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Transitions from deterministic to stochastic diffusion

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Abstract. – We examine characteristic properties of deterministic and stochastic diffusion in low-dimensional chaotic dynamical systems. As an example, we consider a periodic array of scatterers defined by a simple chaotic map on the line. Adding different types of time-dependent noise to this model we compute the diffusion coefficient from simulations. We find that there is a crossover from deterministic to stochastic diffusion under variation of the perturbation strength related to different asymptotic laws for the diffusion coefficient. Typical signatures of this scenario are suppression and enhancement of normal diffusion. Our results are explained by a simple theoretical approximation.

Understanding diffusion in *noisy maps*, that is, in time-discrete dynamical systems where the deterministic equations of motion are perturbed by noise, figures as a prominent problem in the recent literature. The most simple example of such models are one-dimensional chaotic maps on the line. In seminal contributions by Geisel and Nierwetberg [1], and by Reimann *et al.* [2], scaling laws have been derived for the diffusion coefficient yielding suppression and enhancement of diffusion with respect to the variation the of the noise strength. Related results have been obtained in refs. [3, 4]. However, all these results apply only to the onset of diffusion where the scaling laws are reminiscent of a dynamical phase transition, and not much appears to be known far away from this transition point. In such more general situations, only perturbations by a non-zero average bias have been studied [5]. Related models are deterministic Langevin equations, in which the interplay between deterministic and stochastic chaos has been analyzed [6], however, without focusing on diffusion coefficients. Non-diffusive noisy maps have furthermore been investigated by refinements of cycle expansion methods [7].

Deterministic diffusion refers to the asymptotically linear growth of the mean square displacement in a purely deterministic, typically chaotic dynamical system [1, 2, 4, 8, 9], whereas by *stochastic diffusion* we denote the respective behavior of the same quantity in a system driven by uncorrelated random noise. In this work we study the transition scenario from deterministic to stochastic diffusion in the most simple type of chaotic dynamical systems, which are piecewise linear maps on the line. Particularly, we are searching for signatures of deterministic and stochastic dynamics in the diffusion coefficient as a function of the strength of time-dependent stochastic noise. In this aspect our work appears to be related to the recent dispute on a possible distinction between chaotic and stochastic diffusion in experiments [10], where some of the theoretical models studied are very similar to the one introduced below.

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We define our system as follows: The unperturbed map is given by the equation of motion

$$x_{n+1} = M_a(x_n) \quad , \tag{1}$$

where $a \in \mathbb{R}$ is a control parameter and x_n is the position of a point particle at discrete time n. $M_a(x)$ is continued periodically beyond the interval $[-1/2, 1/2)$ onto the real line by a lift of degree one, $M_a(x+1) = M_a(x)+1$. We assume that $M_a(x)$ is anti-symmetric with respect to $x = 0$, $M_a(x) = -M_a(-x)$. The map we study as an example is defined by $M_a(x) = ax$, where the uniform slope a serves as a control parameter. The Lyapunov exponent of this map is given by $\lambda = \ln a$ implying that for $a > 1$ the dynamics is chaotic. We now apply two types of *annealed* disorder [11] to this map, i) noisy slopes [2,4]: we add the random variable Δa_n, $n \in \mathbb{N}$, to all slopes a making them time dependent in form of

$$M_{a+\Delta a_n}(x) = (a + \Delta a_n)x \ , \tag{2}$$

or ii) noisy shifts [1–3]: we add the random variable Δb_n, $n \in \mathbb{N}$, as a time-dependent uniform bias yielding

$$M_{a,\Delta b}(x) = ax + \Delta b_n \ . \tag{3}$$

In both cases we assume that the random variable $\Delta_n \in \{\Delta a_n, \Delta b_n\}$ is independent and identically distributed according to a distribution $\chi_d(\Delta_n)$, where $d \in \{da, db\}$ is again a control parameter. We consider two different types of such distributions, namely random variables distributed uniformly over an interval of size $[-d, d]$ [3,4],

$$\chi_d(\Delta_n) = \frac{1}{2d}\Theta(d + \Delta_n)\Theta(d - \Delta_n) \quad , \tag{4}$$

and dichotomous or δ-distributed random variables [2,4],

$$\chi_d(\Delta_n) = \frac{1}{2}(\delta(d - \Delta_n) + \delta(d + \Delta_n)) \quad . \tag{5}$$

Since $|\Delta_n| \leq d$, we denote d as the perturbation strength. As an example, we sketch in fig. 1 our model for noisy slopes. We now define the diffusion coefficient as

$$D(a, d) = \lim_{n\to\infty} \frac{1}{2n}(\langle x_n^2\rangle_{\rho_0} - \langle x_n\rangle_{\rho_0}^2) \ , \tag{6}$$

with

$$\langle x_n^k\rangle_{\rho_0} = \int \mathrm{d}x \int \mathrm{d}(\Delta_0)\mathrm{d}(\Delta_1)\ldots\mathrm{d}(\Delta_{n-1})\rho_0(x)\chi(\Delta_0)\chi(\Delta_1)\ldots\chi(\Delta_{n-1})x_n^k \ , \tag{7}$$

where $\rho_0(x)$ denotes the initial distribution of an ensemble of moving particles, $x_0 \equiv x$, $k \in \mathbb{N}$, and $\Delta_j, j \in \{1, \ldots, n-1\}$, is the random variable. $D(a, d)$ has been computed by iterating eqs. (2),(3) numerically for an ensemble of moving particles. Because of self-averaging [11], it suffices to generate single series of random variables from eqs. (4),(5) instead of evaluating all the integrals in eq. (7). To obtain better numerical convergence for noisy shifts the first moment squared in eq. (6) was subtracted, while eqs. (4),(5) imply that the long-time average over the random variable Δ_n does not yield any bias. In refs. [8,9] it was shown that the unperturbed map, eq. (1), exhibits normal diffusion if $a > 2$, and the same was found by adding a bias b [12]. Correspondingly, for the types of perturbations defined above diffusion should always be normal if $(a - da) > 2$, as was confirmed in simulations. Hence, the central question

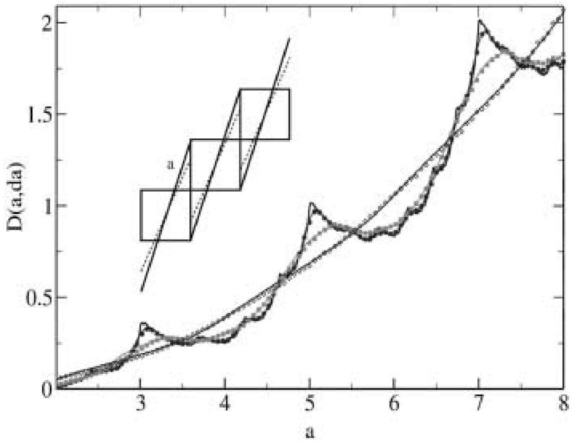

Fig. 1 – Diffusion coefficient $D(a, da)$ for the piecewise linear map shown in the figure. The slope a is perturbed by the uniform noise of maximum strength da of eq. (4). The bold black line depicts numerically exact results for the unperturbed diffusion coefficient at $da = 0$. Computer simulation results for $da \neq 0$ are marked with symbols, the corresponding lines are obtained from the approximation, eq. (8). The parameter values are: $da = 0.1$ (circles), $da = 0.4$ (squares), $da = 1.0$ (diamonds).

is what happens to the parameter-dependent diffusion coefficient $D(a, d)$ under variation of the two control parameters a and d in case of the above two types of noise.

For $da = 0$ it was shown that $D(a, 0)$ is a fractal function of the slope a as a control parameter [8, 9], see fig. 1. Included are results from computer simulations for uniformly distributed noisy slopes at different values of the perturbation strength da [13]. As expected, the irregular structure gradually disappears by increasing da. Qualitatively the same result is obtained by applying noisy shifts [14]. Figure 1 may be compared to the corresponding result for *quenched* slopes, fig. 1 in ref. [15]. Apart from numerical uncertainties, there are clear differences in the critical behavior close to the onset of diffusion. However, for small enough perturbation strength and large enough a the results look qualitatively similar indicating that, in this limit, quenched and annealed diffusion may be treated on the same footing.

This statement is corroborated by a trivial approximation for the perturbed diffusion coefficient, which we motivate starting from dichotomous noisy slopes. Naive reasoning suggests that, at arbitrary fixed parameters a and da, the *perturbed* diffusion coefficient $D(a, da)$ can be approximated by simply averaging over the *unperturbed* diffusion coefficient $D(a, 0)$ at respective values of the slopes $a-da$ and $a+da$ yielding $D_{\mathrm{app}}(a, da) = (D(a-da, 0)+D(a+da, 0))/2$. It is straightforward to extend this heuristic argument to any other type of uncorrelated noise yielding the generalized expression

$$D_{\mathrm{app}}(\boldsymbol{p}, \boldsymbol{d}) = \int \mathrm{d}(\boldsymbol{\Delta})\, \chi_d(\boldsymbol{\Delta}) D(\boldsymbol{p} + \boldsymbol{\Delta}, 0)\ . \tag{8}$$

Here $\boldsymbol{p}$ is a vector of control parameters such as $\boldsymbol{p} = \{a, b\}$ in case of the map above, $\boldsymbol{d}$ is the corresponding vector of perturbation strengths, and $\boldsymbol{\Delta}$ is the vector of perturbations such as $\boldsymbol{\Delta} = \{\Delta a, \Delta b\}$ for noisy shifts and slopes. Further generalizations of this equation, for example, to arbitrary moments as defined in eq. (7), are straightforward. Applying this formula to the case of quenched slopes discussed in ref. [15] reproduces the diffusion coefficient approximation eq. (6) therein. The corresponding approximations for uniform noisy slopes are depicted in fig. 1 as lines. They show that even for $da = 1$ the agreement between theory

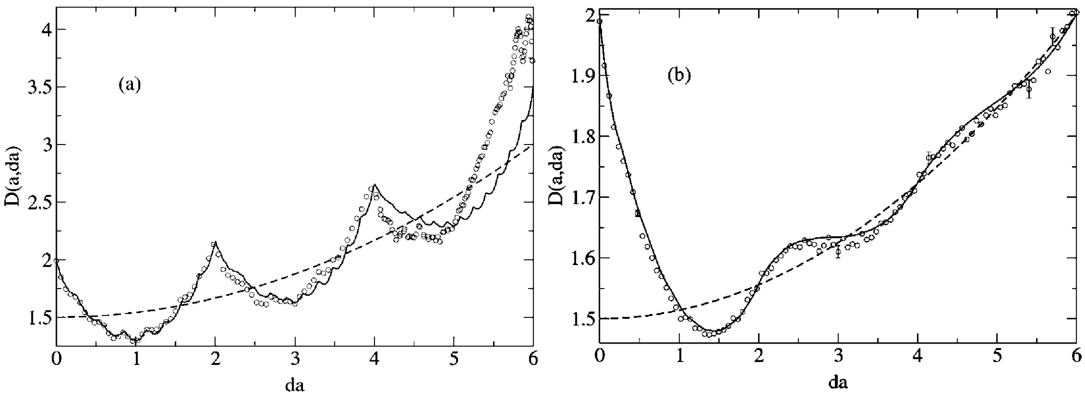

Fig. 2 – Diffusion coefficient $D(a, da)$ as a function of the perturbation strength da at slope $a = 7$ for noisy slopes distributed according to: (a) dichotomous noise, eq. (5), (b) uniform noise, eq. (4). The circles represent results from computer simulations, the bold lines are obtained from the approximation, eq. (8), the dashed lines represent the stochastic limit for the diffusion coefficient, eq. (13).

and simulations is excellent. This confirms that, in the limit described above, quenched and annealed disorder generating normal diffusion can indeed approximately be treated in the same way.

Let us now look at the diffusion coefficient for a given value of a as a function of da. Figure 1 shows that approximately at integer slopes the fractal diffusion coefficient $D(a, 0)$ exhibits local extrema. Since eq. (8) represents an average over the unperturbed solution in a local environment $[a - da, a + da]$, it predicts local suppression and enhancement of diffusion at odd and even integer slopes, respectively, under variation of the perturbation strength da. This has already been conjectured in ref. [9] and has been verified in ref. [15] for quenched slopes. We first check this hypothesis for noisy slopes around the local maximum of $D(a, 0)$ at $a = 7$ distributed according to eqs. (4),(5). Figures 2 (a), (b) depict again results obtained from computer simulations in comparison to eq. (8). As predicted, in both cases there is suppression of diffusion for small enough da. For dichotomous noise the perturbed diffusion coefficient increases on a coarse scale by exhibiting multiple, fractal-like suppression and enhancement on finer scales. For uniform perturbations there is a pronounced crossover from suppression to enhancement on a coarse scale, by again exhibiting oscillations on a fine scale. In both cases the agreement between simple theory and simulations is excellent for small enough da, whereas systematic deviations particularly in case of dichotomous noise are visible for larger da. Note that if $a - \Delta a_n < 2$, particles are getting trapped within a box at a respective time step n, and that for $a - \Delta a_n < 1$, the map is non-chaotic. In the first case simulations and simple reasoning suggest that the perturbed map still exhibits normal diffusion. However, as soon as $a - da < 1$, numerical results indicate that there is no normal diffusion anymore [14]. This appears to be due to the contracting behavior of the non-chaotic map resulting in localization of particles. The oscillatory behavior of the diffusion coefficient in fig. 2 (a) just below this transition point is not yet understood.

Employing eq. (8) we now analyze noisy shifts. The unperturbed two-parameter diffusion coefficient $D(a, b, 0)$ has been calculated numerically exactly for the map under consideration in ref. [12]. Results for the perturbed diffusion coefficient $D(a, db) \equiv D(a, 0, db)$ are presented in fig. 3 (a) for dichotomous noise and in (b) for uniform perturbations, both starting from $D(a, 0)$ at $a = 6$. In both cases the perturbed diffusion coefficient exhibits strong enhancement of diffusion for $da \to 0$ due to the fact that the unperturbed diffusion coefficient at $a = 6$

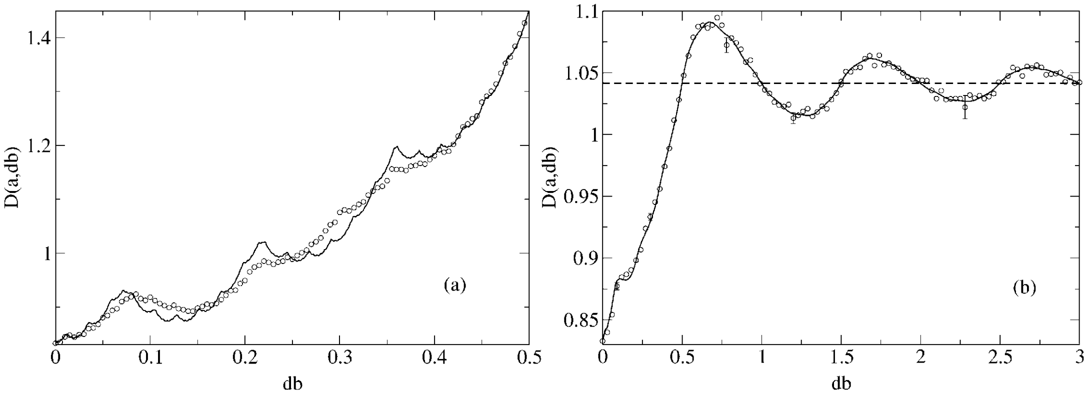

Fig. 3 – Diffusion coefficient $D(a, db)$ as a function of the perturbation strength db at slope $a = 6$ for noisy shifts distributed according to: (a) dichotomous noise, eq. (5), (b) uniform noise, eq. (4). The circles represent results from computer simulations, the bold lines are obtained from the approximation, eq. (8), the dashed line represents the stochastic limit for the diffusion coefficient, eq. (12).

is approximately identical with a local maximum in the (a, b) parameter plane [12]. For dichotmous perturbations it suffices to show results for $0 < db < 0.5$ only. Translation and reflection symmetry of the map imply that this function is mirrored in the interval from $0.5 < db < 1$, and that the full sequence in $0 < db < 1$ is periodically repeated for higher values of db. As in the corresponding case of noisy slopes, the perturbed diffusion coefficient increases on a coarse scale by exhibiting multiple fractal-like suppression and enhancement on a fine scale. In case of uniform perturbations there is a pronounced crossover to an approximately constant diffusion coefficient for larger db.

Before calculating the stochastic limit of the diffusion coefficient, we provide an analytical justification for the heuristic approximation, eq. (8). For the sake of simplicity, we demonstrate it only for noisy slopes, $\Delta_n \equiv \Delta a_n$. Noisy shifts as well as quenched disorder can be treated along the same lines [14]. Let us start from the definition of the diffusion coefficient, eq. (6), where $\langle x_n \rangle = 0$. Let Δa_n be uniformly distributed in $[-da, da]$, $\Delta a_0 \equiv \Delta a$. In case of $da \to 0$ all random variables are bounded by $\Delta a_n = \Delta a + \epsilon$, $-2da \le \epsilon \le 2da$. We now put this expression into the perturbed equation of motion, eqs. (2),(3), as contained in eq. (6), which we write as $x_{n+1,a+\Delta a_n} = M_{a+\Delta a_n}(x_n)$. As a first step, we now take the limit $\epsilon \to 0$ resulting in the expression for the mean square displacement

$$\begin{aligned}\langle x_n^2 \rangle &= \int \mathrm{d}x \int \mathrm{d}(\Delta a)\mathrm{d}(\Delta a_1)\ldots\mathrm{d}(\Delta a_{n-1})\rho_0(x)\chi(\Delta a)\chi(\Delta a_1)\ldots\chi(\Delta a_{n-1})x^2_{n,a+\Delta a_{n-1}} \\ &= \int \mathrm{d}x \int \mathrm{d}(\Delta a)\rho_0(x)\chi(\Delta a)x^2_{n,a+\Delta a}\ (\epsilon \to 0)\ . \qquad (9)\end{aligned}$$

As a second step, we exchange the time limit contained in eq. (6) with the integration over $\mathrm{d}(\Delta a)$ yielding

$$\begin{aligned}D_{\mathrm{app}}(a, da) &= \lim_{n\to\infty} \frac{\langle x_n^2 \rangle}{2n} \\ &= \int \mathrm{d}(\Delta a)\chi_{da}(\Delta a) \lim_{n\to\infty} \int \mathrm{d}x \rho_0(x) \frac{x^2_{n,a+\Delta a}}{2n} \\ &= \int \mathrm{d}(\Delta a)\chi_{da}(\Delta a) D(a + \Delta a, 0)\ , \qquad (10)\end{aligned}$$

where we have used that the unperturbed diffusion coefficient was defined as

$$D(a,0) = \lim_{n\to\infty} \int \mathrm{d}x \rho_0(x) x_{n,a}^2 \ . \tag{11}$$

We have thus verified our previous approximation, eq. (8), for noisy slopes in the limit of small perturbation strength. A similar derivation can be carried out for noisy shifts arriving again at eq. (8) in case of very small perturbation strength. For quenched shifts it is known that a normal diffusion coefficient does not exist [16], thus any approximation by eq. (8) must fail. Indeed, it turns out that in this case taking the limit $\epsilon \to 0$ fundamentally changes the properties of the dynamical system and is thus no valid operation [14].

Finally, we calculate the parameter-dependent stochastic diffusion coefficient related to the map with noisy slopes. Starting from the definition, eq. (11), the complete loss of memory in the unperturbed map is modeled by [9, 17] i) replacing the distance x_n a particle travels by n times the distance a particle travels at any single time step, $n\Delta x = n(M_a(x) - x)$, and ii) neglecting any memory effects in the probability density on the unit interval by assuming $\rho_0(x) = 1$. Then eq. (11) yields

$$D_{rw}(a) = \frac{(a-1)^2}{24} \ . \tag{12}$$

As was shown in refs. [9, 17], this equation correctly describes the asymptotic parameter dependence of the deterministic diffusion coefficient for $a \to \infty$ thus explaining the increase of $D(a,0)$ in fig. 1 on a coarse scale. On this basis, the corresponding result for noisy slopes is easily calculated by using eq. (12) as the functional form for $D(a+\Delta a, 0)$ in the approximation eq. (10) reading

$$D_{rw}(a, da) = D_{rw}(a,0) + \Delta a^2/c \ , \tag{13}$$

where $c = 24$ for dichotomous noise, eq. (5), and $c = 72$ for uniform noise, eq. (4). Equation (13) thus confirms the common sense expectation that noise should typically enhance diffusion and represents the *stochastic limit* of the diffusion coefficient. This equation is depicted in fig. 2 (a), (b) in form of dashed lines. In case of dichotomous noise the correlations are obviously large enough such that, even for large perturbation strength da, there is no transition to the stochastic limit, whereas in case of uniform noisy slopes the diffusion coefficient approaches the stochastic solution asymptotically in da, thus verifying the existence of a transition from deterministic to stochastic diffusion. That such a distinct transition behavior exists in these models was already conjectured in ref. [9]. Analogous calculations for noisy shifts yield eq. (12) for all values of db reflecting the fact that, for large enough a, the stochastic diffusion coefficient should not depend on the bias. This result is shown in fig. 3 (b) and again confirms an asymptotic approach of the diffusion coefficient to the stochastic limit under variation of db. Based on the known result of the existence of a fractal diffusion coefficient for the unperturbed $D(a, b, 0)$, we conjecture that the typical transition scenario in this type of systems consists of (multiple) suppression and enhancement of diffusion. We finally note that eqs. (12),(13) are closely related to the approximation outlined in ref. [1], and to the simple heuristic argument given by Reimann [2] by which he explains the suppression of deterministic diffusion by noise in the climbing sine map near a crisis; more details will be discussed elsewhere [14].

We conclude with a few remarks: 1) It would be interesting to study the problem of noisy maps with non-zero average bias along the same lines. Reference [12] shows that the unperturbed map does not exhibit a linear response for $b \to 0$, thus we conjecture that adding noise generates a transition to Ohm's law. 2) Whether there is a close connection between the mechanism of noise suppression outlined in ref. [18] and the phenomena discussed here, is currently an open question. 3) Our approach could be applied as well to more complex models such as the standard map, particle billiards, or inertia ratchets, where irregular transport coefficients have already been reported and studied under the impact of noise [19]. 4) In physical experiments on classical diffusive transport in low-dimensional periodic arrays of scatterers such as antidot lattices [20], a control parameter like the temperature may mimic the strength of random perturbations. Our results suggest that measuring respective parameter-dependent transport coefficients in such systems may reveal analogous transition scenarios as the ones described in our paper.

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