \geq \sqrt{\frac{r}{2(1+r)(1+R)}} > 0$$

for all $\nu \geq \nu_0$, which contradicts $\lim_{\nu\to\infty} \sigma_\Omega(p_\nu) = 0$ and completes the proof.

□

References

[1] M. Abate, Interation of holomorphic maps on taut manifolds, Mediterranean Press, Consenza (1989)

[2] H. Alexander, *Extremal holomorphic embeddings between the ball and polydisc*, Proc. Amer. Math. Soc. **68**, 200–202 (1978)

[3] J.M. Borwein and S. Fitzpatrick, *Existence of nearest points in Banach spaces*, Can. J. Math. **41**, 702-720 (1989)

[4] C-H. Chu, Jordan structures in geometry and analysis, Cambridge Univ. Press, Cambridge (2012)

[5] C-H. Chu, *Iteration of holomorphic maps on Lie balls*, Adv. Math. **264**, 114–154 (2014)

[6] C-H. Chu and L. J. Bunce, *Compact operations, multipliers and Radon-Nikodym property in JB*-triples* Pacific J. Math. **153**, 249–265 (1992)

[7] F. Deng, Q. Guan and L. Zhang, *Some properties of squeezing functions on bounded domains*, Pacific J. Math. **257**, 319–341 (2012)

[8] F. Deng, Q. Guan, and L. Zhang, *Properties of squeezing functions and global transformations of bounded domains*, Trans. Amer. Math. Soc. **368**, 2679–2696 (2016)

[9] J. E. Fornaess and F. Rong, *Estimate of the squeezing function for a class of bounded domains*, arXiv:1606.01335 (2016)

[10] T. Franzoni and E. Vessentini, *Holomorphic maps and invariant distances*, Math. Studies 40, North-Holland, Amsterdam (1980)

[11] L.A. Harris, *Schwarz's lemma in normed linear spaces*, Proc. Nat. Acad. Sci. USA. **62**, 1014–1017 (1969)

[12] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183**, 503–529 (1983)

[13] W. Kaup, *On a Schwarz lemma for bounded symmetric domains*, Math. Nachr. **197**, 51–60 (1999)

[14] W. Kaup and H. Upmeir, *Banach spaces with biholomorphically equivalent unit balls are isomorphic*, Proc. Amer. Math. Soc. **58**, 129–133 (1976)

[15] K-T. Kim and L. Zhang, *On the uniform sequeezing property of bounded convex domains in* $\mathbb{C}^n$, Pacific J. Math. **282**, 341–358 (2016)

[16] S. Kobayashi, *Intrinsic distances, measure and geometric function theory*, Bull. Amer. Math. Soc. **82**, 357–416 (1976)

[17] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, World Scientific, Singapore (2005)

[18] Y. Kubota, *A note on holomorphic imbeddings of the classical Cartan domains into the unit ball*, Proc. Amer. Math. Soc. **85**, 65–68 (1982)

[19] S. Kwapien, *Isomorphic characterizations of Hilbert spaces by orthogonal series with vector valued coefficients*, Séminaire d'analyse fonctionnelle (Polytechnique) exp. no.8, 1–7 (1972–1973)

[20] K. Liu, X. Sun and S.-T. Yau, *Canonical metrics on the moduli space of Riemann surfaces I*, J. Diff. Geom. **68**, 571–637 (2004)

[21] K. Liu, X. Sun and S.-T. Yau, *Canonical metrics on the moduli space of Riemann surfaces II*, J. Diff. Geom. **69**, 163–216 (2005)

[22] J. Mujica, Complex analysis in Banach spaces, Math. Studies 120, North Holland, Amsterdam (1986)

[23] A. Pelczynski and C. Bessaga, *Some aspects of the present theory of Banach spaces*, In S. Banach: Travaux sur L'Analyse Fonctionnelle, Warszaw (1979)
[24] G.J. Roos, *Exceptional symmetric domains*, Symmetries in complex analysis, Contemp. Math. **468**, 157-189 (2008)
[25] S-K. Yeung, *Geometry of domains with the uniform squeezing property*, Adv. Math. **221**, 547–569 (2009)