Monomial modular representations and symmetric generation of the Harada–Norton group*

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Abstract

This paper is a sequel to Curtis [7], where the Held group was constructed using a 7-modular monomial representation of $3^{\cdot}A_7$, the exceptional triple cover of the alternating group A_7. In this paper, a 5-modular monomial representation of $2^{\cdot}HS{:}2$, a double cover of the automorphism group of the Higman–Sims group, is used to build an infinite semi-direct product $\mathcal{P}$ which has HN, the Harada–Norton group, as a 'natural' image. This approach assists us in constructing a 133-dimensional representation of HN over $\mathbb{Q}(\sqrt{5})$, which is the smallest degree of a 'true' characteristic 0 representation of $\mathcal{P}$. Thus an investigation of the low degree representations of $\mathcal{P}$ produces HN. As in the Held case, extension to the automorphism group of HN follows easily.

KEYWORDS: sporadic group, symmetric presentation, modular representation, matrix group construction.

1 Introduction

In 1973, Bernd Fischer found evidence for the Monster and Baby Monster simple groups, usually denoted M and B respectively; the Monster was discovered independently by Robert L. Griess. If $g \in M$ then $C_M(g)$, the centraliser in M of g, often contains a unique non-abelian composition factor, and in many cases this composition factor is a

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sporadic simple group. Indeed, using the notation for conjugacy classes given in the ATLAS [4], when $g \in$ 2A, 2B, 3A, 3B, 5B, 7A, 7B, 11A or 13A this composition factor is the Baby Monster B, the Conway group Co_1, the Fischer group Fi_{24}', the sporadic Suzuki group Suz, the Hall–Janko group J_2, the Held group He, the alternating group A_7, the Mathieu group M_{12} or the linear group $\mathrm{L}_3(3)$ respectively. These groups were all known at the time; however, both $\mathrm{C_M(3C)} \cong 3 \times \mathrm{Th}$ and $\mathrm{C_M(5A)} \cong 5 \times \mathrm{HN}$ had non-abelian (sporadic) composition factors, now known as the Thompson group and the Harada–Norton group respectively, whose existence was previously unknown. Note that, following the ATLAS, we write $\mathrm{C_M(3C)}$ to mean the centraliser in M of an element in class 3C, and so on. Koichiro Harada deduced much information about HN from knowledge of the involution centralisers in this putative group, namely $2^{\cdot}\mathrm{HS}{:}2$ and $2_+^{1+8}.(\mathrm{A}_5 \times \mathrm{A}_5).2$, and Simon Norton [9] constructed HN as a permutation group on 1140000 points 'by hand', besides conducting a thorough investigation of its structure.

Curtis [7] showed how consideration of a 15-dimensional 7-modular monomial representation of $3^{\cdot}\mathrm{A}_7$, the triple cover of the alternating group A_7, leads, in a well-motivated manner, to a definition of the Held group. As will be explained in more detail below, he obtained He as a homomorphic image of a split extension of a free product of 15 cyclic groups of order 7 by the group $3^{\cdot}\mathrm{A}_7$, an infinite semi-direct product which he denoted by $7^{\star 15}{:_{\mathrm{m}}}\, 3^{\cdot}\mathrm{A}_7$. It was found to be more natural to extend this so-called *progenitor* to a group of shape $7^{\star(15+15)}{:_{\mathrm{m}}}\, 3^{\cdot}\mathrm{S}_7$ in which both actions of A_7 on 15 letters are realised, and the two sets of 15 elements of order 7 are interchanged by outer elements of S_7. A relation which forces certain pairs of these 30 *symmetric generators* of order 7 to generate copies of the simple group $\mathrm{L}_2(7)$ is sufficient to define He. It turns out that the much larger Harada–Norton group HN can be defined in a remarkably similar manner, this time working with generators of order 5. Specifically, we consider a 176-dimensional 5-modular representation of $2^{\cdot}\mathrm{HS}$, the double cover of the Higman–Sims group, to construct a progenitor of shape $5^{\star 176}{:_{\mathrm{m}}}\, 2^{\cdot}\mathrm{HS}$. As in the Held case, it is more natural to 'double up' to a split extension of shape $5^{\star(176+176)}{:_{\mathrm{m}}}\, 2^{\cdot}\mathrm{HS}{:}2$ in which both actions of HS on 176 letters are realised, and outer elements interchange the two sets of 176 symmetric generators of order 5. As above, it turns out that a single relation which forces certain pairs of symmetric generators to generate copies of the simple group $\mathrm{L}_2(5)$ is sufficient to define HN. In both cases, He and HN, the action of the outer automorphism is easily described.

This approach is described below, and an outline of how it can be used to construct the 133-dimensional representation of HN over $\mathbb{Q}(\sqrt{5})$ is given. When referring to this representation, the ATLAS states that "explicit matrices have been computed"; however, such an explicit construction does not seem to occur in the literature. In his thesis, Norton [9] constructed 1140000 vectors which lay in a 133-dimensional subspace of a $(1 + 462)$-dimensional vector space over $\mathbb{Q}(\sqrt{5})$. He specified the action of both A_{12}, which was reasonably straightforward, and an element $g \notin \mathrm{A}_{12}$, which was much harder, on these 1140000 vectors. In principle, we can convert the information in Norton's thesis into 133×133 matrices generating HN, but this would involve considerably more work than what follows. Moreover, the construction employed here is well-motivated and could have resulted in the discovery of the group independently of the Monster. In effect, Norton constructed HN in the form $\mathrm{HN} = \langle \mathrm{A}_{12}, g\rangle$, using an amalgam of A_{12} and $\mathrm{N}_{\mathrm{HN}}(\mathrm{A}_6) \cong (\mathrm{A}_6 \times \mathrm{A}_6).2^2$ over $\mathrm{N}_{\mathrm{A}_{12}}(\mathrm{A}_6) \cong \frac{1}{2}(\mathrm{S}_6 \times \mathrm{S}_6)$. However, we construct the group in the form $\mathrm{HN} = \langle 2^{\cdot}\mathrm{HS}{:}2, t\rangle$ as an amalgam of $2^{\cdot}\mathrm{HS}{:}2$ and $(\mathrm{D}_{10} \times \mathrm{U}_3(5))^{\cdot}2$ over $\mathrm{U}_3(5){:}4 \cong (\mathrm{U}_3(5) \times 2)^{\cdot}2$. In our approach, it turns out that maximal subgroups isomorphic to A_{12} emerge easily as subgroups generated by certain subsets of our symmetric generators.

The definition of HN as a homomorphic image of a progenitor of shape $5^{\star(176+176)}{:_{\mathrm{m}}}\, 2^{\cdot}\mathrm{HS}{:}2$ will be helpful, both in the construction of HN, and for verifying that we do have a representation of it. By-products are the permutation representations of HN of degrees 1140000 and 1539000 obtained by coset enumeration over A_{12} and $2^{\cdot}\mathrm{HS}{:}2$ respectively.

In Section 2, we introduce *symmetric presentations*, giving symmetric presentations of $\mathrm{PGL}_2(p)$ and $\mathrm{L}_2(p)$ for p a prime. We then describe how factoring 'larger' *progenitors* a relation that forces a certain 'small' subprogenitor to be $\mathrm{PGL}_2(p)$ or $\mathrm{L}_2(p)$ gives rise to the sporadic Held and Harada–Norton groups (just stated at this stage). This is intended to cover the analogy between the symmetric presentations of the Held group, already covered by Curtis [7], and the Harada–Norton group, which we discuss in this paper. It is also intended to convey the fact that it is a very 'local' relation that produces these symmetric presentations. In Section 3, we give presentations of HS:2 and $2^{\cdot}\mathrm{HS}{:}2$ that we shall need later. We also justify the fact that certain relations in these presentations are redundant.

In Section 4, we write down a presentation of the Harada–Norton group based on our [at this stage still conjectured] symmetric presentation. Coset enumeration proves the correctness of this presentation, and thus also of our symmetric presentation. This presentation also resolves the ambiguity that was present when we initially introduced the progenitor $5^{\star(176+176)}{:_{\mathrm{m}}}\, 2^{\cdot}\mathrm{HS}{:}2$ in Section 2.3 by providing a presentation for the progenitor. We end this section by deriving and describing all possible outer automorphisms of HN using our symmetric presentation along with some information about the conjugacy classes of HN.

In Section 5, we give alternative presentations of HN based on our symmetric presentation, and also related presentations of S_7, $\mathrm{U}_3(5){:}2$ and HS:2. In Section 6, we determine all subgroups of HN generated subsets of the symmetric generators. We finish with Section 7 in which we briefly describe how to use our symmetric presentation to construct a 133-dimensional representation of HN over $\mathbb{Q}(\sqrt{5})$.

2 Symmetric generation and progenitor groups

Following Curtis [6, 7], we adopt the notation $m^{\star n}$ to mean $\mathrm{C}_m \star \mathrm{C}_m \star \cdots \star \mathrm{C}_m$ (n times), a free product of n copies of the cyclic group of order m. Let $\mathcal{F} = T_0 \star T_1 \star \cdots \star T_{n-1}$ be such a group, with $T_i = \langle t_i\rangle \cong \mathrm{C}_m$. Certainly permutations of the set of *symmetric* *generators* $\mathscr{T} = \{t_0, t_1, \dots t_{n-1}\}$ induce automorphisms of $\mathcal{F}$. But raising a given t_i to a power of itself coprime to m, while fixing the others, also gives rise to an automorphism of $\mathcal{F}$. Together these generate the group M of all *monomial automorphisms* of $\mathcal{F}$ which is a wreath product $H_r \wr \mathrm{S}_n$, where H_r is an abelian group of order $r = \phi(m)$, the number of positive integers less than m and coprime to it. A semi-direct product of the form:

$$\mathcal{P} \cong m^{\star n}{:}N,$$

where N is a subgroup of M which acts transitively on the set of cyclic subgroups $\bar{\mathscr{T}} = \{T_0, T_1, \dots, T_{n-1}\}$, is called a *progenitor*. We call N the *control subgroup* and its elements *monomial permutations* or, more informally, *monomials*. Of course, N may simply permute the set of elements $\mathscr{T}$, as will always be the case when $m = 2$, and a wealth of interesting homomorphic images arise from this case. The more general case involving proper monomial action allows further fascinating possibilities. Note that, since $\mathcal{P} = \langle N, \mathscr{T} \rangle$ and the action of N on $\mathscr{T}$ by conjugation is well-defined, elements of $\mathcal{P}$ may be put into a canonical form by gathering the elements of N on the left-hand side. Thus every element of $\mathcal{P}$ may be written, essentially uniquely, as an element of N followed by a word in the symmetric generators $\mathscr{T}$. In particular, if we seek homomorphic images of $\mathcal{P}$, as we shall be doing, the relators by which we must factor will have form πw, for $\pi \in N$ and w a word in the elements of $\mathscr{T}$.

As a classical example, we let p be a prime and consider

$$\mathcal{F} = \langle t_1 \rangle \star \langle t_2 \rangle \cong \mathrm{C}_p \star \mathrm{C}_p \cong p^{\star 2}.$$

Then if λ is a generator of the cyclic group $\mathbb{Z}_p^\times$, the group of monomial automorphisms of $\mathcal{F}$ is isomorphic to $\mathrm{C}_{p-1} \wr 2$ and is generated by

$$\begin{aligned} &\pi : t_1 \mapsto t_1^\lambda, t_2 \mapsto t_2, \text{ and} \\ &\sigma : t_1 \leftrightarrow t_2. \end{aligned}$$

We abbreviate these monomial actions as

$$\pi \sim \begin{pmatrix} \lambda & \cdot \\ \cdot & 1 \end{pmatrix} \quad \text{and} \quad \sigma \sim \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix},$$

with the obvious meaning. Now it turns out that the projective general linear group $\mathrm{PGL}_2(p)$ is an image of

$$\mathcal{P} \cong p^{\star 2}{:_{\mathrm{m}}}\mathrm{D}_{2(p-1)},$$

where

$$N = \langle \pi\pi^{-\sigma}, \sigma \rangle = \left\langle \begin{pmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \right\rangle \cong \mathrm{D}_{2(p-1)}.$$

Indeed, the classical presentations of Todd [11], see also Stanley [10, page 130], in our language take the form:

$$\frac{p^{\star 2}{:_{\mathrm{m}}}\mathrm{D}_{2(p-1)}}{\left[\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} t_1\right]^3} \cong \mathrm{PGL}_2(p), \tag{1}$$

where the left-hand side denotes the progenitor $p^{\star 2} :_{\rm m} \mathrm{D}_{2(p-1)}$ factored by the relator

$$\left[\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} t_1\right]^3,$$

which may be rewritten as the relation

$$\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} = t_1 t_2 t_1.$$

We realise the image $\mathrm{PGL}_2(p)$ regarded as the group of linear fractional transformations of $\mathrm{PG}_1(p) = \mathbb{Z}_p \cup \{\infty\}$ as follows:

$$\pi\pi^{-\sigma} = \begin{pmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{pmatrix} \sim \eta \mapsto \lambda\eta \quad \text{and} \quad \sigma = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \sim \eta \mapsto -\frac{1}{\eta}$$

and

$$t_1 \sim \eta \mapsto \eta + 1 \quad \text{and} \quad t_2 \sim \eta \mapsto \frac{\eta}{1-\eta}.$$

For p odd, in order to obtain the simple group, we let

$$N = \langle (\pi\pi^{-\sigma})^2, \sigma \rangle = \left\langle \begin{pmatrix} \mu & \cdot \\ \cdot & \mu^{-1} \end{pmatrix}, \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \right\rangle \cong \mathrm{D}_{p-1},$$

where μ (which we take, without loss of generality, to be λ^2) is a generator for the quadratic residues of $\mathbb{Z}_p^\times$. We then get:

$$\frac{p^{\star 2} :_{\rm m} \mathrm{D}_{p-1}}{\left[\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} t_1\right]^3} \cong \begin{cases} \mathrm{L}_2(p) \times 2 & \text{if } p \equiv 1 \pmod 4, \\ \mathrm{L}_2(p) & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2}$$

To quotient out the central involution in the case when $p \equiv 1 \pmod 4$, we add the additional relator

$$\left[\begin{pmatrix} \cdot & \lambda^{-2} \\ \lambda^2 & \cdot \end{pmatrix} t_1^\lambda\right]^3.$$

Note that in all of the above presentations, whether the group we have presented is $\mathrm{PGL}_2(p)$, $\mathrm{L}_2(p) \times 2$ or $\mathrm{L}_2(p)$, we have $\langle t_1, t_2 \rangle \cong \mathrm{L}_2(p)$.

Thus, if the two symmetric generators of order p are denoted by t_1 and t_2, the control subgroup is generated by automorphisms of $p^{\star 2} = \langle t_1 \rangle \star \langle t_2 \rangle$. In Presentation 1 the element $\pi\pi^{-\sigma}$ conjugates t_1 to t_1^λ and t_2 to $t_2^{\lambda^{-1}}$ and so acts as

$$(t_1, t_1^\lambda, t_1^{\lambda^2}, \ldots, t_1^{\lambda^{-2}}, t_1^{\lambda^{-1}})(t_2, t_2^{\lambda^{-1}}, t_2^{\lambda^{-2}}, \ldots, t_2^{\lambda^2}, t_2^\lambda)$$

on the non-trivial powers of t_1 and t_2; the involution σ interchanges t_1 and t_2 by conjugation, so we write that it has action (t_1, t_2). Note that we can determine the action of the given automorphisms on t_i^j for all i and j; in particular, (t_1, t_2) is an abbreviation for $(t_1, t_2)(t_1^2, t_2^2) \ldots (t_1^{p-1}, t_2^{p-1})$. The subscript 'm' on the colons in the progenitors above conveys the fact that the action is properly monomial.

Note that given an $n \times n$ monomial matrix A over $\mathbb{Z}_m$ where the nonzero entries of A are units, we can define the (group) action of A on $F \cong m^{\star n}$ by $t_i{}^A = t_j^{a_{ij}}$ where a_{ij} is the unique nonzero entry in the ith row of A.

2.1 Some notes and notation

We define N^i to be $\mathrm{C}_N(t_i)$ and N_i to be $\mathrm{N}_N(\langle t_i \rangle)$. Note that N_i is the stabiliser of T_i in N, when we consider N acting as a permutation group on the T_i. Extending this notion, we define:

$$N^{i_1 i_2 \ldots i_r} = N^{i_1} \cap N^{i_2} \cap \cdots \cap N^{i_r} = \mathrm{C}_N(\langle t_{i_1}, t_{i_2}, \ldots, t_{i_r} \rangle)$$

and

$$N_{i_1 i_2 \ldots i_r} = N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_r}.$$

Define a *monomial matrix* to be an $m \times m$ matrix over a ring R such that there is just one nonzero entry in each row and column, and a *monomial group* to be a group of monomial matrices. Note that in a monomial group the nonzero entries must be units of the ring R. Let $\{e_i\}$ be a 'basis' for R^m. If our monomial group, G say, acts transitively on the $\langle e_i \rangle$ then this monomial representation of G is equivalent (up to conjugation by an invertible monomial matrix) to a representation induced up from a linear (i.e. 1-dimensional) representation of $H = \mathrm{Stab}_G(\langle e_1 \rangle)$ which has index m in G. Note that the different choices of transversal of H in G correspond to conjugations by monomial matrices (and in general, different transversals for induced representations correspond to conjugation by a 'block-monomial' matrix). If our monomial group G does not act transitively on the $\langle e_i \rangle$ then this monomial representation is the direct sum of representations induced from linear representations of two or more subgroups.

Often, we consider monomial representations where the nonzero entries are various roots of unity in $\mathbb{C}$. We then reduce these modulo p for various primes p in order to obtain representations over finite fields (which in the main cases of interest in this paper are always the prime fields $\mathbb{Z}_p$). In general, the nonzero entries will correspond to elements of $\mathrm{Aut}(T)$, which we conveniently embed in the group ring $\mathbb{Z}\mathrm{Aut}(T)$. The only such case when T is non-cyclic that we have considered extensively is $T \cong 2^2$ when we are sometimes led to consider monomial matrices over $\mathbb{Z}\mathrm{S}_3$.

2.2 The Held progenitor and relation

In Curtis [7] we took $N \cong 3^{\cdot}\mathrm{S}_7$ as our control subgroup. This group possesses subgroups of index 30 of shape $3 \times \mathrm{L}_2(7)$, and if we induce a non-trivial linear representation of such a subgroup up to N we obtain a $(15+15)$-dimensional faithful monomial representation of $3^{\cdot}S_7$ over any field which contains non-trivial cube roots of unity. We choose the field $\mathbb{Z}_7$ of integers modulo 7, and use this representation to define the action of N on the free product $7^{\star(15+15)}$. It can be shown that the two progenitors of shape $7^{\star(15+15)}{:}3^{\cdot}S_7$ obtained by choosing 2 or 4 as our primitive cube root of unity are non-isomorphic. The subgroup of N fixing one of the symmetric generators, which is isomorphic to $L_2(7)$, acts with orbits $(1+14)+(7+8)$ on the 30 cyclic subgroups of order 7. Fixing a further symmetric generator in the 7-orbit is a subgroup isomorphic to S_4 which acts with orbits $(1+6+8)+(1+6+8)$. Normalising the subgroup generated by these two symmetric generators of order 7, r_0 and s_0 say, we have the 'central' element of order 3, which may be taken to square the r_i and fourth power the s_i, and an involution, commuting with the aforementioned S_4, interchanging them (replacing s_0 by a power if necessary). The subgroup of N isomorphic to S_4 mentioned above centralises r_0 and s_0. Thus, in the notation used in Presentation 2, these two automorphisms of $7^{\star 2}$ are denoted by

$$\begin{pmatrix} 2 & \cdot \\ \cdot & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}.$$

Therefore factoring by the relator

$$\left[\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} r_0\right]^3$$

ensures that $\langle r_0, s_0\rangle \cong L_2(7)$, or an image thereof. Factoring one of the two progenitors of shape $7^{\star(15+15)}{:}_{\mathrm{m}}3^{\cdot}S_7$ by a relator corresponding to this results in the Held group He, a sporadic simple group of order 4030 387200; with the other progenitor we get the trivial group. The outer automorphism of He is obtained by adjoining an element of order 2 which commutes with the control subgroup and inverts all the symmetric generators.

2.3 The Harada–Norton progenitor and relation

In order to define the Harada–Norton group HN in an analogous manner, using a progenitor with symmetric generators of order 5, we take as our control subgroup the group N of shape $2^{\cdot}HS{:}2$ in which the outer involutions lift to elements of order 2. [The isoclinic variant of this group, namely $2^{\cdot}HS^{\cdot}2$, has no outer involutions.] This group contains a subgroup $H \cong (2\times U_3(5))^{\cdot}2 \cong U_3(5){:}4$, which is generated by $U_3(5)$ together with an element of order 4 acting on it as on outer automorphism and squaring to the central involution. Thus, $H/H' \cong C_4$. In the usual way, we map a generator of H/H' onto a primitive fourth root of unity in an appropriate field. We induce the corresponding linear representation of H up to N to obtain a faithful monomial $(176+176)$-dimensional representation of N. Over the complex numbers $\mathbb{C}$, this gives an irreducible representation whose restriction to $2^{\cdot}HS$ has character which is the sum of the two 176-dimensional characters given in Table 2 at the end of the paper. Of course, the field with fourth roots of unity which interests us is $\mathbb{Z}_5$, which enables us to define a progenitor of shape $5^{\star(176+176)}{:}_{\mathrm{m}}2^{\cdot}HS{:}2$. As in the

Held case, there are two non-isomorphic progenitors of this shape, depending on whether we choose 2 or 3 as our primitive fourth root of unity.

Let $\mathscr{T} = \mathscr{R} \cup \mathscr{S} = \{r_0, r_1, \ldots, r_{175}, s_0, s_1, \ldots, s_{175}\}$ be our set of symmetric generators, and let $\bar{\mathscr{T}} = \{\langle r_i\rangle, \langle s_i\rangle : i = 0, 1, \ldots, 175\}$ be the set of cyclic subgroups they generate, arranged so that $N' \cong 2^{\cdot}\mathrm{HS}$ has orbits $\bar{\mathscr{R}} = \{\langle r_0\rangle, \langle r_1\rangle, \ldots, \langle r_{175}\rangle\}$ and $\bar{\mathscr{S}} = \{\langle s_0\rangle, \langle s_1\rangle, \ldots, \langle s_{175}\rangle\}$ on $\bar{\mathscr{T}}$. Then H may be chosen to normalise $\langle r_0\rangle$, whence H' commutes with $\langle r_0\rangle$, and both H and H' have orbits $(1 + 175) + (50 + 126)$ on $\bar{\mathscr{T}} = \bar{\mathscr{R}} \cup \bar{\mathscr{S}}$. We now choose $\langle s_0\rangle$ to lie in the 50-orbit of H (which is in $\bar{\mathscr{S}}$). Then $\langle r_0, s_0\rangle$ is centralised (in N) by a subgroup K of H' isomorphic to A_7, which has orbits $(1 + 7 + 42 + 126) + (1 + 7 + 42 + 126)$ on $\bar{\mathscr{T}}$. The subgroup $\langle r_0, s_0\rangle$ is normalised in N by an element of order 4 which squares the r_i and cubes the s_i, and an involution interchanging r_0 and s_0. Together, these two elements extend K to $\mathrm{N}_N(K) \cong (2^2 \times \mathrm{A}_7){:}2$. Thus, the subgroup $\langle r_0, s_0\rangle \cong 5^{\star 2} \cong \langle r_0\rangle \star \langle s_0\rangle$ is normalised in N by a subgroup isomorphic to D_8, which in the notation used in Presentation 1 acts as

$$\begin{pmatrix} 2 & \cdot \\ \cdot & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}.$$

Therefore factoring by the relator

$$\left[\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} r_0\right]^3,$$

where the matrix is one of the non-central involutions of $\mathrm{N}_N(K) \cong (2^2 \times \mathrm{A}_7){:}2$ that commutes with $K \cong \mathrm{A}_7$, ensures that $\langle r_0, s_0\rangle \cong \mathrm{L}_2(5)$, or an image thereof. Factoring one of the two progenitors of shape $5^{\star(176+176)}{:}_{\mathrm{m}}\,2^{\cdot}\mathrm{HS}{:}2$ by a relator corresponding to this results in the Harada–Norton group HN, a sporadic simple group of order 273 030912 000000. As before, factoring the other progenitor $5^{\star(176+176)}{:}_{\mathrm{m}}\,2^{\cdot}\mathrm{HS}{:}2$ by the corresponding relation gives rise to the trivial group. The outer automorphism of HN is obtained by adjoining an element of order 4 which commutes with N' and squares all the symmetric generators (and is thus inverted by the elements in $N \setminus N'$).

2.4 Non-isomorphic progenitors

In Bray [1, 2], it is shown that the two (possible) progenitors $5^{\star(176+176)}{:}_{\mathrm{m}}\,2^{\cdot}\mathrm{HS}{:}2$ in which the central involution of N inverts all the symmetric generators are non-isomorphic. This involved a double coset count, and also shows that for each $p \geqslant 13$, with p prime and $p \equiv 1 \pmod 4$ there are two non-isomorphic progenitors $p^{\star(176+176)}{:}_{\mathrm{m}}\,2^{\cdot}\mathrm{HS}{:}2$ in which $N \cong 2^{\cdot}\mathrm{HS}{:}2$ acts faithfully. The same technique also shows that for each prime $p \geqslant 7$ with $p \equiv 1 \pmod 6$ there are two isomorphism types of progenitor $p^{\star(15+15)}{:}_{\mathrm{m}}\,3^{\cdot}\mathrm{S}_7$ in which the control subgroup acts faithfully.

Naturally, the two similar-looking progenitors are rather difficult to distinguish. One technique to do this might be to introduce an invariant. The 'invariant' we want to take is the 'abelianisation' of the progenitor. The version of 'abelianisation' we want is rather weaker than the usual one, namely we just make T_i and T_j commute whenever $i \neq j$; thus the 'abelianisation' of the progenitor $T^{\star n}{:}N$ in this sense is $T^n{:}N$. Unfortunately, this definition can depend on the generating set chosen and so is not an invariant of the group. This problem is also addressed in Bray [2].

In our case, we can rescue the situation, for if $\mathcal{P} \cong p^{\star(176+176)}{:_\mathrm{m}}2^{\cdot}\mathrm{HS}{:}2$ and $\phi : \mathcal{P} \to N$ is an epimorphism then $\ker\phi$ is forced to be the $p^{\star 352}$ we first thought of. To see this, let ζ be the central involution of N and firstly suppose that $\zeta\phi = 1$. Since ζ inverts all the symmetric generators this forces $t\phi = 1$ for all symmetric generators t and thus forces $\mathcal{P}\phi$ to be an image of HS:2, a contradiction. Thus $\zeta\phi \neq 1$ and $N\phi \cong N$, which forces $\mathcal{P}\phi = N\phi = N$. But now $N^0\phi \cong \mathrm{U}_3(5)$ has no elements of order p in its N-centraliser; thus $t_0\phi = 1$ since $t_0\phi$ commutes with $N^0\phi$ and $t_0\phi$ has order 1 or p. Therefore $t\phi = 1$ for all symmetric generators t. Similar arguments show that the only epimorphisms from $p^{\star(15+15)}{:_\mathrm{m}}3^{\cdot}\mathrm{S}_7$ onto $3^{\cdot}\mathrm{S}_7$ have the visible $p^{\star(15+15)}$ in their kernels. In these cases, our abelianisation consists of performing the usual group theoretic abelianisation on the $\ker\phi$'s, which we have now established to be unique.

If $p \geqslant 13$, with p prime and $p \equiv 1 \pmod 4$, then both progenitors $p^{\star(176+176)}{:_\mathrm{m}}2^{\cdot}\mathrm{HS}{:}2$ have isomorphic abelianisations $p^{352}{:}2^{\cdot}\mathrm{HS}{:}2$, and if $p \geqslant 13$, with p prime and $p \equiv 1 \pmod 6$, then both progenitors $p^{\star(15+15)}{:_\mathrm{m}}3^{\cdot}\mathrm{S}_7$ have isomorphic abelianisations $p^{30}{:}3^{\cdot}\mathrm{S}_7$. In each case, $N \cong 2^{\cdot}\mathrm{HS}{:}2$ acts irreducibly on the p^{352} regarded as an $\mathbb{Z}_pN$-module; also $3^{\cdot}\mathrm{S}_7$ acts irreducibly on the p^{30} regarded as an $\mathbb{Z}_p3^{\cdot}\mathrm{S}_7$-module. The $5^{\star(176+176)}{:_\mathrm{m}}2^{\cdot}\mathrm{HS}{:}2$ giving rise to the Harada–Norton group abelianises as $5^{240\cdot 56\cdot 56}{:}2^{\cdot}\mathrm{HS}{:}2$, with submodules of dimensions 0, 240, 296 and 352 only; the other $5^{\star(176+176)}{:_\mathrm{m}}2^{\cdot}\mathrm{HS}{:}2$ abelianises as $5^{56\cdot 56\cdot 240}{:}2^{\cdot}\mathrm{HS}{:}2$ [with the dual module]. The abelianisation of the progenitor $7^{\star(15+15)}{:_\mathrm{m}}3^{\cdot}\mathrm{S}_7$ giving rise to the Held group is $7^{12\cdot 18}{:}3^{\cdot}\mathrm{S}_7$ with a single non-trivial proper submodule of dimension 12. The other $7^{\star(15+15)}{:_\mathrm{m}}3^{\cdot}\mathrm{S}_7$ has abelianisation $7^{18\cdot 12}{:}3^{\cdot}\mathrm{S}_7$.

3 The Higman–Sims group

We must first find a presentation for our proposed control subgroup $2^{\cdot}\mathrm{HS}{:}2$. Our starting point is the symmetric presentation of Curtis [6] given below:

$$\frac{2^{\star 50}{:}(\mathrm{U}_3(5){:}2)}{t_it_kt_it_j = (i,k)} \cong \mathrm{HS}{:}2, \tag{3}$$

where i—j—k is a path in the Hoffman–Singleton graph and (i,k) is the unique non-trivial element of $\mathrm{U}_3(5){:}2$ which commutes with the stabiliser of i and k. This is an involution, and is also the unique element of $\mathrm{U}_3(5){:}2$ that fixes j, swaps i and k, and fixes the other five points joined to j. Thus (i,k) is a transposition in the copy of S_7 fixing j.

So that we can successfully complete the required coset enumeration for HN, we have opted to create an alternative presentation of HS:2 based on the above symmetric presentation, rather than the one given in [6]. As generators of $H \cong \mathrm{U}_3(5){:}2$, we choose x of order 2 and y of order 6 such that x is an outer involution playing the rôle of (i,k) in Presentation 3, and $\langle x, y^2\rangle \cong \mathrm{S}_7$. These can be chosen to satisfy the presentation:

$$\langle x, y \mid x^2 = y^6 = [x,y]^3 = (xy^2xy^3)^3 = (xy)^8 = [x, y^{-2}xyxyxy^{-2}] = (xyxy^3)^4 = 1\rangle, \quad (4)$$

which can readily be shown to define $\mathrm{U}_3(5){:}2$. If the relation $[x, y^{-2}xyxyxy^{-2}] = 1$ is omitted, the resulting presentation defines $3^{\cdot}\mathrm{U}_3(5){:}2$, in which the outer elements invert the 'central' elements of order 3. In order to try to keep the length of the additional relation $t_it_kt_it_j = (i,k)$ to a minimum, we choose z to correspond to t_j, whence its centraliser in H is $\langle x, yxyxy^2xy^{-1}\rangle \cong \mathrm{S}_7$. The symmetric generator t_i is then given by z^{y^3xyxy}, and the additional relation of Presentation 3 reduces to $(z^{y^3xyxy}x)^3 = z$. This leads to the following presentation for HS:2:

$$\begin{aligned} N = \langle x, y, z \mid x^2 = y^6 = [x,y]^3 = (xy^2xy^3)^3 = (xy)^8 = [x, y^{-2}xyxyxy^{-2}] = (xyxy^3)^4 \\ = z^2 = (xz)^2 = [z, yxyxy^2xy^{-1}] = z(z^{y^3xyxy}x)^3 = 1\rangle \cong \mathrm{HS}{:}2, \end{aligned} \quad (5)$$

which can be verified by performing a coset enumeration over $H = \langle x, y\rangle$, to obtain the index 352. In fact, the relation $[x, y^{-2}xyxyxy^{-2}] = 1$ of Presentation 5 is redundant. This comes about because coset enumeration over $\langle x, y\rangle$ still yields the index 352, and so the derived subgroup of N, which is still perfect, has shape 3.HS or HS. In the former case, the central 3 of N' would be generated by the image of $[x, y^{-2}xyxyxy^{-2}]$, which is inverted by x. But this is a contradiction since $\langle x, y\rangle \leqslant N'$. Thus, the latter case (i.e. $N' \cong \mathrm{HS}$) must hold, and since it is easily seen, by abelianising the presentation, that N' has index 2 in N, we must have $N \cong \mathrm{HS}{:}2$. Thus, we have shown:

$$\begin{aligned} N = \langle x, y, z \mid x^2 = y^6 = [x,y]^3 = (xy^2xy^3)^3 = (xy)^8 = (xyxy^3)^4 \\ = z^2 = (xz)^2 = [z, yxyxy^2xy^{-1}] = z(z^{y^3xyxy}x)^3 = 1\rangle \cong \mathrm{HS}{:}2. \end{aligned} \quad (6)$$

3.1 The double cover of HS:2

We must now construct a presentation of the double cover $2^{\cdot}\mathrm{HS}{:}2$. We shall let x, y, z, H and N refer to preimages in $2^{\cdot}\mathrm{HS}{:}2$ (Atlas version) of what they were in HS:2. Now the image of x in HS is a 2B-involution, so x must have order 4 and $H = \langle x, y\rangle \cong (\mathrm{U}_3(5)\times 2)^{\cdot}2$, which we shall often write as $\mathrm{U}_3(5){:}4$. So x^2 must be the central involution of $2^{\cdot}\mathrm{HS}{:}2 = N$. Thus, to derive a presentation of $2^{\cdot}\mathrm{HS}{:}2$ from our presentation of HS:2, we must first ensure that x^2 is central of order 2, and then set each of the relators in Presentation 5 equal to 1 or x^2 as appropriate. Without loss of generality, we may assume that $y \in H'$ (since this happens in the image HS:2). Since $H/H' = \langle xH'\rangle$, with $yH' = H'$, we can determine whether a relator which is the identity in the image $\mathrm{U}_3(5){:}2$ is still the identity in H, or whether it is the central involution of H. Thus, we easily determine that

$$\begin{aligned} \langle x, y \mid x^4 = [x^2, y] = y^6 = [x,y]^3 = (xy^2xy^3)^3x^{-2} \\ = (xy)^8 = [x, y^{-2}xyxyxy^{-2}] = (xyxy^3)^4 = 1\rangle, \end{aligned}$$

is a presentation for $H \cong \mathrm{U}_3(5){:}4$. Now z is the preimage of an element of class 2C and so has order 2. The centraliser of the image of z in the image of H is a subgroup isomorphic to S_7. It turns out that the preimage of this copy of S_7, which has shape $(\mathrm{A}_7 \times 2)^{\cdot}2$, has elements that conjugate z to zx^2. Thus z commutes with the unique subgroup of index 2 in this preimage, which has shape $\mathrm{A}_7 \times 2$. This implies that z inverts x, i.e. $(zx)^2 = 1$ holds, and also that $yxyxy^2xy^{-1}$ conjugates z to zx^2 (since this element is in the outer half of the copy of $(\mathrm{A}_7 \times 2)^{\cdot}2$ referred to above), and so we have $z^2 = (zx)^2 = [z, yxyxy^2xy^{-1}]x^{-2} = 1$. These relations imply that $[z, x^2] = 1$ and $[z, x] = x^2$. The map $(x, y, z) \mapsto (x^{-1}, y, z)$ fixes all the previous relations but interchanges the two preimages of the relation $z(z^{y^3xyxy}x)^3 = 1$ of HS:2, and so $z(z^{y^3xyxy}x)^3$ can be set equal to either of the central elements; naturally we choose $z(z^{y^3xyxy}x)^3 = 1$. We may check that we have not factored out the central element of $2^{\cdot}\mathrm{HS}{:}2$ by taking a permutation representation of degree 1408 over $\langle y, y^x\rangle \cong \mathrm{U}_3(5)$. Thus we have:

$$\begin{aligned}\langle x, y, z \mid x^4 = [x^2, y] = y^6 = [x, y]^3 = (xy^2xy^3)^3x^{-2} = (xy)^8 = [x, y^{-2}xyxyxy^{-2}] \\ = (xyxy^3)^4 = z^2 = (zx)^2 = [z, yxyxy^2xy^{-1}]x^{-2} = z(z^{y^3xyxy}x)^3 = 1\rangle \cong 2^{\cdot}\mathrm{HS}{:}2.\end{aligned}$$

In order to shorten the above presentation for $2^{\cdot}\mathrm{HS}{:}2$, we first replace $x^4 = [x^2, y] = 1 = (xy^2xy^3)^3x^{-2}$ by $x^2yx^2y^{-1} = 1 = (x^{-1}y^2xy^3)^3$ in the preceding presentation of $(\mathrm{U}_3(5) \times 2)^{\cdot}2$. Furthermore, in either of these presentations of $(\mathrm{U}_3(5) \times 2)^{\cdot}2$, omitting $[x, y^{-2}xyxyxy^{-2}] = 1$ gives rise to $(3^{\cdot}\mathrm{U}_3(5) \times 2)^{\cdot}2$. In particular,

$$\langle x, y, z \mid x^2yx^2y^{-1} = y^6 = [x, y]^3 = (x^{-1}y^2xy^3)^3 = (xy)^8 = (xyxy^3)^4 = 1\rangle \tag{7}$$

is a presentation for $(3^{\cdot}\mathrm{U}_3(5) \times 2)^{\cdot}2$, as can verified by coset enumeration over $\langle xy\rangle \cong \mathrm{C}_8$. We then verify equality between $\langle x, yxyxy^2xy^{-1}\rangle$ and $\langle x, yxy^{-1}xy, (yxy)^2, y^2xy^{-2}xy^2\rangle$ by using coset enumeration with Presentation 7. An argument similar to the one we employed in the HS:2 case shows that:

$$\begin{aligned}\langle x, y, z \mid x^2yx^2y^{-1} = y^6 = [x, y]^3 = (x^{-1}y^2xy^3)^3 = (xy)^8 = (xyxy^3)^4 = z^2 \\ = (zx)^2 = [z, yxyxy^2xy^{-1}]x^{-2} = z(z^{y^3xyxy}x)^3 = 1\rangle \cong 2^{\cdot}\mathrm{HS}{:}2.\end{aligned} \tag{8}$$

It will be convenient in what follows to use the notation $N_0 = H$ and $N^0 = H'$ to denote the normaliser of $\langle r_0\rangle$ and the centraliser of r_0 respectively.

4 The progenitor and extra relation

We are now in position to write down a presentation for a progenitor of shape

$$5^{\star(176+176)}{:_{\mathrm{m}}}\, 2^{\cdot}\mathrm{HS}{:}2,$$

in which the central element of $N \cong 2^{\cdot}\mathrm{HS}{:}2$ inverts all the symmetric generators. This is done by adding a generator $t\,(= r_0)$ of order 5 to Presentation 8, and requiring that

t commutes with $N^0 = \langle y, y^x \rangle$ and $t^x = t^2$ or t^3. This last choice must result in non-isomorphic progenitors, since we know that there are two possible such progenitors up to isomorphism.

So far, we have the additional relations $t^5 = [t, y] = t^x t^{\pm 2} = 1$; we must now seek an extra relation that forces $\langle r_0, s_0 \rangle \cong \mathrm{L}_2(5)$, where $\langle s_0 \rangle$ lies in the 50-orbit of $\bar{\mathscr{T}}$ under N_0. We may take $s_0 = r_0^z$. Referring to Lemma 1 of Curtis [5], we find that $\langle r_0, s_0 \rangle \cap N \leqslant \mathrm{C}_N(\mathrm{C}_N(r_0) \cap \mathrm{C}_N(s_0)) = \langle z, x^2 \rangle$. Thus our two choices for the extra relation are $(zt)^3 = 1$ or $(zx^2t)^3 = 1$, since x^2 inverts t. But $(zt)^3 = 1$ implies that $(zx^2t^2)^3 = 1$, for there are elements of N that conjugate z to zx^2 and t to t^2, and we have only specified t up to powering. So we may take $(zt)^3 = 1$ as our final relation.

For both of the progenitors $5^{\star(176+176)} {:_{\mathrm{m}}} 2^{\cdot}\mathrm{HS}{:}2$ in which we were interested, we factored out by the extra relation $(zt)^3 = 1$ and enumerated the cosets of $N \cong 2^{\cdot}\mathrm{HS}{:}2$ in the resulting group G using Magma. In the case when $t^x = t^2$ we obtain $|G : N| = 1$; this is sufficient to show that $G = 1$ in this case. In the case when $t^x = t^{-2}$ we find that $|G : N| = 1539000$, which is the index we seek. In performing these enumerations, we appended some redundant relations. The actual presentations we used are:

$$\begin{aligned}\langle x, y, z, t \mid x^4 = [x^2, y] = y^6 = [x, y]^3 = (xy^2xy^3)^3x^{-2} = (xy)^8 = [x, y^{-2}xyxyxy^{-2}] \\ = (xyxy^3)^4 = z^2 = (xz)^2 = z(z^{y^3xyxy}x)^3 = [z, yxy^{-1}xy] = [z, (yxy)^2] \\ = [z, y^2xy^{-2}xy^2] = t^5 = t^x t^{\pm 2} = [t, y] = (zt)^3 = 1\rangle.\end{aligned}$$

Putting all of this together, we obtain that:

$$\begin{aligned}\langle x, y, z, t \mid y^6 = x^2yx^2y^{-1} = [x, y]^3 = (x^{-1}y^2xy^3)^3 = (xy)^8 = (xyxy^3)^4 = z^2 = (zx)^2 \\ = [z, yxyxy^2xy^{-1}]x^{-2} = z(z^{y^3xyxy}x)^3 = t^5 = t^x t^2 = [t, y] = (zt)^3 = 1\rangle.\end{aligned}$$

is a presentation for the Harada–Norton group HN.

Note. We invariably performed coset enumerations in Magma [3] with `Hard:=true` set. The coset enumerations we have been required to perform up to this point (the HN-enumerations described above, the enumeration of the cosets of $\langle x, y \rangle$ in Presentation 8, and so on) have not required more than 2 million cosets to complete. Some other presentations, such as those for HN with the redundant relations removed, might require more cosets to be defined in their enumerations. The enumerations associated with Section 5 can also be performed using relatively small tables, with the enumerations of the HN groups over their visible $2^{\cdot}\mathrm{HS}{:}2$ subgroups requiring no more than 2 million cosets (and no redundant relations).

4.1 The automorphism group

We shall now consider possible outer automorphisms of HN. Let a be such an automorphism. We know that HN has just one class of subgroups of shape $2^{\cdot}\mathrm{HS}{:}2$, for such groups must be the centralisers of 2A involutions, and so we may assume that a normalises $N = \langle x, y, z\rangle$. Furthermore, we know that Aut(2˙HS:2) ≅ HS:2×2 and so, multiplying by a suitable inner automorphism, we may assume that $[x, a] = [y, a] = 1$ and that $z^a = z$ or zx^2. Thus a now centralises N^0 and so a normalises $\mathrm{C}_{\mathrm{HN}}(N^0) = \langle t, x^2\rangle \cong \mathrm{D}_{10}$. Therefore $t^a \in \{t, t^2, t^{-2}, t^{-1}\}$. Now zt, zt^{-1}, zx^2t^2 and zx^2t^{-2} have order 3 and zt^2, zt^{-2}, zx^2t and zx^2t^{-1} have order 5. Thus if $z^a = z$ then $t^a = t$ or t^{-1} and (conjugation by) a is realised as conjugation by 1 or x^2 respectively, so a is inner. Therefore $z^a = zx^2$ and $t^a = t^2$ or t^{-2}, so a has order 4 and squares to (conjugation by) x^2. Replacing a by a^{-1} if necessary, we may assume that $t^a = t^2$. Note that za has order 2 so that our extension of HN to HN.2 is split.

In summary, to obtain Aut(HN) ≅ HN:2 from HN, we adjoin a such that $a^2 = x^2$, $[a, x] = [a, y] = 1$, $z^a = zx^2$ and $t^a = t^2$. Thus we easily get two maximal subgroups of HN:2 as follows: $\langle N, a\rangle \cong 4^{\cdot}\mathrm{HS}{:}2$ and $\langle N_0, t, a\rangle \cong 5{:}4 \times \mathrm{U}_3(5){:}2$. The automorphism a squares the r_i and cubes the s_i.

We remark that the above derives the form of the outer automorphism of HN and we remark also that the element we give above is indeed an outer automorphism. This uses the class list of HN and information we provide via our symmetric presentation. Thus we have shown that Out(HN) ≅ 2. Note that similar arguments, applied to the Held symmetric presentation of Curtis [7], show that Out(He) ≅ 2 and moreover derive the outer automorphism exhibited in that paper.

5 Another presentation of HN

More recently, we have produced another presentation of HN which is also derived from our symmetric presentation and which may be of interest to the reader. We include presentations of HS:2, $\mathrm{U}_3(5)$:2 and S_7 as well. Implicit within this series of presentations are HS:2 as an image of $2^{\star 50}{:}(\mathrm{U}_3(5){:}2)$ in the aforementioned manner, and $\mathrm{U}_3(5)$:2 as an image of a progenitor of shape $2^{\star 42}{:}\mathrm{S}_7$. We remark that in each of the presentations of HN given below a subgroup isomorphic to A_{12} is given by $\langle \alpha, \beta, \delta, \delta^{\gamma\beta\gamma}, \tau\rangle$; in the first presentation, this subgroup also contains ζ.

$$\begin{aligned}\langle \alpha, \beta, \gamma, \delta, \zeta, \tau \mid \zeta^2 &= [\zeta, \alpha] = [\zeta, \beta] = [\zeta, \gamma] = [\zeta, \delta] = \alpha^2\zeta = \beta^7 = (\alpha\beta^2)^4\\ &= (\alpha\beta\alpha\beta^3)^3\zeta = ((\alpha\beta)^3\alpha\beta^{-3})^2 = \gamma^2\zeta = [\alpha, \gamma] = [\beta\alpha\beta, \gamma]\\ &= (\beta\alpha\beta^3\gamma)^3 = \delta^2 = (\alpha\delta)^2 = [\beta, \delta] = \delta^{\gamma\beta\gamma\beta^{-1}}(\gamma\delta)^3\\ &= \tau^5 = (\zeta\tau)^2 = \tau^\alpha\tau^2 = \tau^\gamma\tau^{-2} = [\tau, \beta] = (\delta\tau)^3 = 1\rangle \cong \mathrm{HN}.\end{aligned}$$

$$\begin{aligned}\langle \alpha, \beta, \gamma, \delta, \tau \mid \alpha^4 &= [\alpha^2, \beta] = \beta^7 = (\alpha\beta^2)^4 = \alpha^{-2}(\alpha\beta\alpha\beta^3)^3 = ((\alpha\beta)^3\alpha\beta^{-3})^2\\ &= \gamma^2\alpha^2 = [\alpha, \gamma] = [\beta\alpha\beta, \gamma] = (\beta\alpha\beta^3\gamma)^3 = \delta^2 = (\alpha\delta)^2 = [\beta, \delta]\\ &= \delta^{\gamma\beta\gamma\beta^{-1}}(\gamma\delta)^3 = \tau^5 = \tau^\alpha\tau^2 = \tau^\gamma\tau^{-2} = [\tau, \beta] = (\delta\tau)^3 = 1\rangle \cong \mathrm{HN}.\end{aligned}$$

$$\langle \alpha, \beta, \gamma, \delta \mid \alpha^2 = \beta^7 = (\alpha\beta^2)^4 = (\alpha\beta\alpha\beta^3)^3 = ((\alpha\beta)^3\alpha\beta^{-3})^2 = \gamma^2 = [\beta\alpha\beta, \gamma] \\ = [\alpha, \gamma] = (\beta\alpha\beta^3\gamma)^3 = \delta^2 = [\alpha, \delta] = [\beta, \delta] = \delta^{\gamma\beta\gamma\beta^{-1}}(\gamma\delta)^3 = 1\rangle \cong \mathrm{HS}{:}2.$$

$$\langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^7 = (\alpha\beta^2)^4 = (\alpha\beta\alpha\beta^3)^3 = ((\alpha\beta)^3\alpha\beta^{-3})^2 \\ = \gamma^2 = [\alpha, \gamma] = [\beta\alpha\beta, \gamma] = (\beta\alpha\beta^3\gamma)^3 = 1\rangle \cong \mathrm{U}_3(5){:}2.$$

$$\langle \alpha, \beta \mid \alpha^2 = \beta^7 = (\alpha\beta^2)^4 = (\alpha\beta\alpha\beta^3)^3 = ((\alpha\beta)^3\alpha\beta^{-3})^2 = 1\rangle \cong \mathrm{S}_7.$$

6 Subgroups of HN generated by subsets of the symmetric generators

In this section, we are interested in subgroups generated by subsets of the symmetric generators. Since for t a symmetric generator, the four elements t, t^2, t^3 and t^4 each generate the cyclic subgroup $\langle t\rangle$, the relevant objects to consider are $\langle t\rangle$ for t a symmetric generator. Up to now, we have not given a name to the subgroups $\langle t\rangle$; in this paper, we shall refer to them as *cogs*. The progenitor $5^{\star(176+176)}:_{\mathrm{m}} 2^{\cdot}\mathrm{HS}{:}2$ contains 352 cogs, and each cog contains 4 symmetric generators. We shall label a cog by a symmetric generator it contains.

6.1 Subgroups generated by up to 2 cogs

Firstly, we note that 0 and 1 cogs generate the cyclic groups of orders 1 and 5 respectively. Given a cog t, $\mathrm{N}_N(t) \cong 2.(\mathrm{U}_3(5){:}2)$ has orbits $1 + 175 + 50 + 126$ on cogs, the orbit of size 1 being $\{t\}$.

If u is in the orbit of size 50 under $\mathrm{N}_N(t)$, we say that t and u are *50-joined* and we have $\langle t, u\rangle \cong \mathrm{A}_5$. This subgroup contains no further cogs. If u is in the orbit of size 126 under $\mathrm{N}_N(t)$, we say that t and u are *126-joined* and we have $\langle t, u\rangle \cong 5_+^{1+2}$. This subgroup contains no further cogs.

If u is in the orbit of size 175 under $\mathrm{N}_N(t)$, we say that t and u are *175-joined* and we have $\langle t, u\rangle \cong \mathrm{A}_6$. In this case, unlike the two cases above, t and u are cogs from the same *side*, that is N'-orbit; thus t and u are both in $\bar{\mathscr{R}}$ or both in $\bar{\mathscr{S}}$. The subgroup $\langle t, u\rangle$ contains two cogs from the other side, v and w say, which we call the *mates* of the first two. The two mates v and w are 175-joined; the four joins between $\{t, u\}$ and $\{v, w\}$ are all 50-joins. The group $\langle t, u\rangle$ contains just 4 cogs.

Note that if t and u are on the same side then the stabiliser in N of $\{t, u\}$, a group of shape $2.(2 \times \mathrm{A}_6.2^2)$, has two orbits of size 2, namely $\{t, u\}$ and $\{v, w\}$, with $\{v, w\}$ being the mates of $\{t, u\}$ as above.

6.2 Subgroups generated by 3 or more cogs

In each of the cases below, the subgroup generated by the cogs under consideration contains the '*matiness closure*' of those cogs, as it must, and no further cogs.

If a subgroup contains 3 or more cogs then two of them are on the same side, so that the subgroup contains a $\{t, u, v, w\}$ configuration generating an A_6 as detailed above. The stabiliser in N of such a configuration is a maximal subgroup of shape $2.(2 \times A_6.2^2).2$. This subgroup has orbits of lengths $4 + 24 + 144 + 180$ on cogs, with the orbit of length 4 being $\{t, u, v, w\}$. All of these orbits split equally across $\bar{\mathscr{R}}$ and $\bar{\mathscr{S}}$; thus the extra cog we add, s say, can be assumed to be on the same side as t and u.

If s is in the 180-orbit, so that s is 126-joined to both v and w, then $\langle t, u, v, w, s\rangle \cong 2^{\cdot}\mathrm{HS}$. This subgroup contains $32 = 16 + 16$ cogs, and these happen to be the matiness closure of the 5 given cogs. The stabiliser in N of such a configuration of 32 cogs is a maximal subgroup of shape $2.(2_+^{1+6}{:}S_5)$. If we add in yet another cog not in these 32 then we obtain the whole of HN, which of course contains all the cogs, and thus all of the symmetric generators. In fact, the stabiliser of these 32 cogs acts transitively on the remaining 320 cogs; adding one of these to our 32 and taking the matiness closure gives all the cogs.

If s is in the 144-orbit, so that s is 50-joined to one of v and w, and 126-joined to the other, then $\langle t, u, v, w, s\rangle \cong U_3(5)$. This subgroup contains $12 = 6 + 6$ cogs, and these are the matiness closure of the 5 given cogs. The stabiliser in N of this configuration of 12 cogs is a maximal subgroup of shape $2.(5{:}4 \times S_5)$, and this has orbits $12+100+240$ on the cogs, with 12-orbit being the subgroup generating $U_3(5)$. Taking these 12 cogs plus a cog from the 240-orbit yields a configuration generating the whole of HN; indeed the matiness closure of this configuration contains all cogs. Taking the 12 cogs plus a cog from the 100-orbit yields a configuration generating $2^{\cdot}\mathrm{HS}$; we have already seen this configuration above, and the matiness closure of the $12 + 1$ cogs consists of all 32 cogs that lie in this particular copy of $2^{\cdot}\mathrm{HS}$. Note that both the 100-orbit and the 240-orbit intersect the aforementioned 180-orbit non-trivially, so that no new cases can arise by extending these 12 cogs.

In the final case, s is in the 24-orbit, so that s is 50-joined to both v and w. Thus any case in which 126-joins occur has already been dealt with in one of the two cases above, so that from now on we may assume that all of our joins are 50-joins or 175-joins. Then we have $\langle t, u, v, w, s\rangle \cong A_7$, and this contains just one more cog, s' say. In fact, there are just 10 cogs that can 'validly' extend these 6; adding all 10 of them gives A_{12} (with no 126-joins present). Adding the cogs one at a time to A_7 gives A_8, A_9, A_{10}, A_{11} and A_{12}. At each stage another cog lies in the subgroup generated by the previous configuration and the just-added cog; thus we append two cogs to the configuration at each stage. For $n \in \{7, 8, 9, 10, 11, 12\}$, A_n is generated (in a unique way) by a configuration of $2(n-4)$ cogs whose [set-wise] stabiliser in N, namely $2.(2 \times S_{12-n} \times S_{n-4})$, acts transitively upon them, and also acts transitively upon the $2(12-n)$ cogs that can be validly added to the configuration. This completes the analysis of all subgroups generated by subsets of the symmetric generators.

6.3 Table of results

In Table 1 below we display information about all subsets of symmetric generators (up to N-conjugacy) whose image in HN contains no further symmetric generators. In the table below, H denotes the subgroup generated by the symmetric generators, and the "maximal" column refers to the maximality of $\langle H, \mathrm{N}_N(H)\rangle$ in HN. An $m_1 + m_2$ in the "config[uration] code" column tells how many cogs H has on each side. An entry $m + m$ (50) in the "config. code" column indicates that there are no 126-joins between the cogs of H; an entry $m + m$ (126) indicates that there is at least one 126-join.

In the $6 + 6$ (126) configuration, each cog is 126-joined to just one other [the '126-graph' of this configuration is six copies of K_2]. In the $16 + 16$ (126) configuration, each cog is 126-joined to just six others. The associated 126-graph has subsets of $\{0, 1, 2, 3, 4, 5\}$ modulo complementation as vertices, and S is joined to $S \cup \{i\}$ if $i \notin S$ and to $S \setminus \{i\}$ if $i \in S$.

Table 1: Subgroups of HN generated by subsets of the symmetric generators

config. code	H	$\mathrm{N}_N(H)$	$H \cap N$	$\langle H, \mathrm{N}_N(H)\rangle$	maximal
$0+0$	1	$2.(\mathrm{HS}{:}2)$	1	$2^{\cdot}\mathrm{HS}{:}2$	yes
$1+0$	5	$2.(\mathrm{U}_3(5){:}2)$	1	$(\mathrm{D}_{10} \times \mathrm{U}_3(5))^{\cdot}2$	yes
$1+1$ (50)	A_5	$2.(2 \times \mathrm{S}_7)$	2^2	$(\mathrm{A}_5 \times \mathrm{A}_7){:}2$	no
$1+1$ (126)	5_+^{1+2}	$2.(5_+^{1+2}{:}[2^5])$	5	$5_+^{1+4}{:}[2^6]$	no
$2+2$ (50)	A_6	$2.(2 \times \mathrm{A}_6.2^2).2$	D_8	$(\mathrm{A}_6 \times \mathrm{A}_6).2^2$	no
$3+3$ (50)	A_7	$2.(2 \times \mathrm{S}_5 \times \mathrm{S}_3)$	$(2^2 \times 3){:}2$	$(\mathrm{A}_7 \times \mathrm{A}_5){:}2$	no
$4+4$ (50)	A_8	$2.(2 \times \mathrm{S}_4 \times \mathrm{S}_4)$	$(2^2 \times \mathrm{A}_4){:}2$	$(\mathrm{A}_8 \times \mathrm{A}_4){:}2$	no
$5+5$ (50)	A_9	$2.(2 \times \mathrm{S}_3 \times \mathrm{S}_5)$	$(2^2 \times \mathrm{A}_5){:}2$	$(\mathrm{A}_9 \times 3){:}2$	no
$6+6$ (50)	A_{10}	$2.(2^2 \times \mathrm{S}_6)$	$(2^2 \times \mathrm{A}_6){:}2$	S_{10}	no
$7+7$ (50)	A_{11}	$2.(2 \times \mathrm{S}_7)$	$(2^2 \times \mathrm{A}_7){:}2$	A_{11}	no
$8+8$ (50)	A_{12}	$2.(2 \times \mathrm{S}_8)$	$(2^2 \times \mathrm{A}_8){:}2$	A_{12}	yes
$6+6$ (126)	$\mathrm{U}_3(5)$	$2.(5{:}4 \times \mathrm{S}_5)$	$2^{\cdot}\mathrm{S}_5^+$	$(\mathrm{D}_{10} \times \mathrm{U}_3(5))^{\cdot}2$	yes
$16+16$ (126)	$2^{\cdot}\mathrm{HS}$	$2.(2_+^{1+6}{:}\mathrm{S}_5)$	$2.4.2^4.\mathrm{S}_5$	$2^{\cdot}\mathrm{HS}{:}2$	yes
$176+176$ (all)	HN	$2.(\mathrm{HS}{:}2)$	$2^{\cdot}\mathrm{HS}{:}2$	HN	no

The symmetric presentation of $\mathrm{U}_3(5){:}2$ related to the above $6 + 6$ (126) configuration is given by Curtis [6] in his Higman–Sims paper. Jabbar [8, pp. 133–134] has produced a presentation for HS as a homomorphic image of $5^{\star(16+16)}{:}_{\mathrm{m}}4.2^4{:}\mathrm{S}_5$; this is the symmetric presentation satisfied by the $16 + 16$ (126) configuration above.

The $8 + 8$ (50) configuration that generates A_{12} gives rise to a symmetric presentation of A_{12} as follows. Let $t_i \sim (i, 8, 9, \mathrm{X}, \infty)$ and $u_i \sim (i, 9, 8, \infty, \mathrm{X})$ for $0 \leqslant i \leqslant 7$. Then the t_i

and u_i generate A_{12}. The symmetries in S_{12} of the t_i and u_i consist of: $\mathrm{Sym}(\{0,\dots,7\})$, permuting the t_i and u_i naturally; $\sigma \sim (8,9)(\mathrm{X},\infty)$, which swaps t_i and u_i (for all i); and $\iota \sim (8,9,\infty,\mathrm{X})$, which squares the t_i and cubes the u_i. These symmetries generate a subgroup isomorphic to $\mathrm{S}_8 \times \mathrm{D}_8$, of which $(\mathrm{A}_8 \times 2^2).2 \cong \frac{1}{2}(\mathrm{S}_8 \times \mathrm{D}_8)$ lies in A_{12}. Thus A_{12} is an image of the progenitor $5^{\star(8+8)}:_{\mathrm{m}}(\mathrm{A}_8 \times 2^2).2$. Using mechanical coset enumeration we verify that:

$$\frac{5^{\star(8+8)}:_{\mathrm{m}}(\mathrm{A}_8 \times 2^2).2}{(\sigma t_0)^3 = ((0,1)\iota\sigma t_0^2)^3 = 1} \cong \mathrm{A}_{12}. \tag{9}$$

In fact, the extra relation $(zt)^3$ we append to $5^{\star(176+176)}:_{\mathrm{m}}2^{\cdot}\mathrm{HS}{:}2$ implies that both of the extra relations $(\sigma t_0)^3 = ((0,1)\iota\sigma t_0^2)^3 = 1$ hold in a subprogenitor $5^{\star(8+8)}$ with $8+8$ (50) configuration; thus coset enumeration over this subgroup, which has index 1140000, is also a valid method to establish that we have a symmetric presentation of HN. In the notation of Section 4, a copy of A_{12} generated in the above manner is given by $\langle x, yxyxy^2xy^{-1}, z, z^{yxy}, t\rangle$.

The above symmetric presentation of A_{12} belongs to a series of symmetric presentations of the alternating groups by replacing 8 by n and 12 by $n+4$ for $n \geqslant 4$, see Bray [1]. We shall investigate this series of symmetric presentations, along with related ones, elsewhere.

The essence of the calculations in this section is to find all matiness closed sets of cogs. These calculations were done (with the aid of a computer) using the 352-point permutation action of HS:2 (the central involution of N fixes all of the cogs). The cog-wise stabiliser of two 175-joined cogs has orbits $1+1+12+72+90+2+12+36+36+90$ on cogs; the mates of the two given cogs form the orbit of size 2. The $\mathrm{N}_N(H)$ column of Table 1 was also compiled using this permutation representation. Calculations within subprogenitors, in particular to verify that $\langle x, yxyxy^2xy^{-1}, z, z^{yxy}, t\rangle$ (notation as in Section 4) satisfies Symmetric Presentation 9, can be done using the 1408 point action of $N \cong 2^{\cdot}\mathrm{HS}{:}2$. We then work in A_{12} to verify the assertion that the subgroup generated by two 175-joined cogs contains just two further cogs. We use our knowledge of A_{12} to fill in most of the rest of the table. We can use, say, a 133-dimensional representation of HN over $\mathbb{F}_9$ to help us determine what the configurations $1+1$ (126), $6+6$ (126) and $16+16$ (126) generate. (The 133-dimensional representation we make below is written over the ring $\mathbb{Z}[\sqrt{5}, \frac{1}{2}, \frac{1}{5}]$, and so can be directly reduced modulo 3.)

7 The matrix construction

We now give an outline of how to use our symmetric presentation to construct the faithful 133-dimensional representation(s) of HN over $\mathbb{Q}(\sqrt{5})$; these are interchanged by the field automorphism $\sqrt{5} \mapsto -\sqrt{5}$. Such a representation restricts to $2^{\cdot}\mathrm{HS}{:}2$ as $77\mathrm{a}^+ \oplus 56\mathrm{a}^{\pm}$. The two possible 56-dimensional representations are faithful and differ by only the sign of $\sqrt{5}$ and so we may choose either one.

The 77-dimensional representation of [2˙]HS:2 was chopped out of a 100-dimensional permutation module (along with a 22-dimensional representation), so this was relatively easy to obtain. Constructing the 56-dimensional representation of 2˙HS:2 was much harder. We shall omit the details of how this was done, but note that it was aided by the symmetric presentation of HS:2 as an image of $2^{\star 50}{:}(\mathrm{U}_3(5){:}2)$ that we have already mentioned in this paper. The details are contained in the first author's PhD thesis, see Bray [1]. Care should be taken when reading Bray [1] since the generators that were chosen for N_0 then are different from those used in this paper. A website available at:

http://web.mat.bham.ac.uk/spres/

contains, among other things, a 56-dimensional representation of 2˙HS:2 over $\mathbb{Z}[\sqrt{5}, \frac{1}{5}]$ and a 133-dimensional representation of HN over $\mathbb{Z}[\sqrt{5}, \frac{1}{2}, \frac{1}{5}]$. These representations are given on the generators x, y, z and t used in this paper.

The restriction to $N^0 \cong \mathrm{U}_3(5)$ is as $(21\mathrm{a} \oplus 28\mathrm{b} \oplus 28\mathrm{c}) \oplus (28\mathrm{b} \oplus 28\mathrm{c})$. (The labelling of the 28-dimensional irreducibles of N^0 is chosen so that the permutation module of N^0 over $\mathrm{C}_{N^0}(z) \cong \mathrm{A}_7$ is $1\mathrm{a} \oplus 21\mathrm{a} \oplus 28\mathrm{a}$.) With respect to a well-chosen basis, N^0 and x have the forms shown below:

$$N^0 \sim \begin{bmatrix} 21\mathrm{a} & & & & \\ & 28\mathrm{b} & & & \\ & & 28\mathrm{c} & & \\ & & & 28\mathrm{b} & \\ & & & & 28\mathrm{c} \end{bmatrix} \quad \text{and} \quad x \sim \begin{bmatrix} X & & & & \\ & & \mathrm{I}_{28} & & \\ & \mathrm{I}_{28} & & & \\ & & & & \mathrm{I}_{28} \\ & & & -\mathrm{I}_{28} & \end{bmatrix},$$

and N has block diagonal form $77 \oplus 56$. It is to be understood that, for each element of N^0, the submatrices in each of the 28b blocks are the same; the same applies to the 28c blocks. Again, the first author's PhD thesis tells us how to produce such a desirable basis.

Now the t that commute with N^0 have the form:

$$t \sim \begin{bmatrix} \kappa\mathrm{I}_{21} & & & & \\ & \alpha\mathrm{I}_{28} & & \beta\mathrm{I}_{28} & \\ & & \epsilon\mathrm{I}_{28} & & \zeta\mathrm{I}_{28} \\ & \gamma\mathrm{I}_{28} & & \delta\mathrm{I}_{28} & \\ & & \eta\mathrm{I}_{28} & & \theta\mathrm{I}_{28} \end{bmatrix},$$

when written with respect to this basis. The equations $t \neq 1 = t^5$, $t^x = t^3$ have just two solutions (for t) up to elements centralising N and these differ only by the sign of $\sqrt{5}$. In fact, we find that $\kappa = 1$, $\alpha = \delta = b/2$, $\epsilon = \theta = c/2$, $\beta\gamma = (-b-3)/4$, $\zeta = b\beta$ and $\eta = b\gamma$ where $\{b, c\} = \{\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})\}$ and without loss of generality we may set

$\beta = 1$, since we may conjugate t by elements centralising N. However, our matrices for N are in a reasonably 'nice' form and the map induced by applying the field automorphism $\sqrt{5} \mapsto -\sqrt{5}$ to matrix entries fixes elements of $N' \cong 2^{\cdot}\mathrm{HS}$ and multiplies outer elements of N (i.e., those in $N \setminus N'$) by the central involution x^2 of N. This map preserves N set-wise (and in fact is an automorphism of N) and so the two possibilities give rise to isomorphic groups (which must be HN). However, for only one of these possibilities do we have $\mathrm{o}(zt) = 3$; for the other one we have $\mathrm{o}(zt^2) = 3$ and $\mathrm{o}(zt) = 5$.

We remark (without proof) that the minimal degree of a true representation (see next sentence) of our progenitor $5^{\star(176+176)}{:_{\mathrm{m}}}\,2^{\cdot}\mathrm{HS}{:}2$ in characteristic other than 2 or 5 is indeed 133, the two possibilities of this degree giving rise to representations of the Harada–Norton group HN. Our definition of a true representation is motivated by our notion of a true image of a progenitor; thus we define a true representation of a progenitor $T^{\star n}{:}N$ to be a representation ρ for which $N\rho \cong N$, $T_0\rho \cong T_0$ and all the $T_i\rho$ are distinct. But in characteristic other than 2 or 5 the other progenitor $5^{\star(176+176)}{:_{\mathrm{m}}}\,2^{\cdot}\mathrm{HS}{:}2$ also has two true representations of degree 133 (with the minimum being 133 also); these representations are not representations of HN.

In characteristic 5 the possible representations of the two progenitors $5^{\star(176+176)}{:_{\mathrm{m}}}\,2^{\cdot}\mathrm{HS}{:}2$ are markedly different. Our progenitor has true representations of degree 57 [including irreducible ones]; the other progenitor does not have true representations of degree 57, or in smaller degrees.

7.1 Remarks on the representation

Presently, our representation of HN is written over $\mathbb{Z}[\mathrm{b}_5, \frac{1}{2}, \frac{1}{5}]$, where b_5 denotes the irrationality $\frac{1}{2}(-1 + \sqrt{5})$. Since $\mathbb{Z}[\mathrm{b}_5]$ is a Euclidean domain, it is possible to convert our representation into a $\mathbb{Z}[\mathrm{b}_5]$-integral representation of HN. We have not done this, and the 'integer explosion' that would almost certainly result has made us reluctant to try.

It is impossible to eliminate the $\frac{1}{2}$'s from the representation without destroying the block diagonal form of $N \cong 2^{\cdot}\mathrm{HS}{:}2$. For if we insist on block diagonal form for $2^{\cdot}\mathrm{HS}{:}2$, then the involution x^2 reduced modulo 2 would become the identity, whereas it is not the identity in HN. We do not know whether we can eliminate $\frac{1}{5}$'s from the representation while still preserving the block diagonal form of $2^{\cdot}\mathrm{HS}{:}2$. We have not tried very hard to do this, but the main sticking point appears to be in finding a $\mathbb{Z}[\mathrm{b}_5]$-integral representation of HS:2 of type 77a whose 5-modular reduction has the required structure, namely $1 \oplus 21 \oplus 55$.

Some effort has been expended to make sure that certain key representations of subgroups of N were expressed with respect to a 'nice' basis. In particular, the irreducible 77-dimensional representation of HS:2 and the irreducible 56-dimensional representation of $(\mathrm{U}_3(5) \times 2)^{\cdot}2$ (the latter needed when constructing $2^{\cdot}\mathrm{HS}{:}2$) have sparse matrices and require just the integers in $\{-2, -1, 0, 1, 2\}$. [This applies to any element of the groups, not just their generators.] At the moment, a typical entry of our HN-representation, when written as $\alpha + \beta\sqrt{5}$, for $\alpha, \beta \in \mathbb{Q}$, would have the numerators and denominators of α and β being anything up to (and beyond?) 1000 in modulus; the generators themselves are much nicer.

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Table 2: Some characters of 2˙HS (taken from the ATLAS [4])

	44352000	7680	2880	360	3840	256	64	500	300	25	36	24	7	16	16	16	20	20	11	11	12	15	20	20
p power		A	A	A	A	A	A	A	A	A	AB	AA	A	B	C	C	AA	BB	A	A	BA	BA	AA	AA
p' part		A	A	A	A	A	A	A	A	A	AB	AA	A	A	A	A	AA	BB	A	A	AA	BA	AA	AA
ind	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	11A	B∗∗	12A	15A	20A	B∗∗
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
+	22	6	−2	4	−6	2	2	−3	2	2	−2	0	1	0	0	0	1	−2	0	0	0	−1	−1	−1
+	77	13	1	5	5	5	1	2	−3	2	1	1	0	1	−1	−1	−2	1	0	0	−1	0	0	0
+	154	10	10	1	−2	6	−2	4	4	−1	1	1	0	0	0	0	0	0	0	0	1	1	−2	−2
+	154	10	−10	1	−10	−2	2	4	4	−1	−1	1	0	0	2	−2	0	0	0	0	−1	1	0	0
+	154	10	−10	1	−10	−2	2	4	4	−1	−1	1	0	0	−2	2	0	0	0	0	−1	1	0	0
+	175	15	11	4	15	−1	3	0	5	0	2	0	0	−1	1	1	0	1	−1	−1	0	−1	0	0
+	231	7	−9	6	15	−1	−1	6	1	1	0	−2	0	−1	−1	−1	2	1	0	0	0	1	0	0
ind	1	2	4	3	4	4	4	5	5	5	12	6	7	8	8	8	10	20	11	11	12	15	20	20
	2	2		6	4			10	10	10	12	6	14	8			10		22	22	12	30	20	20
+	56	8	0	2	0	0	0	6	−4	1	0	2	0	0	0	0	−2	0	1	1	0	2	0	0
◦	176	16	0	5	16i	0	0	1	6	1	3i	1	1	0	0	0	1	0	0	0	−i	0	i	i
◦	176	16	0	5	−16i	0	0	1	6	1	−3i	1	1	0	0	0	1	0	0	0	i	0	−i	−i