A Version of Hilbert’s axioms for the Euclidean plane

Axioms of Incidence

I. 1 To each pair of (distinct) points \( p \) and \( q \), there is a unique line passing through \( p \) and \( q \).
I. 2 Every line contains at least two distinct points.
I. 3 There exist three distinct points which are not collinear.

Axioms of Betweenness

B. 1 If \( p, q, r \) are points with \( p \neq q \neq r \), then \( p, q, r \) are distinct, collinear and \( r \neq q \neq p \).
B. 2 Given distinct points \( q \) and \( s \), there exist points \( p, r, t \) on \( \overleftrightarrow{qs} \) such that \( p \neq q \neq s \), \( q \neq r \neq s \) and \( q \neq s \neq t \).
B. 3 If \( p, q, r \) are distinct collinear points then exactly one is between the other two.
B. 4 (Separation Axiom) Let \( \ell \) be any line and \( A, B, C \) points not on \( \ell \).
   (i) If \( A, B \) and \( B, C \) are on the same side of \( \ell \) then \( A, C \) are on the same side of \( \ell \).
   (ii) If \( A, B \) and \( B, C \) are on opposite sides of \( \ell \) then \( A, C \) are on the same side of \( \ell \).

Axioms of Congruence

C. 1 If \( A, B \) are distinct points and \( \overrightarrow{A'X} \) is a ray, there is a unique point \( B' \) on \( \overrightarrow{A'X} \) with \( AB \cong A'B' \).
C. 2 Congruence is an equivalence relation on segments. That is,

\[
AB \cong AB, \quad AB \cong CD \Rightarrow CD \cong AB, \quad AB \cong CD \cong EF \Rightarrow AB \cong EF.
\]
C. 3 Let \( A \neq B \neq C \) and \( A' \neq B' \neq C' \) with \( AB \cong A'B' \) and \( BC \cong B'C' \). Then \( AC \cong A'C' \).
C. 4 Given any angle \( \angle BAC \), any ray \( \overrightarrow{A'B'} \) and any side of \( \overline{A'B'} \), there is a unique ray \( \overrightarrow{A'C'} \) on this side of \( \overline{A'B'} \) such that \( \angle BAC \cong \angle B'A'C' \).

C. 5 Congruence is an equivalence relation on angles. That is,

\[
\angle A \cong \angle A, \quad \angle A \cong \angle B \Rightarrow \angle B \cong \angle A, \quad \angle A \cong \angle B \cong \angle C \Rightarrow \angle A \cong \angle C.
\]

C. 6 (SAS) Let \( \triangle ABC \) and \( \triangle A'B'C' \) be two triangles. If \( AB \cong A'B' \), \( \angle BAC \cong \angle B'A'C' \) and \( AC \cong A'C' \), then \( \triangle ABC \cong \triangle A'B'C' \). (Meaning all pairs of corresponding sides and angles are congruent.)

**Dedekind’s Axiom**

Suppose that the set of points on a line \( \ell \) is the disjoint union of two non-empty sets \( \Sigma_1 \) and \( \Sigma_2 \) such that no point of \( \Sigma_1 \) lies between two points of \( \Sigma_2 \) and vice versa. Then there is a unique point \( P \) on \( \ell \) such that, for all \( P_1, P_2 \) on \( \ell \),

\[
P_1 \ast P \ast P_2 \iff P \neq P_1, P_2 \text{ and } P_1, P_2 \text{ are not in the same set } \Sigma_i.
\]

**Parallel Axiom**

Given any line \( \ell \) and any point \( P \) not on \( \ell \), there is at most one line through \( P \) parallel to \( \ell \).