A Version of Hilbert's axioms for the Euclidean plane

Axioms of Incidence

I. 1 To each pair of (distinct) points p and q, there is a unique line passing through p and q.

I. 2 Every line contains at least two distinct points.

I. 3 There exist three distinct points which are not collinear.

Axioms of Betweenness

B. 1 If p, q, r are points with $p * q * r$, then p, q, r are distinct, collinear and $r * q * p$.

B. 2 Given distinct points q and s, there exist points p, r, t on $\overparen{qs}$ such that $p * q * s$, $q * r * s$ and $q * s * t$.

B. 3 If p, q, r are distinct collinear points then exactly one is between the other two.

B. 4 (Separation Axiom) Let ℓ be any line and A, B, C points not on ℓ.

(i) If A, B and B, C are on the same side of ℓ then A, C are on the same side of ℓ.

(ii) If A, B and B, C are on opposite sides of ℓ then A, C are on the same side of ℓ.

Axioms of Congruence

C. 1 If A, B are distinct points and $\overrightarrow{A'X}$ is a ray, there is a unique point B' on $\overrightarrow{A'X}$ with $AB \cong A'B'$.

C. 2 Congruence is an equivalence relation on segments. That is,

$$AB \cong AB, \quad AB \cong CD \Rightarrow CD \cong AB, \quad AB \cong CD \cong EF \Rightarrow AB \cong EF.$$

C. 3 Let $A * B * C$ and $A' * B' * C'$ with $AB \cong A'B'$ and $BC \cong B'C'$. Then $AC \cong A'C'$.

C. 4 Given any angle $\angle BAC$, any ray $\overrightarrow{A'B'}$ and any side of $\overline{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on this side of $\overline{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$.

C. 5 Congruence is an equivalence relation on angles. That is,

$$\angle A \cong \angle A, \quad \angle A \cong \angle B \Rightarrow \angle B \cong \angle A, \quad \angle A \cong \angle B \cong \angle C \Rightarrow \angle A \cong \angle C.$$

C. 6 (SAS) Let ABC and $A'B'C'$ be two triangles. If $AB \cong A'B'$, $\angle BAC \cong \angle B'A'C'$ and $AC \cong A'C'$, then $\triangle ABC \cong \triangle A'B'C'$. (Meaning all pairs of corresponding sides and angles are congruent.)

Dedekind's Axiom

Suppose that the set of points on a line ℓ is the disjoint union of two non-empty sets Σ_1 and Σ_2 such that no point of Σ_1 lies between two points of Σ_2 and vice versa. Then there is a unique point P on ℓ such that, for all P_1, P_2 on ℓ,

$$P_1 * P * P_2 \quad \Longleftrightarrow \quad P \neq P_1,\ P_2 \text{ and } P_1,\ P_2 \text{ are not in the same set } \Sigma_i.$$

Parallel Axiom

Given any line ℓ and any point P not on ℓ, there is at most one line through P parallel to ℓ.