ORDERING GRAPH PRODUCTS OF GROUPS

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1. INTRODUCTION

Let Γ be a finite simple graph with vertex set V, so the edges may be taken to be unordered pairs of distinct elements of V. Assume that, for every $v \in V$, there is assigned a group G_v. The *graph product* $G\Gamma$ is the quotient of the free product $\ast_{v\in V} G_v$ by the normal subgroup generated by all $[G_u, G_v]$ for which $\{u, v\}$ is an edge. (As usual, $[G_u, G_v]$ means the subgroup generated by $\{x^{-1}y^{-1}xy \mid x \in G_u,\ y \in G_v\}$.) These groups were studied by E. R. Green in her thesis [6], and have since attracted considerable attention (see, for example, [7] and the references cited there).

In this paper, it is shown that, if all G_v are right orderable, then $G\Gamma$ is right orderable, and if all G_v are (two-sided) orderable then $G\Gamma$ is orderable. Recall that a right order on a group G is a total order $\leq$ such that $a \leq b$ and $c \in G$ implies $ac \leq bc$. The *strictly positive cone* is the set $P := \{g \in G \mid 1 < g\}$. It has the properties

$$PP \subseteq P,\ G \setminus \{1\} = P \cup P^{-1} \text{ and } P \cap P^{-1} = \emptyset.$$

Conversely, given a subset P of G satisfying these conditions, defining $a < b$ to mean $ba^{-1} \in P$ gives a right order on G, with strictly positive cone P. A right order is called a two-sided order (abbreviated to *bi-order*) if in addition it is a left order, that is, $a \leq b$ and $c \in G$ implies $ca \leq cb$. A necessary and sufficient condition for this is that the strictly positive cone P is closed under conjugation: $x^{-1}Px \subseteq P$ for all $x \in G$.

Some elementary facts about graph products will be used.

(1) If Δ is a full subgraph of Γ with vertex set U, then $G\Delta$ embeds naturally in $G\Gamma$, and there is a retraction $\rho_{\Gamma,\Delta} : G\Gamma \to G\Delta$ such that $x\rho_{\Gamma,\Delta} = x$ for $x \in G\Delta$ and $x\rho_{\Gamma,\Delta} = 1$ for all $x \in G\Delta'$, where Δ' is the full subgraph of Γ with vertex set $V \setminus U$.

(2) For any $v \in V$, there is a decomposition

$$G\Gamma = (G_v \times GE) \ast_{GE} GZ \qquad (\ast)$$

where Z is the graph obtained by removing the vertex v and all edges incident with it from Γ, and E is the full subgraph of Γ whose vertices are the vertices of Γ adjacent to v (a full subgraph of Z).

To see (1), the inclusion map $\ast_{v\in U} G_v \to \ast_{v\in V} G_v$ and the projection map $\ast_{v\in V} G_v \to \ast_{v\in U} G_v$ (which is the identity on G_v for $v \in U$ and trivial on G_v for $v \notin U$) induce homomorphisms $\iota : G\Delta \to G\Gamma$ and $\rho : G\Gamma \to G\Delta$ with $\iota\rho = \mathrm{id}_{G\Delta}$. Thus ι is injective and $\rho_{\Gamma,\Delta} = \rho$ is the required retraction. To prove (2), there is an obvious map $\ast_{v\in V} G_v \to$

$(G_v \times GE) *_{GE} GZ$, which induces a homomorphism $G\Gamma \to (G_v \times GE) *_{GE} GZ$, and the inverse homomorphism can be defined using the universal property of free products with amalgamation. It is also proved as part of [6, Lemma 3.20].

In particular, taking Δ in (1) to have a single vertex, the groups G_v embed naturally in $G\Gamma$ and will be viewed as subgroups of $G\Gamma$.

In fact the result on right orderability can be easily proved using results in [2], and this will be done immediately.

Theorem A. *If all G_v are right orderable, then $G\Gamma$ is right orderable, and if Δ is any full subgraph of Γ, then every right order on $G\Delta$ extends to a right order on $G\Gamma$.*

Proof. The proof is by induction on n, the number of vertices of Γ. There is nothing to prove if $n \leq 1$, otherwise choose a vertex v and use the decomposition $(*)$ in (2):

$$G\Gamma = (G_v \times GE) *_{GE} GZ$$

where Z is the graph obtained by removing the vertex v and all edges incident with it from Γ, and E is the full subgraph of Γ whose vertices are the vertices of Γ adjacent to v. By induction, GZ is right orderable, and any right order on GE extends to a right order on GZ. Since GE is a subgroup of GZ, it is right orderable, hence $G_v \times GE$ is right orderable and any right order on GE extends to a right order on $G_v \times GE$, because of the exact sequence

$$1 \longrightarrow GE \longrightarrow G_v \times GE \xrightarrow{p} G_v \longrightarrow 1$$

where p is projection onto the first coordinate. See, for example, [2, Lemma 2.1]. By [2, Corollary 5.1], $G\Gamma$ is right orderable.

Suppose Δ is a full subgraph of Γ. If $\Gamma = \Delta$, obviously every right order on $G\Delta$ extends to $G\Gamma$, so assume $\Delta \neq \Gamma$, and choose a vertex v of Γ not in Δ. Let $\leq$ be a right order on $G\Delta$. In the decomposition $(*)$, Δ is a full subgraph of Z, so by induction $\leq$ extends to a right order on GZ, which induces a right order on GE by restriction. This right order extends to a right order on $G_v \times GE$, as observed above. By [2, Corollary 5.1], the right orders on $G_v \times GE$ and GZ extend to a right order on $G\Gamma$. This order extends $\leq$, as required. □

To prove the analogous result on bi-ordering graph products, more work is needed. It is a well-known theorem of Vinogradov [9] that free products of bi-orderable groups are bi-orderable. However, given a family of bi-ordered groups with a total order on the index set, a canonical way is needed to bi-order the free product of the family. This is dealt with in the next section, and the result on bi-ordering graph products will be proved in the third and final section.

2. Ordering Free Products

To make precise the statement that free products of bi-ordered groups with a totally ordered index set can be canonically ordered, a functor will be defined, from a certain category $\mathbf{G}$ to the category $\mathbf{O}$ of bi-ordered groups and order-preserving homomorphisms.

The objects of $\mathbf{G}$ are pairs $\mathcal{G} = (G, \{G_i \mid i \in I\})$ where G is a group, $\{G_i \mid i \in I\}$ is a family of bi-ordered subgroups of G, I is a totally ordered set and $G = \mathop{\ast}_{i \in I} G_i$. The set I

is called the *index set* of $\mathcal{G}$ and G_i is called a *free factor* of $\mathcal{G}$. Also, G is denoted by $\langle\mathcal{G}\rangle$. Use will be made of the *projection map* $e_{\mathcal{G},i} : \langle\mathcal{G}\rangle \to G_i$, for $i \in I$ (the unique map which is the identity on G_i and trivial on G_j, for $j \in I$, $j \neq i$).

A G-morphism from $\mathcal{G} = (G, \{G_i \mid i \in I\})$ to $\mathcal{H} = (H, \{H_j \mid j \in J\})$ is a pair

$$\mathsf{f} = (\lambda, \{f_i \mid i \in I\})$$

where $\lambda : I \to J$ is an order isomorphism, and for each $i \in I$, $f_i : G_i \to H_{i\lambda}$ is an isomorphism of bi-ordered groups. If $\mathsf{g} = (\mu, \{g_j \mid j \in J\}) : \mathcal{H} \to \mathcal{K}$ is a morphism, fg is defined to be $(\lambda\mu, \{f_i g_{i\lambda} \mid i \in I\})$, and the identity morphism $1_{\mathcal{G}}$ is $(id_I, \{id_{G_i} \mid i \in I\})$. Clearly this makes G into a category.

Theorem 2.1. *There is a functor $Q : \mathsf{G} \to \mathsf{O}$ such that, for every object $(G, \{G_i \mid i \in I\})$ of G, the underlying group of $(G, \{G_i \mid i \in I\})Q$ is G, and such that, for every morphism $\mathsf{f} = (\lambda, \{f_i \mid i \in I\})$ from $(G, \{G_i \mid i \in I\})$ to $(H, \{H_j \mid j \in J\})$, $\mathsf{f}Q$ is the isomorphism $G \to H$ whose restriction to G_i is f_i composed with the inclusion map $H_{i\lambda} \to H$.*

The proof of Theorem 2.1 takes up the rest of this section. It is modelled on the method of bi-ordering free groups given by Bergman ([1]). Given an object $\mathcal{G} = (G, \{G_i \mid i \in I\})$ in G, a bi-order needs to be defined on G. Before defining the order, an auxiliary construction will be introduced.

Let $l \in I$, and let

$$L = \langle a^{-1} G_i a \mid a \in G_l, i \in I, i > l\rangle.$$

Then $L = \mathop{\ast}_{(i,a)} a^{-1}G_i a$, where $a \in G_l$ and $i > l$. To see this, if $u = a_1^{-1} g_1 a_1 \dots a_n^{-1} g_n a_n$, where $a_j \in G_l$, $g_j \in G_{i_j}$, $i_j > l$ for $1 \le j \le n$ and $(i_j, a_j) \neq (i_{j+1}, a_{j+1})$ for $1 \le j < n$, then viewing this as a word in $\bigcup_{i\in I} G_i$ and cancelling/consolidating to obtain a reduced word, the letters g_j ($1 \le j \le n$) and the final letter a_n remain. This follows by induction on n. Thus $u \neq 1$, hence L is a free product as claimed.

Bi-order $a^{-1}G_i a$ by: $a^{-1}ga < a^{-1}ha$ if and only if $g < h$ in G_i. Then totally order the index set $I_l := \{i \in I \mid i > l\} \times G_l$ lexicographically: $(i,a) < (i_1, a_1)$ if and only if $i < i_1$ or $i = i_1$ and $a < a_1$. This gives a new object in G, namely

$$\mathcal{G}_l := (L, \{G_j \mid j \in I_l\})$$

where, for $j = (i,a) \in I_l$, $G_j = a^{-1}G_i a$. Thus $L = \langle\mathcal{G}_l\rangle$.

Take $1 \neq g \in G$ and write g as a reduced word relative to the decomposition $\mathop{\ast}_{i\in I} G_i$, say $g = g_1 \dots g_k$, where $g_j \in G_{i_j}$. The *length* of g is defined to be k.

Let $l = \min\{i_j \mid 1 \le j \le k\}$; l will be denoted by $gm_{\mathcal{G}}$. Rewrite the expression for g as $g = a_0 b_1 a_1 \dots a_{n-1} b_n a_n$, where $b_j \in G_l \setminus \{1\}$, and $a_j \in \mathop{\ast}_{i>l} G_i$ with $a_j \neq 1$ for $1 \le j \le n-1$, which in turn can be rewritten as

$$g = (b_1 \dots b_n) \prod_{j=0}^{n} (b_{j+1} \dots b_n)^{-1} a_j (b_{j+1} \dots b_n) = g' g^*$$

where $g' = b_1 \dots b_n \in G_l$ and $g^* \in \langle \mathcal{G}_l \rangle$ (note that $b_{j+1} \dots b_n$ means 1 when $j = n$). This decomposition is unique: if $g = h'h^*$, where $h' \in G_l$ and $h^* \in \langle \mathcal{G}_l \rangle$, then $g\,e = g' = h'$, where $e = e_{\mathcal{G},l}$ is the projection map, hence also $g^* = h^*$.

Note that g^* has shorter length relative to the free product decomposition of $\langle \mathcal{G}_l \rangle$ than k, the length of g relative to $\langle \mathcal{G} \rangle$. (The second length equals the first length plus n, and $n \neq 0$ by definition of l.) Therefore, a subset $P_{\mathcal{G}}$ of $\langle \mathcal{G} \rangle$ can be defined for all objects $\mathcal{G}$ of G recursively, as follows. Let $g \in \langle \mathcal{G} \rangle$, $g \neq 1$, and let $l = gm_{\mathcal{G}}$.

(1) If $g' \neq 1$, then $g \in P_{\mathcal{G}}$ if and only if $g' > 1$ in the given bi-order on the free factor G_l.
(2) If $g' = 1$, then $g \in P_{\mathcal{G}}$ if and only if $g^* \in P_{\mathcal{G}_l}$.

Eventually, it will be shown that $P_{\mathcal{G}}$ is the strictly positive cone for the required bi-order on $\langle \mathcal{G} \rangle$. Note that $g^{-1}m_{\mathcal{G}} = gm_{\mathcal{G}}$, $g^{-1} = (g')^{-1}(g'(g^*)^{-1}(g')^{-1})$ and $(g')^{-1} \in G_l$, $g'(g^*)^{-1}(g')^{-1} \in \langle \mathcal{G}_l \rangle$. Hence, if $g' \neq 1$, then exactly one of g, $g^{-1} \in P_{\mathcal{G}}$. If $g' = 1$, $g = g^*$, $g^{-1} = (g^*)^{-1} = (g^{-1})^*$, and by induction on length, exactly one of g, $g^{-1} \in P_{\mathcal{G}}$. Thus $G \setminus \{1\} = P_{\mathcal{G}} \cup P_{\mathcal{G}}^{-1}$ and $P_{\mathcal{G}} \cap P_{\mathcal{G}}^{-1} = \emptyset$.

The next thing to show is that $P_{\mathcal{G}} P_{\mathcal{G}} \subseteq P_{\mathcal{G}}$, and to do so it is necessary to look at the recursive definition in greater detail. Let $\mathcal{G} = (G, \{G_i \mid i \in I\})$ be an object of G and let $i_1 \in I$. One can form $\mathcal{G}_{i_1}$, with index set I_{i_1}. Given $i_2 \in I_{i_1}$, the construction can be repeated, obtaining $(\mathcal{G}_{i_1})_{i_2}$ with index set $(I_{i_1})_{i_2}$. Performing this operation n times (and omitting parentheses) gives an object $\mathcal{G}_{i_1 \dots i_n}$ of G with index set $I_{i_1 \dots i_n}$.

Definition. A sequence of indices $(i_1, \dots, i_n)$ arising in this way is called a *$\mathcal{G}$-descent sequence*.

Note that the empty sequence is allowed as a $\mathcal{G}$-descent sequence, the corresponding object of G being $\mathcal{G}$ with index set I. Also, an initial subsequence (prefix) of a $\mathcal{G}$-descent sequence is also a $\mathcal{G}$-descent sequence.

Remark 2.1. If $(i_1, \dots, i_n)$ is a $\mathcal{G}$-descent sequence and $i \in I_{i_1 \dots i_n}$ then $i = (i_0, a_1, \dots, a_n)$ for some $i_0 \in I$ and $a_1, \dots, a_n \in \langle \mathcal{G} \rangle$, and the free factor G_i of $\mathcal{G}_{i_1 \dots i_n}$ is $(a_1 \dots a_n)^{-1} G_{i_0} (a_1 \dots a_n)$. Moreover, the bi-order on G_i is given by: $a^{-1}ga < a^{-1}ha$ if and only if $g < h$ in G_{i_0}, where $a = a_1 \dots a_n$. This follows by induction on n. For later use, define, for $1 \leq j \leq n+1$

$$i^{(j)} := (i_0, a_1, \dots, a_{j-1}, 1, a_j, \dots, a_n).$$

Definition. Let $g \in \langle \mathcal{G} \rangle$. A $\mathcal{G}$-descent sequence $\mathbf{i} = (i_1, \dots, i_n)$ is *discriminating for g* if there exists j, $1 \leq j \leq n$ such that

(1) $g \in \langle \mathcal{G}_{i_1 \dots i_{j-1}} \rangle$;
(2) $i_j = gm_{\mathcal{G}_{i_1 \dots i_{j-1}}}$;
(3) $g\,e \neq 1$, where $e = e_{\mathcal{G}_{i_1 \dots i_{j-1}}, i_j} : \langle \mathcal{G}_{i_1 \dots i_{j-1}} \rangle \to G_{i_j}$ is the projection map.

If (1)–(3) hold, $\mathbf{i}$ is said to be discriminating for g at j, and $g\,e$ is called the $\mathbf{i}$-*signature* of g. This is justified because there is only one value of j such that $\mathbf{i}$ is discriminating for g at j. To see this, take j as small as possible. Then by (3), $g \notin \langle \mathcal{G}_{i_1 \dots i_j} \rangle$, because all free factors of $\langle\mathcal{G}_{i_1\ldots i_j}\rangle$ have the form $a^{-1}G_k a$, where $k \in I_{i_1\ldots i_{j-i}}$ and $k > i_j$, which are subgroups of $\ker(e)$, hence $\langle\mathcal{G}_{i_1\ldots i_j}\rangle$ is a subgroup of $\ker(e)$. Since

$$\langle\mathcal{G}\rangle \geq \langle\mathcal{G}_{i_1}\rangle \geq \ldots \geq \langle\mathcal{G}_{i_1\ldots i_n}\rangle,$$

$g \notin \langle\mathcal{G}_{i_1\ldots i_p}\rangle$ for $p \geq j$. Thus the value of j is unique.

Definition. If $\mathbf{i}$ is discriminating for g at j, and $ge > 1$ in G_{i_j}, g is called $\mathbf{i}$-*positive* (where G_{i_j} has its bi-order as a free factor of $\mathcal{G}_{i_1\ldots i_{j-1}}$).

For $g \in \langle\mathcal{G}\rangle$, $g \neq 1$, the definition of $P_{\mathcal{G}}$ above will recursively construct a canonical $\mathcal{G}$-descent sequence $\mathbf{i}$ which is discriminating for g, and $g \in P_{\mathcal{G}}$ if and only if g is $\mathbf{i}$-positive.

Lemma 2.2. *If $\mathbf{i} = (i_1,\ldots,i_n)$ is a $\mathcal{G}$-descent sequence, $1 \leq j \leq n$, $i \in I_{i_1\ldots i_{j-1}}$ and $i < i_j$, then*

(1) $\mathbf{i}^{(j)} := (i_1,\ldots i_{j-1}, i, i_j^{(j)},\ldots,i_n^{(j)})$ *is also a $\mathcal{G}$-descent sequence.*
(2) *If $\mathbf{i}$ is discriminating for g, then so is $\mathbf{i}^{(j)}$, and the $\mathbf{i}$-signature of g equals the $\mathbf{i}^{(j)}$-signature of g.*
(3) *If g is $\mathbf{i}$-positive, then g is $\mathbf{i}^{(j)}$-positive.*

Proof. First, it will be shown that, for $j \leq l \leq n$,

(i) $i_l^{(j)} \in I_{i_1\ldots i_{j-1} i i_j^{(j)}\ldots i_{l-1}^{(j)}}$;
(ii) if $k \in I_{i_1\ldots i_l}$, then $k^{(j)} \in I_{i_1\ldots i_{j-1} i i_j^{(j)}\ldots i_l^{(j)}}$;
(iii) the map $I_{i_1\ldots i_l} \to I_{i_1\ldots i_{j-1} i i_j^{(j)}\ldots i_l^{(j)}}$, $k \mapsto k^{(j)}$ preserves the strict order on these sets.

This will be proved by induction on l. (It is implicit in (i) that $(i_1,\ldots,i_{j-1},i,i_j^{(j)},\ldots,i_{l-1}^{(j)})$ is a $\mathcal{G}$-descent sequence, and it follows from (i) that $(i_1,\ldots,i_{j-1},i,i_j^{(j)},\ldots,i_l^{(j)})$ is a $\mathcal{G}$-descent sequence, so (ii) and (iii) make sense.) First note that, if $k \in I_{i_1\ldots i_l}$, then $k = (m,a)$ for some $m \in I_{i_1\ldots i_{l-1}}$ with $m > i_l$ and $a \in G_{i_l}$. Also, $k^{(j)} = (m^{(j)},a)$, and since $i \in I_{i_1\ldots i_{j-1}}$, $(i_1,\ldots,i_{j-1},i)$ is a $\mathcal{G}$-descent sequence.

Suppose $l = j$. Then $i_j \in I_{i_1\ldots i_{j-1}}$ and $i_j > i$, $1 \in G_i$, so $i_j^{(j)} = (i_j,1) \in I_{i_1\ldots i_{j-1}i}$ and (i) holds in this case. If $k = (m,a) \in I_{i_1\ldots i_j}$ then $m > i_j$, so $m > i$ and similarly, $m^{(j)} = (m,1) \in I_{i_1\ldots i_{j-1}i}$. Also, $m^{(j)} > (i_j,1) = i_j^{(j)}$, and $a \in G_{i_j} = G_{i_j^{(j)}}$, by Remark 2.1. Hence $k^{(j)} \in I_{i_1\ldots i_{j-1} i i_j^{(j)}}$, and (ii) holds. Suppose also $k_1 = (m_1,a_1) \in I_{i_1\ldots i_j}$. By Remark 2.1, the bi-order on G_{i_j} as a free factor of $\mathcal{G}_{i_1\ldots i_{j-1}}$ is the same as its bi-order as the free factor $G_{i_j^{(j)}}$ of $\mathcal{G}_{i_1\ldots i_{j-1}i}$. Thus if $k < k_1$ then either $m < m_1$, whence $m^{(j)} = (m,1) < (m_1,1) = m_1^{(j)}$ and so $k^{(j)} = (m^{(j)},a) < (m_1^{(j)},a_1) = k_1^{(j)}$, or $m = m_1$ and $a < a_1$, whence $m^{(j)} = m_1^{(j)}$ and again $k^{(j)} < k_1^{(j)}$, hence (iii) holds.

Now suppose $l > j$ and (i)–(iii) hold for $l-1$. Then by (ii) for the case $l-1$, with $k = i_l$, (i) holds for l. Suppose $k \in I_{i_1 \dots i_l}$ and write $k = (m,a)$ as above, with $i_l < m$. By the induction hypothesis, $m^{(j)} \in I_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}$ and in the order on this set, $i_l^{(j)} < m^{(j)}$. Also, $G_{i_l} = G_{i_l^{(j)}}$ by Remark 2.1, and the bi-orders of this group as a free factor of $\mathcal{G}_{i_1 \dots i_{l-1}}$ and as a free factor of $\mathcal{G}_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}$ are the same. It follows that $k^{(j)} \in I_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_l^{(j)}}$, so (ii) holds for l. Suppose also $k_1 = (m_1, a_1) \in I_{i_1 \dots i_l}$ and $k < k_1$. If $m < m_1$ then by the induction hypothesis $m^{(j)} < m_1^{(j)}$, and the argument in the case $l = j$ shows that $k^{(j)} < k_1^{(j)}$ (in the case $m = m_1$ and $a < a_1$ as well), hence (iii) holds for l. This establishes (i), (ii) and (iii).

Part (1) of the lemma now follows from (i) (with $j = n$).

Suppose $\mathbf{i}$ is discriminating for g at l. If $l < j$ then clearly $\mathbf{i}^{(j)}$ is discriminating for g at l and (2) and (3) hold in this case.

Suppose $l = j$. In the expression for g as a reduced word in the free factors of $\mathcal{G}_{i_1 \dots i_{j-1}}$, say $g = g_1 \dots g_p$, let

$$K = \left\{ k \in I_{i_1 \dots i_{j-1}} \mid \text{at least one of } g_1, \dots, g_p \text{ belongs to } G_k \right\}$$

a finite subset of $I_{i_1 \dots i_{j-1}}$ with least element i_j. Hence if $k \in K$, $k > i$, so $k^{(j)} = (k,1) \in I_{i_1 \dots i_{j-1} i}$. By Remark 2.1, $G_k = G_{k^{(j)}}$ is also a free factor of $\mathcal{G}_{i_1 \dots i_{j-1} i}$. Also, the map $K \to I_{i_1 \dots i_{j-1} i}$, $k \mapsto k^{(j)} = (k,1)$ is order preserving, hence $gm_{\mathcal{G}_{i_1 \dots i_{j-1} i}} = i_j^{(j)}$ and $ge_{i_j} = ge_{i_j^{(j)}}$, where

$$e_{i_j} = e_{\mathcal{G}_{i_1 \dots i_{j-1}}, i_j}, \quad e_{i_j^{(j)}} = e_{\mathcal{G}_{i_1 \dots i_{j-1} i}, i_j^{(j)}}.$$

Thus $\mathbf{i}^{(j)}$ is discriminating for g at $j+1$ and (2) holds. By Remark 2.1, the bi-order on G_{i_j} as a free factor of $\mathcal{G}_{i_1 \dots i_{j-1}}$ is the same as its bi-order as the free factor $G_{i_j^{(j)}}$ of $\mathcal{G}_{i_1 \dots i_{j-1} i}$, hence (3) holds in this case.

Finally suppose $l > j$. If $k \in I_{i_1 \dots i_{l-1}}$ then by (ii) of the claim, $k^{(j)} \in I_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}$ and by Remark 2.1, $G_k = G_{k^{(j)}}$, so every free factor of $\mathcal{G}_{i_1 \dots i_{l-1}}$ is a free factor of $\mathcal{G}_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}$. It follows from (iii) that $gm_{\mathcal{G}_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}} = i_l^{(j)}$, and $ge_{i_l} = ge_{i_l^{(j)}}$, using similar abbreviations for the projection maps to those in the previous case. Thus $\mathbf{i}^{(j)}$ is discriminating for g at $l+1$ and (2) holds. By Remark 2.1, the order on G_{i_l} as a free factor of $\mathcal{G}_{i_1 \dots i_{l-1}}$ is the same as its order as the free factor $G_{i_l^{(j)}}$ of $\mathcal{G}_{i_1 \dots i_{j-1} i i_j^{(j)} \dots i_{l-1}^{(j)}}$, hence (3) holds. This completes the proof. □

Lemma 2.3. *Let $\mathbf{i}_1$ be a $\mathcal{G}$-descent sequence discriminating for g, and let $\mathbf{i}_2$ be a $\mathcal{G}$-descent sequence discriminating for h. Then there is a $\mathcal{G}$-descent sequence $\mathbf{i}$ such that*

(1) $\mathbf{i}$ *is discriminating for g and for h;*
(2) *the $\mathbf{i}$-signature of g equals the $\mathbf{i}_1$-signature of g and the $\mathbf{i}$-signature of h equals the $\mathbf{i}_2$-signature of h;*

(3) *if g is $\mathbf{i}_1$-positive, then g is $\mathbf{i}$-positive and if h is $\mathbf{i}_2$-positive, then h is $\mathbf{i}$-positive.*

Proof. Put $\mathbf{i}_1 = (i_{11}, i_{12}, \ldots, i_{1m})$, $\mathbf{i}_2 = (i_{21}, i_{22}, \ldots, i_{2n})$ and suppose $\mathbf{i}_1$, $\mathbf{i}_2$ agree in the first p places, where $p \geq 0$ and p is maximal subject to this.

Case 1. If $p = m$ then $\mathbf{i} = \mathbf{i}_2$ is the desired sequence, and if $p = n$ then $\mathbf{i} = \mathbf{i}_1$ is the desired sequence.

Case 2. Otherwise, either $i_{1,p+1} > i_{2,p+1}$ or $i_{1,p+1} < i_{2,p+1}$. In the first case, replace $\mathbf{i}_1$ by

$$(i_{11}, \ldots, i_{1p}, i_{2,p+1}, i_{1,p+1}^{(p+1)}, \ldots, i_{1m}^{(p+1)}),$$

leaving $\mathbf{i}_2$ unchanged, and in the second case, replace $\mathbf{i}_2$ by

$$(i_{21}, \ldots, i_{2p}, i_{1,p+1}, i_{2,p+1}^{(p+1)}, \ldots, i_{2n}^{(p+1)})$$

without changing $\mathbf{i}_1$. By Lemma 2.2, the new sequences are $\mathcal{G}$-descent sequences and it suffices to prove the lemma for the new pair of sequences. The new sequences agree in at least the first $p+1$ places, so this reduces the non-negative integer $(m-p)+(n-p)$. Thus repetition of this procedure will terminate eventually in Case 1, giving the required sequence. (In fact, it must terminate with two sequences of length at most $m+n$.) $\square$

Corollary 2.4. *If $g \in \langle \mathcal{G} \rangle \setminus \{1\}$ and $\mathbf{i}$ is a $\mathcal{G}$-descent sequence which is discriminating for g, then the $\mathbf{i}$-signature of g is independent of $\mathbf{i}$. The following are equivalent:*

(1) *$g \in P_{\mathcal{G}}$;*
(2) *g is $\mathbf{i}$-positive for some $\mathcal{G}$-descent sequence $\mathbf{i}$ discriminating for g;*
(3) *g is $\mathbf{i}$-positive for all $\mathcal{G}$-descent sequences $\mathbf{i}$ discriminating for g.*

Proof. This follows from Lemma 2.3, applied with $g = h$, and the observation preceding Lemma 2.2. $\square$

Corollary 2.5. *If g, $h \in P_{\mathcal{G}}$ then $gh \in P_{\mathcal{G}}$.*

Proof. By Lemma 2.3 and Cor. 2.4, there is a $\mathcal{G}$-descent sequence $\mathbf{i}$ such that both g and h are $\mathbf{i}$-positive. Suppose $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{i}$ is discriminating for g at l and discriminating for h at p. Abbreviate $e_{\mathcal{G}_{i_1 \ldots i_{j-1}}, i_j}$ to e_{i_j}.

Case 1. $p > l$. Then g, $h \in \langle \mathcal{G}_{i_1 \ldots i_{l-1}} \rangle$, and $h \in \langle \mathcal{G}_{i_1 \ldots i_l} \rangle \subseteq \ker(e_{i_l})$, so $(gh)e_{i_l} = ge_{i_l} > 1$. Also, $hm_{\mathcal{G}_{i_1 \ldots i_{l-1}}} \geq i_l$, hence $(gh)m_{\mathcal{G}_{i_1 \ldots i_{l-1}}} \geq i_l$, and since $(gh)e_{i_l} \neq 1$, $(gh)m_{\mathcal{G}_{i_1 \ldots i_{l-1}}} = i_l$. Hence $\mathbf{i}$ is discriminating for gh at l and gh is $\mathbf{i}$-positive, therefore $gh \in P_{\mathcal{G}}$.

Case 2. $p < l$. Similarly $(gh)e_{i_p} = he_{i_p} > 1$, $\mathbf{i}$ is discriminating for gh at p and gh is $\mathbf{i}$-positive, so $gh \in P_{\mathcal{G}}$.

Case 3. $p = l$. Then $ge_{i_p} > 1$, $he_{i_p} > 1$, so $(gh)e_{i_p} = (ge_{i_p})(he_{i_p}) > 1$, and $(gh)m_{\mathcal{G}_{i_1 \ldots i_{l-1}}} \geq i_p$. Again it follows that $\mathbf{i}$ is discriminating for gh at p and gh is $\mathbf{i}$-positive, so $gh \in P_{\mathcal{G}}$. $\square$

Thus $P_{\mathcal{G}}$ is the strictly positive cone for a right order on $\langle \mathcal{G} \rangle$, for all objects $\mathcal{G}$ of $\mathbf{G}$. The next step is to show that if f is a morphism in $\mathbf{G}$, then the group isomorphism $\mathsf{f}Q$ is order-preserving, equivalently, maps the strictly positive cone to the strictly positive cone.

Lemma 2.6. *Let $\mathcal{G} = (G, \{G_i \mid i \in I\})$, $\mathcal{H} = (H, \{H_j \mid j \in J\})$ be objects of* $\mathbf{G}$, *and let* $\mathsf{f} = (\lambda, \{f_i \mid i \in I\})$ *be a morphism from $\mathcal{G}$ to $\mathcal{H}$. Then $P_{\mathcal{G}}(\mathsf{f}Q) \subseteq P_{\mathcal{H}}$.*

Proof. Firstly, for $l \in I$, f induces a morphism $\mathsf{f}_l = (\lambda_l, \{f_{i'} \mid i' \in I_l\})$ from $\mathcal{G}_l$ to $\mathcal{H}_{l\lambda}$, as follows. For $(i,a) \in I_l$, where $i \in I$, $i > l$ and $a \in G_l$, define $(i,a)\lambda_l = (i\lambda, af_l)$. It is easily checked that $\lambda_l : I_l \to J_{l\lambda}$ is an order-preserving bijection. For $i' = (i,a) \in I_l$, $f_{i'} : a^{-1}G_ia \to (af_l)^{-1}H_{i\lambda}(af_l)$ is defined by $a^{-1}ga \mapsto (af_l)^{-1}(gf_i)(af_l)$. This is clearly an isomorphism of bi-ordered groups, as f_i is order-preserving.

Note that $f_{i'}$ is $\mathsf{f}Q$ restricted to $a^{-1}G_la$, and it follows that f_lQ is $\mathsf{f}Q$ restricted to $\langle \mathcal{G}_l \rangle$. To prove the lemma, it will be shown, by induction on n, that for all n and any morphism $\mathsf{f} = (\lambda, \{f_i \mid i \in I\})$ of $\mathbf{G}$, say from $\mathcal{G} = (G, \{G_i \mid i \in I\})$ to $\mathcal{H} = (H, \{H_j \mid j \in J\})$, if $g \in \langle \mathcal{G} \rangle$, $g \neq 1$, has length n relative to the free product decomposition of $\mathcal{G}$, then $g \in P_{\mathcal{G}}$ implies $g(\mathsf{f}Q) \in P_{\mathcal{H}}$.

Assume then, that g has length n and $g \in P_{\mathcal{G}}$. Write $g = g'g^*$ as in the recursive definition, so $g' \in G_l$, $g^* \in \langle \mathcal{G}_l \rangle$, where $l = gm_{\mathcal{G}}$. Then

$$\begin{aligned} h := g(\mathsf{f}Q) &= (g'(\mathsf{f}Q))\,(g^*(\mathsf{f}Q)) \\ &= (g'f_l)\,(g^*(\mathsf{f}_lQ)) \end{aligned}$$

and $g'f_l \in H_{l\lambda}$, $g^*(\mathsf{f}_lQ) \in \langle \mathcal{H}_{l\lambda} \rangle$. Let $g = g_1 \dots g_n$ be the expression of g as a reduced word relative to the decomposition $\ast_{i \in I} G_i$, where $g_k \in G_{i_k}$. Then $h = h_1 \dots h_n$ is the expression of h as a reduced word relative to the decomposition $\ast_{j \in J} H_j$, where $h_k = g_kf_{i_k} \in H_{i_k\lambda}$. Since λ is order-preserving, it follows that $hm_{\mathcal{H}} = l\lambda$. Therefore, $h' = g'f_l$ and $h^* = g^*(\mathsf{f}_lQ)$. Thus, if $g' \neq 1$, then $g' > 1$ in G_l, hence $h' > 1$ in $H_{l\lambda}$, so $h \in P_{\mathcal{H}}$. If $g' = 1$ then $h' = 1$ and g^* has shorter length than g (relative to the free product decomposition of $\langle \mathcal{G}_l \rangle$), so by induction

$$g \in P_{\mathcal{G}} \Rightarrow g^* \in P_{\mathcal{G}_l} \Rightarrow h^* = g^*(\mathsf{f}_lQ) \in P_{\mathcal{H}_{l\lambda}} \Rightarrow h \in P_{\mathcal{H}}.$$

This completes the proof. □

The final step is to show that $P_{\mathcal{G}}$ is closed under conjugation, so is the strictly positive cone for a bi-order on $\mathcal{G}$; this will use the following remark.

Remark 2.2. Let $\mathcal{G} = (G, \{G_i \mid i \in I\})$ be an object of $\mathbf{G}$, let $i \in I$ and suppose $g \in \langle \mathcal{G}_i \rangle$. Then $g \in P_{\mathcal{G}}$ if and only if $g \in P_{\mathcal{G}_i}$.

For let $(i_1, \dots, i_n)$ be a $\mathcal{G}_i$-descent sequence which is discriminating for g at j, say. Then $(i, i_1, \dots, i_n)$ is a $\mathcal{G}$-descent sequence discriminating for g at $j+1$, with exactly the same signature in the group G_{i_j}, which is bi-ordered in the same way for both descent sequences. By Cor. 2.4, $g \in P_{\mathcal{G}}$ if and only if $g \in P_{\mathcal{G}_i}$.

Lemma 2.7. *Let $\mathcal{G} = (G, \{G_i \mid i \in I\})$ be an object of* $\mathbf{G}$ *and let $x \in \langle \mathcal{G} \rangle$. Then $x^{-1}P_{\mathcal{G}}x \subseteq P_{\mathcal{G}}$.*

Proof. It will be shown that, for all $\mathcal{G}$, all $x \in \langle \mathcal{G} \rangle$ and all $g \in P_{\mathcal{G}}$ of length n, $x^{-1}gx \in P_{\mathcal{G}}$, by induction on n. Since $\langle \mathcal{G} \rangle$ is generated by $\bigcup_{i \in I} G_i$, it can be assumed that $x \in G_i$ for some i. Write $g = g'g^*$ as in the recursive definition, so $g' \in G_l$, $g^* \in \langle \mathcal{G}_l \rangle$, where $l = gm_{\mathcal{G}}$.

Case 1: $i < l$. Then $g \in \langle \mathcal{G}_i \rangle = \ast_{(j,a)\in I_i} a^{-1}G_j a$. Conjugation by x induces a morphism $\mathsf{f} = (\lambda, \{f_{(j,a)} \mid (j,a) \in I_i\}) : \mathcal{G}_i \to \mathcal{G}_i$, where $(j,a)\lambda = (j,ax)$ and $yf_{(j,a)} = x^{-1}yx$ for $y \in G_{(j,a)} = a^{-1}G_j a$. By Remark 2.2 and Lemma 2.6,

$$g \in P_{\mathcal{G}} \Rightarrow g \in P_{\mathcal{G}_i} \Rightarrow x^{-1}gx = g(\mathsf{f}Q) \in P_{\mathcal{G}_i} \Rightarrow x^{-1}gx \in P_{\mathcal{G}}$$

as required.

Case 2: $i = l$. If $g' \neq 1$ then $(x^{-1}gx)' = x^{-1}g'x$ and $(x^{-1}gx)m_{\mathcal{G}} = l$. Then $g \in P_{\mathcal{G}}$ implies $g' > 1$ in G_i, so $x^{-1}g'x > 1$ as G_i is bi-ordered, hence $x^{-1}gx \in P_{\mathcal{G}}$ by definition. Otherwise $g = g^* \in \langle \mathcal{G}_i \rangle$, and as in the previous case, $x^{-1}gx \in P_{\mathcal{G}}$.

Case 3: $i > l$. Then $(x^{-1}gx)' = g'$, so if $g' \neq 1$ then $x^{-1}gx \in P_{\mathcal{G}}$ by definition. Otherwise, $g = g^* \in P_{\mathcal{G}_l}$ by definition, and $x \in \langle \mathcal{G}_l \rangle$, so by induction $x^{-1}gx = (x^{-1}gx)^* \in P_{\mathcal{G}_l}$, hence $x^{-1}gx \in P_{\mathcal{G}}$ by definition, since $l = (x^{-1}gx)m_{\mathcal{G}}$.

This completes the inductive proof. □

Thus for an object $\mathcal{G}$ of $\mathbf{G}$, $\mathcal{G}Q$ is $\langle \mathcal{G} \rangle$ with the bi-order having $P_{\mathcal{G}}$ as strictly positive cone, which has been shown to be a bi-order. It follows from Lemma 2.6 that if f is a morphism then $\mathsf{f}Q$ is order-preserving. It is routine to check that Q satisfies the conditions for a functor (it preserves multiplication and identity morphisms), so Theorem 2.1 is proved.

Note that, in the recursive definition of the order on $\langle \mathcal{G} \rangle$, if g has length 1, then clause (1) applies, hence the order on $\langle \mathcal{G} \rangle$ extends the given orders on the free factors.

3. Bi-ordering Graph Products

Let $G\Gamma$ be a graph product, let v be a vertex of Γ and consider the decomposition $(*)$ in (2) of §1:

$$G\Gamma = (G_v \times GE) *_{GE} GZ.$$

where Z is the graph obtained by removing the vertex v and all edges incident with it from Γ, and E is the full subgraph of Γ whose vertices are the vertices of Γ adjacent to v.

Lemma 3.1. *Let $\rho_{\Gamma,Z} : G\Gamma \to GZ$ be the retraction defined in §1, and let K be the kernel of $\rho_{\Gamma,Z}$. Then $K = \ast_{g\in R}\, gG_v g^{-1}$, where R is any transversal for the cosets $\{gGE \mid g \in GZ\}$.*

Proof. Corresponding to the decomposition $(*)$, let X be the usual Bass-Serre tree on which $G\Gamma$ acts ([8, Ch.I, §4, Theorem 7]). The vertex set is $(G\Gamma/(G_v \times GE)) \coprod (G\Gamma/GZ)$ and the edge set is $(G\Gamma/GE) \coprod \overline{(G\Gamma/GE)}$. Although these are disjoint unions, it will cause no confusion to view the vertices and edges corresponding to $G\Gamma/GE$ just as cosets. For $g \in G\Gamma$, the directed edge gGE starts at $g(G_v \times GE)$ and ends at gGZ, and for each such edge there is an oppositely oriented edge $\overline{gGE}$. Thus $\overline{(G\Gamma/GE)} = \{\overline{gGE} \mid g \in G\Gamma\}$. The action of $G\Gamma$ on X (on the left) is via the usual action on cosets.

Now K acts on X by restriction, and the action is transitive on the vertices gGZ. For $G\Gamma = K \rtimes GZ$ (because $\rho_{\Gamma,Z}$ is a retraction), so if $g \in G\Gamma$, $g = kz$ for some (unique) $k \in K$ and $z \in GZ$, and $gGZ = kGZ$. The edges ending at the vertex GZ are the cosets gGE for $g \in GZ$. If gGE, g_1GE are two such distinct edges, then their endpoints $g(G_v \times GE)$,

$g_1(G_v \times GE)$ are in distinct K-orbits, hence so are the edges themselves. For if $kg(G_v \times GE) = g_1(G_v \times GE)$, where $k \in K$, then $g_1 = kgae$ for some $a \in G_v$ and $e \in GE$. Applying $\rho_{\Gamma,Z}$ to this, $g_1 = ge$, so $gGE = g_1GE$.

Therefore, if R is a transversal for GZ/GE, the set of edges $\{gGE, \overline{gGE} \mid g \in R\}$ is the set of edges incident with GZ, and these edges, together with their endpoints, form a fundamental domain T for the action of K on X, in the sense of [8, Ch.I, §4, Definition 7]. There is an associated tree of groups (K,T) with K isomorphic to $K_T := \varinjlim(K,T)$ (see Ch. I, §4, Theorem 10 and the remarks preceding it in [8]).

The K-stabilizer of the common endpoint GZ of the edges of T is $K \cap GZ = 1$, and the stabilizer of the edge gGE is $K \cap gGEg^{-1} = g(K \cap GE)g^{-1} = 1$. The stabilizer of $g(G_v \times GE)$ is

$$K \cap g(G_v \times GE)g^{-1} = g(K \cap (G_v \times GE))g^{-1}.$$

If $k = ae$, where $k \in K$, $a \in G_v$ and $e \in GE$, then applying $\rho_{\Gamma,Z}$ gives $e = 1$, so $k \in G_v$. Hence the stabilizer of $g(G_v \times GE)$ is gG_vg^{-1}.

Therefore $K_T = \ast_{g \in R}\, gG_vg^{-1}$ (cf Example (c), §4.4, Chapter I in [8], with $A = 1$) and the lemma follows. □

Remark 3.1. If $gGE = g_1GE$ then $gag^{-1} = g_1ag_1^{-1}$ for all $a \in G_v$, so $gG_vg^{-1} = g_1G_vg_1^{-1}$. Thus changing the transversal in Lemma 3.1 does not change the decomposition of K, and $C := \{gG_vg^{-1} \mid g \in GZ\} = \{gG_vg^{-1} \mid g \in R\}$, for any transversal R. Conversely, if $gG_vg^{-1} = g_1G_vg_1^{-1}$, where g, $g_1 \in GZ$, then $gGE = g_1GE$. This follows because if $xG_vx^{-1} = G_v$, where $x \in GZ$, then $x \in GE$ by the normal form theorem for free products with amalgamation.

Let $\rho_{Z,E} : GZ \to GE$ be the retraction defined in §1 and let $L = \ker(\rho_{Z,E})$. Then $GZ = L \rtimes GE$, so $R = L$ is a valid choice for R in Lemma 3.1 and $L = L^{-1}$, hence $K = \ast_{l \in L}\, l^{-1}G_vl$.

Suppose both GZ and G_v are bi-ordered. For $l \in L$, bi-order $l^{-1}G_vl$ by: $l^{-1}gl \le l^{-1}g_1l$ if and only if $g \le g_1$ in G_v. As a subgroup of GZ, L is bi-ordered by restriction, in particular is totally ordered. Therefore $\mathcal{K} := (K, \{l^{-1}G_vl \mid l \in L\})$ is an object of the category $\mathbf{G}$. Thus $K = \mathcal{K}Q$ is bi-ordered.

Also, GZ acts on C by conjugation, so for $g \in GZ$, $l \in L$, $g^{-1}(l^{-1}G_vl)g = l_1^{-1}G_vl_1$ for some unique $l_1 \in L$, by Remark 3.1. The map $\lambda_g : l \mapsto l_1$ is a permutation of L, giving an action of $G\Gamma$ on L. That is, the map $g \mapsto \lambda_g$ is a homomorphism from GZ to the symmetric group on L. Further, the map $f_l^g : l^{-1}G_vl \to (l\lambda_g)^{-1}G_v(l\lambda_g)$, $l^{-1}xl \mapsto (l\lambda_g)^{-1}x(l\lambda_g)$ is an isomorphism of bi-ordered groups, and is conjugation by g, by Remark 3.1, because GE commutes with G_v.

Lemma 3.2. *In these circumstances, for all $g \in GZ$, $\mathsf{f}_g := (\lambda_g, \{f_l^g \mid l \in L\})$ is a morphism from $\mathcal{K}$ to $\mathcal{K}$ in* $\mathbf{G}$.

Proof. The only thing to check is that λ_g is order-preserving. Since GZ is generated by G_u, where u runs through the vertices of Z, it suffices to show this when $g \in G_u$, where u is a vertex of Z.

Case 1. u is not a vertex of E. Then $g \in L$, so $l\lambda_g = lg$, for all $l \in L$, and $l < l_1$ implies $lg < l_1 g$ because L is bi-ordered.

Case 2. u is a vertex of E. Then g commutes with all elements of G_v, so for $l \in L$,

$$g^{-1}l^{-1}G_v lg = (g^{-1}l^{-1}g)G_v(g^{-1}lg)$$

and $g^{-1}lg \in L$ as L is normal in GZ. Hence $l\lambda_g = g^{-1}lg$ for all $l \in L$, and $l < l_1$ implies $g^{-1}lg < g^{-1}l_1 g$ since GZ is bi-ordered. □

Theorem B. *If G_v is bi-orderable, for all vertices v of Γ, then $G\Gamma$ is bi-orderable, and if Δ is a full subgraph of Γ, then any bi-order on $G\Delta$ extends to a bi-order on $G\Gamma$.*

Proof. The proof is by induction on the number n of vertices of Γ, and there is nothing to prove when $n \leq 1$, so assume $n > 1$. Choose a vertex v and consider the decomposition $(*)$

$$G\Gamma = (G_v \times GE) *_{GE} GZ.$$

By induction GZ can be bi-ordered, and $K = \ker(\rho_{\Gamma,Z})$ can be bi-ordered (as $\mathcal{K}Q$) as in the discussion preceding Lemma 3.2. Since $G\Gamma = K \rtimes GZ$, the orders on K and GZ extend to a right order on $G\Gamma$ (see [2, Lemma 2.1]). To show that this right order is a bi-order, it suffices to show that the bi-order on K is invariant under conjugation by elements of GZ. But for $g \in GZ$, there is a morphism $\mathfrak{f}_g$ given by Lemma 3.2. Then $\mathfrak{f}_g Q$ is an order-preserving automorphism of K, and is conjugation by g, since it acts on the free factors of K as conjugation by g. Any bi-order on GZ can be extended in this way to a bi-order on $G\Gamma$, and the last part of the theorem follows by induction (cf the proof of Theorem A). □

In Theorem B it was assumed that the graph Γ is finite, but it is possible to consider $G\Gamma$ when Γ is an infinite simple graph; the definition is the same and properties (1) and (2) of graph products in §1 remain valid, as does Lemma 3.1.

Theorem C. *Theorems A and B remain true if Γ is infinite.*

Proof. If all G_v are bi-orderable, then it follows from Theorem B that $G\Gamma$ is bi-orderable. This is because bi-orderability is a local property: a group is bi-orderable if and only if every finitely generated subgroup is bi-orderable, and a finitely generated subgroup of $G\Gamma$ is contained in $G\Delta$ for some finite full subgraph Δ of Γ.

To obtain the second part of Theorem B when Γ is infinite requires a little more work. Given a full subgraph Δ of Γ, and a bi-order $\leq$ on $G\Delta$, let Ω be the set of all pairs $(B, \leq')$, where B is a full subgraph of Γ containing Δ and $\leq'$ is a bi-order on GB extending $\leq$. Partially order Ω by: $(B_1, \leq_1) \leq (B_2, \leq_2)$ if and only if B_1 is a subgraph of B_2 and $\leq_2$ extends $\leq_1$. Then $(\Delta, \leq) \in \Omega$, and a non-empty chain in Ω has an upper bound (by taking unions), so by Zorn's Lemma Ω has a maximal element, say $(Z, \leq_0)$. Suppose $Z \neq \Gamma$, and choose a vertex v of Γ not in Z, and let Γ' be the full subgraph of Γ whose vertices are those of Z together with v. Let E be the full subgraph of Γ' whose vertices are the vertices of Γ adjacent to v, and let $K = \ker(\rho_{\Gamma',Z})$. Then $G\Gamma' = K \rtimes GZ$, and arguing as in Theorem B, $\leq_0$ extends to a bi-order $\leq'$ on $G\Gamma'$. Thus $(\Gamma', \leq') \in \Omega$, contradicting the maximality of $(Z, \leq_0)$. Hence $\leq_0$ is an extension of $\leq$ to $G\Gamma$, as required.

Similarly, right orderability is a local property, so if Γ is infinite and all G_v are right orderable, then $G\Gamma$ is right orderable by Theorem A. The second part of Theorem A is also valid when Γ is infinite, using Zorn's Lemma, replacing "bi-order" by "right order" in the argument above. This works because a free product of right orderable groups is right orderable (see, for example, [2, Corollary 5.11]), and an extension of right ordered groups is right ordered ([2, Lemma 2.1]). Details are left to the reader. □

Another addition to Theorems A and B is the following.

Theorem D. *Let $G\Gamma$ be a graph product, where Γ is a (possibly infinite) simple graph.*
(1) *If G_v is right ordered, for all vertices v of Γ, then $G\Gamma$ has a right order extending all the given right orders on the groups G_v.*
(2) *If G_v is bi-ordered, for all vertices v of Γ, then $G\Gamma$ has a bi-order extending all the given bi-orders on the groups G_v.*

Proof. The proof of (2) needs another argument using Zorn's Lemma, considering pairs $(B, \leq)$, where B is a full subgraph of Γ and $\leq$ is a bi-order on GB extending the given right order on G_v, for all vertices v of B. It works because, in Lemma 3.1, if G_v is bi-ordered then K has a bi-order extending the bi-order on G_v, by the observation at the end of §2. Part (1) can be proved by a similar argument. It works because a free product of right ordered groups has a right order extending the orders on the free factors (again see [2, Corollary 5.11]). Once more, the details are left to the reader. □

A graph product with all vertex groups infinite cyclic is called a right-angled Artin group, a free partially commutative group or a graph group. Theorem B generalises the known result that these groups are bi-orderable (see [4], [5]). This special case also follows from a result in the thesis of C. Droms ([3, Chapter III, Theorem 1.1]), that these groups are residually torsion-free nilpotent.

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