Supporting Information
Weak Galilean invariance as a selection principle for coarse-grained diffusive models

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The SI is organized as follows. In section 1, we derive the transformation rule between different inertial frames $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, moving at relative velocity v_0, of position and velocity processes satisfying the generalized Langevin equation (LE) (Eq. **4**). This is obtained by only employing the transformation rule of their stochastic equations of motion, that we derive analytically from the Kac-Zwanzig model (main text). This calculation thus provides a derivation of the transformation rule for their joint statistics Eq. **10**. Section 2 contains detailed derivations of the Fokker-Planck (FP) type equations in both frames $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, that are shown in Table S1, for several stochastic processes generating both normal and anomalous diffusion. In particular, we discuss overdamped Gaussian processes, the generalized LE, and the Lévy walk. This discussion highlights that weak GI is indeed satisfied by all such processes. In section 3, we derive analytically the propagator of the incorrect FP equation of a continuous-time random walk (CTRW) in the comoving frame $\widetilde{\mathcal{S}}$, originally proposed in refs. (1–3), which is numerically plotted in Fig. 2A. In section 4, we derive the characteristic functional of the noise $\overline{\xi}$, which is defined as the time derivative of a subordinated Brownian motion. We then use this result to verify that the FP equation of a process X, whose dynamics is described by the LE $\dot{X} = -v_0 + \overline{\xi}$, is the non local advection-diffusion Eq. **13**. In section 5, we provide an alternative derivation that employs the formulation of a CTRW in terms of subordinated processes. This discussion elucidates the effect of the spatio-temporal coupling imposed by weak GI on the subordinated LEs. In section 6, we show that $\overline{\xi}$ can be used to describe more general processes, including superdiffusive ones, that do not possess a formulation in terms of subordination. We then give a proof that their FP equation is still Eq. **13**. Section 7 contains a technical note about the Fox H-function and the three parameter Mittag Leffler function, whose properties are used throughout the main text and SI. Below we denote with X,V position and velocity processes for general dynamics, except for the CTRW whose position is called Y.

1. Solution of the generalized Langevin equations in $\mathcal{S}$ and $\widetilde{\mathcal{S}}$

Let us consider the generalised LE in the laboratory frame $\mathcal{S}$:

$$\dot{X}(t) = V(t), \qquad M\dot{V}(t) = -\int_0^t \Omega(t-t')V(t')\,\mathrm{d}t' + \xi(t). \qquad \text{[S1]}$$

The tracer trajectory $(X(t), V(t))$, with initial condition (X_0, V_0) at time $t = 0$, can be obtained exactly by Laplace transforming Eq. **S1**. For the position, this yields $\lambda X(\lambda) - X_0 = V(\lambda)$, while for the velocity

$$V(\lambda) = \frac{MV_0}{M\lambda + \Omega(\lambda)} + \frac{\xi(\lambda)}{M\lambda + \Omega(\lambda)}. \qquad \text{[S2]}$$

Transforming back these equations in time space, we obtain $X(t) = X_0 + \int_0^t V(t')\,\mathrm{d}t'$ and

$$V(t) = MV_0 w(t) + \int_0^t w(t-t')\xi(t')\,\mathrm{d}t', \qquad \text{[S3]}$$

where the function w is defined in Laplace transform by

$$w(\lambda) = [M\lambda + \Omega(\lambda)]^{-1}. \qquad \text{[S4]}$$

We then consider the corresponding dynamics in the comoving frame $\widetilde{\mathcal{S}}$. These are described by

$$\dot{\widetilde{X}}(t) = \widetilde{V}(t), \qquad M\dot{\widetilde{V}}(t) = -\int_0^t \Omega(t-t')[\widetilde{V}(t') + v_0]\,\mathrm{d}t' + \xi(t). \qquad \text{[S5]}$$

As before, we can derive the exact trajectory $(\widetilde{X}(t), \widetilde{V}(t))$ by taking the Laplace transform of Eqs. **S5**. This yields for the position $\lambda\widetilde{X}(\lambda) - \widetilde{X}_0 = \widetilde{V}(\lambda)$ and for the velocity

$$\widetilde{V}(\lambda) = \frac{M\widetilde{V}_0}{M\lambda + \Omega(\lambda)} - \frac{v_0\Omega(\lambda)}{\lambda[M\lambda + \Omega(\lambda)]} + \frac{\xi(\lambda)}{M\lambda + \Omega(\lambda)}, \qquad \text{[S6]}$$

where $\widetilde{X}_0, \widetilde{V}_0$ are the initial condition in the transformed frame. Employing the relations: $\widetilde{V}_0 = V_0 - v_0$ and $\widetilde{X}_0 = X_0$, that result from the Galilean transformation (GT) Eq. **1**, we find

$$\begin{aligned}\widetilde{V}(\lambda) &= \frac{M(V_0 - v_0)}{M\lambda + \Omega(\lambda)} - v_0 \frac{\Omega(\lambda)}{\lambda[M\lambda + \Omega(\lambda)]} + \frac{\xi(\lambda)}{M\lambda + \Omega(\lambda)} \\ &= \frac{MV_0}{M\lambda + \Omega(\lambda)} + \frac{\xi(\lambda)}{M\lambda + \Omega(\lambda)} - \frac{v_0}{\lambda} \\ &= V(\lambda) - \frac{v_0}{\lambda}.\end{aligned} \quad \text{[S7]}$$

Substituting this equation into that of the position, we can write

$$\lambda\widetilde{X}(\lambda) = X_0 + V(\lambda) - \frac{v_0}{\lambda} = \lambda X(\lambda) - \frac{v_0}{\lambda}. \quad \text{[S8]}$$

Taking their inverse Laplace transforms yields: $\widetilde{V}(t) = V(t) - v_0$ and $\widetilde{X}(t) = X(t) - v_0 t$. These transformation rules for X, V directly provide Eq. **10**.

2. Analysis of weak Galilean invariance for several stochastic coarse-grained models

In this section, we study several different stochastic models (3–13), that are widely used in the literature to model both normal and anomalous diffusion, in terms of weak Galilean invariance (GI). An overview is given in Table S1.

A. Overdamped Gaussian processes: fractional and scaled Brownian motion. General overdamped Gaussian processes are described in the laboratory frame $\mathcal{S}$ by the LE

$$\dot{X}(t) = \xi(t), \quad \text{[S9]}$$

where $\xi(t)$ is a Gaussian coloured noise with $\langle \xi(t) \rangle = 0$ and two-point correlation function $\langle \xi(t)\xi(t') \rangle = C(t, t')$. The time evolution of its position distribution $P(x, t) = \langle \delta(x - X(t)) \rangle$ is given by

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \langle \xi(t)\delta(x - X(t)) \rangle . \quad \text{[S10]}$$

To get a closed equation for P, one needs to compute the averaged quantity in its right-hand side (rhs). For Gaussian noise, one employs Novikov's theorem (14, 15), that yields

$$\begin{aligned}\langle \xi(t)\delta(x - X(t)) \rangle &= \int_0^t C(t, t') \left\langle \frac{\delta[\delta(x - X(t))]}{\delta\xi(t')} \right\rangle \mathrm{d}t' \\ &= -\frac{\partial}{\partial x} \int_0^t C(t, t') \left\langle \delta(x - X(t)) \frac{\delta X(t)}{\delta\xi(t')} \right\rangle \mathrm{d}t' = -\frac{\partial}{\partial x} D(t)P(x, t),\end{aligned} \quad \text{[S11]}$$

where $\delta X(t)/\delta\xi(t') = \Theta(t - t')$ and $D(t) = \int_0^t C(t, t')\,\mathrm{d}t'$. Substituting it in Eq. **S10**, we obtain

$$\frac{\partial}{\partial t} P(x, t) = \frac{\partial^2}{\partial x^2} D(t)P(x, t). \quad \text{[S12]}$$

The previous argument holds for both stationary noises, whose correlation function depends only on the time difference, i.e., $C(t, t') = C(|t - t'|)$, and non-stationary ones. In the former case, an important example is the fractional Brownian motion; in the latter case, the scaled Brownian motion (4–6). These processes are defined by setting the two-point correlation function equal to $C(|t - t'|) = \beta(\beta - 1)|t - t'|^{\beta-2}$ and $C(t, t') = \beta t^{\beta-1}\delta(t - t')$ respectively with $0 < \beta < 2$ (6), that yield the same diffusion coefficient $D(t) = \beta t^{\beta-1}$. Eq. **S12** is easily solved by the Gaussian $P(x, t) = e^{-\frac{x^2}{4\Sigma(t)}} / \sqrt{4\pi\Sigma(t)}$, where $\Sigma(t) = \int_0^t D(t')\,\mathrm{d}t'$. Applying the GT Eq. **1**, we obtain: $\widetilde{P}(x, t) = e^{-\frac{(x+v_0 t)^2}{4\Sigma(t)}} / \sqrt{4\pi\Sigma(t)}$. This is easily shown to satisfy the FP equation:

$$\frac{\partial}{\partial t} \widetilde{P}(x, t) = \left[\frac{\partial}{\partial x} v_0 + \frac{\partial^2}{\partial x^2} D(t) \right] \widetilde{P}(x, t), \quad \text{[S13]}$$

that corresponds to the LE

$$\dot{\widetilde{X}}(t) = -v_0 + \xi(t). \quad \text{[S14]}$$

Therefore, the description of overdamped Gaussian processes satisfies properties $i{-}iii$ (main text), i.e., it exhibits weak GI.

B. Generalised Langevin equation. We write the generalised Langevin Eq. **4** as (we set $M = 1$ without loss of generality)

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -\int_0^t \Omega(t-t')V(t')\,\mathrm{d}t' + \xi(t), \tag{S15}$$

where Ω is a prescribed drag coefficient and the coloured Gaussian noise ξ has the two point correlation function

$$\langle \xi(t)\xi(t')\rangle = C(|t-t'|) = \sigma\Omega(|t-t'|), \tag{S16}$$

with $\sigma = k_B T$ (T is the temperature of the bath at equilibrium). Thus, it satisfies the fluctuation-dissipation relation (16). Relevant examples are (a) underdamped normal diffusion, for which $\Omega(t) = \gamma\delta(t)$ ($\gamma > 0$), and (b) fractional LE (5–7), for which $\Omega(t) = \gamma_\alpha t^{-\alpha}/\Gamma(1-\alpha)$, $0<\alpha<1$, $\gamma_\alpha > 0$. We call $X_0 = X(0), V_0 = V(0)$ the initial conditions. Eq. **S15** has been widely discussed in the main text in terms of weak GI. In particular, the validity of the properties i, iii has been discussed. Here, we show that also property ii holds. First, we derive the Klein-Kramers equation for its joint position-velocity probability density function (PDF) in the laboratory frame $\mathcal{S}$ $P(x, v, t) = \langle\delta(x - X(t))\delta(v - V(t))\rangle$. Due to the Gaussian nature of ξ, and using the exact solution of the dynamics Eq. **S3**, the joint characteristic function is (17, 18)

$$P(k,p,t) = \exp\left\{ i\langle X(t)\rangle k + i\langle V(t)\rangle p - \frac{1}{2}[\sigma_{xx}^2(t)k^2 + 2\sigma_{xv}^2(t)kp + \sigma_{vv}^2(t)p^2]\right\}, \tag{S17}$$

where $\langle X(t)\rangle = V_0\overline{w}(t) + X_0$, $\langle V(t)\rangle = V_0 w(t)$ and we defined the auxiliary function $\overline{w}(t) = \int_0^t w(t')\,\mathrm{d}t'$ and

$$\sigma_{xx}^2(t) = \sigma\left[2\int_0^t \overline{w}(t')\,\mathrm{d}t' - \overline{w}^2(t)\right], \qquad \sigma_{vv}^2(t) = \sigma[1 - w^2(t)], \qquad \sigma_{xv}^2(t) = \sigma\overline{w}(t)[1 - w(t)]. \tag{S18}$$

We take the following partial derivatives in t, p (to ease notation we drop any explicit dependence of P on its variables):

$$\frac{1}{P}\frac{\partial}{\partial t}P = iV_0[w(t)k + \dot{w}(t)p] - \frac{1}{2}\left[2\sigma_{xv}^2(t)k^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{xv}^2(t)kp + \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{vv}^2(t)p^2\right], \tag{S19a}$$

$$\frac{1}{P}\frac{\partial}{\partial p}P = -\sigma_{vv}^2(t)p - \sigma_{xv}^2(t)k + iV_0 w(t), \tag{S19b}$$

where we further used the relation $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{xx}^2 = 2\sigma_{xv}^2$. Eliminating V_0, we derive the following equation:

$$\frac{1}{P}\frac{\partial}{\partial t}P = [k - \Gamma(t)p]\frac{1}{P}\frac{\partial}{\partial p}P - D_{vv}(t)p^2 - D_{xv}(t)kp, \tag{S20}$$

where the drag and diffusion coefficients Γ, D_{vv}, D_{xv} are defined as

$$\Gamma(t) = -\frac{\dot{w}(t)}{w(t)}, \qquad D_{vv}(t) = \sigma\Gamma(t), \qquad D_{xv}(t) = \sigma[-1 + w(t) + \Gamma(t)\overline{w}(t)]. \tag{S21}$$

Taking its inverse Fourier transform yields:

$$\frac{\partial}{\partial t}P(x,v,t) = \left[-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}\Gamma(t)v + \frac{\partial^2}{\partial v^2}\sigma\Gamma(t) + \frac{\partial^2}{\partial x\partial v}D_{xv}(t)\right]P(x,v,t). \tag{S22}$$

For example (a), we find $\Gamma(t) = \gamma$, $D_{xv}(t) = 0$, thus yielding the ordinary Klein-Kramers equation:

$$\frac{\partial}{\partial t}P(x,v,t) = -\frac{\partial}{\partial x}vP(x,v,t) + \gamma\frac{\partial}{\partial v}\left[v + \sigma\frac{\partial}{\partial v}\right]P(x,v,t). \tag{S23}$$

Let us now consider the generalized LE in the comoving frame $\widetilde{\mathcal{S}}$, i.e., Eq. **9**, which we write as

$$\dot{\widetilde{X}}(t) = \widetilde{V}(t), \qquad \dot{\widetilde{V}}(t) = -\int_0^t \Omega(t-t')[\widetilde{V}(t') + v_0]\,\mathrm{d}t' + \xi(t). \tag{S24}$$

We now apply the previous technique to compute its Klein-Kramers equation. Being related by the GT Eq. **1**, only their first moment changes to $\langle\widetilde{X}(t)\rangle = \langle X(t)\rangle - v_0 t$, $\langle\widetilde{V}(t)\rangle = \langle V(t)\rangle - v_0$. Therefore, the joint characteristic function in $\widetilde{\mathcal{S}}$ is

$$\widetilde{P}(k,p,t) = \exp\left\{ i\langle X(t)\rangle k + i\langle V(t)\rangle p - ikv_0 t - ipv_0 - \frac{1}{2}[\sigma_{xx}^2(t)k^2 + 2\sigma_{xv}^2(t)kp + \sigma_{vv}^2(t)p^2]\right\}, \tag{S25}$$

such that Eqs. **S19a**, **S19b** changes to

$$\frac{1}{\widetilde{P}}\frac{\partial}{\partial t}\widetilde{P} = iV_0[w(t)k + \dot{w}(t)p] - ikv_0 - \frac{1}{2}[2\sigma_{xv}^2(t)k^2 + \dot{\sigma}_{xv}^2(t)kp + \dot{\sigma}_{vv}^2(t)p^2], \tag{S26a}$$

$$\frac{1}{\widetilde{P}}\frac{\partial}{\partial p}\widetilde{P} = -\sigma_{vv}^2(t)p - \sigma_{xv}^2(t)k - iv_0 + iV_0 w(t). \tag{S26b}$$

Elimination of the parameter V_0 yields

$$\frac{1}{\widetilde{P}}\frac{\partial}{\partial t}\widetilde{P} = [k-\Gamma(t)p]\frac{1}{\widetilde{P}}\frac{\partial}{\partial p}\widetilde{P} - ip\Gamma(t)v_0 - D_{vv}(t)p^2 - D_{xv}(t)kp, \tag{S27}$$

whose Fourier inverse is given by

$$\frac{\partial}{\partial t}\widetilde{P}(x,v,t) = \left[-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}\Gamma(t)(v+v_0) + \frac{\partial^2}{\partial v^2}\sigma\Gamma(t) + \frac{\partial^2}{\partial x \partial v}D_{xv}(t)\right]\widetilde{P}(x,v,t). \tag{S28}$$

The special case (a) follows straightforwardly, i.e.,

$$\frac{\partial}{\partial t}\widetilde{P}(x,v,t) = -\frac{\partial}{\partial x}v\widetilde{P}(x,v,t) + \gamma\frac{\partial}{\partial v}\left[(v+v_0)+\sigma\frac{\partial}{\partial v}\right]\widetilde{P}(x,v,t). \tag{S29}$$

Clearly, Eqs. **S22**, **S28** satisfy property *ii*.

C. Lévy walk. The Lévy walk model (8–11) is a special class of the spatiotemporally coupled continuous-time random walk (CTRW) (3, 12, 13). This is typically employed to model position mean-square displacement superdiffusive behaviour, and thus has been widely used to describe transport processes in, e.g., biological systems (10). Here, we study only the 1-dim case. In the laboratory frame $\mathcal{S}$ a Lévy walk is mathematically obtained as follows: A particle moves with constant speed $u_\pm = \pm u$, where for later convenience we denote by $u_\pm$ its forward/backward velocity, for a random running time τ sampled by a prescribed distribution ψ, after which it randomly changes its direction of motion. The position distribution of a process X performing this type of dynamics is described in terms of master equations, similar to those of the CTRW (3), but with a coupled transition probability $\phi(y,\tau) = \frac{1}{2}[\delta(y-u_+\tau)+\delta(y-u_-\tau)]\psi(\tau) = \frac{1}{2}\delta(|y|-u\tau)\psi(\tau)$, that relates the walker's position y to the running time τ. The GT to the comoving frame $\widetilde{\mathcal{S}}$ expressed by Eq. **1** only changes the walker's velocity as $u_\pm = \pm u - v_0$. This is shown easily by transforming ϕ, which yields $\widetilde{\phi}(\widetilde{y},\tau) = \frac{1}{2}\delta(|\widetilde{y}+v_0\tau|-u\tau)\psi(\tau) = \frac{1}{2}[\delta(\widetilde{y}+(v_0-u)\tau)+\delta(\widetilde{y}+(v_0+u)\tau)]\psi(\tau)$. Thus, the microscopic dynamics of Lévy walks is Galilean invariant, and we expect its position distribution P_u to correspondingly satisfy weak GI. First, we show that property *iii* is satisfied. Remarkably, its position PDF can be obtained exactly in the laboratory frame (10). In fact, denoting $P_0(x)$ the initial distribution and $\Psi(t) = 1 - \int_0^t \psi(t')\,\mathrm{d}t'$ the probability of sampling a running time larger than t, P_u is given by

$$P_u(k,\lambda) = \frac{[\Psi(\lambda-iuk)+\Psi(\lambda+iuk)]P_0(k)}{2-[\psi(\lambda+iuk)+\psi(\lambda-iuk)]}. \tag{S30}$$

Identifying in the previous eq. left/right velocities $u_\pm$ and substituting for those in the comoving frame $\widetilde{\mathcal{S}}$, we obtain the PDF

$$\begin{aligned}\widetilde{P}_u(k,\lambda) &= \frac{[\Psi(\lambda-iu_+k)+\Psi(\lambda-iu_-k)]P_0(k)}{2-[\psi(\lambda-iu_+k)+\psi(\lambda-iu_-k)]}\\ &= \frac{[\Psi(\lambda+iv_0k-iuk)+\Psi(\lambda+iv_0k+iuk)]P_0(k)}{2-[\psi(\lambda+iv_0k-iuk)+\psi(\lambda+iv_0k+iuk)]} = P(k,\lambda+iv_0k),\end{aligned} \tag{S31}$$

highlighting that the property Eq. **11** holds for Lévy walks (P, $\widetilde{P}$ are related by the Laplace variable change $\lambda \to \lambda + iv_0k$).

Secondly, we show that property *ii* also holds. A FP type equation has recently been proposed for Lévy walks, that has the form in the laboratory frame $\mathcal{S}$ (11)

$$\left[\frac{\partial^2}{\partial t^2} - u^2\frac{\partial^2}{\partial x^2}\right]P_u(x,t) = -\frac{1}{2}\left[\frac{\partial}{\partial t}-u\frac{\partial}{\partial x}\right]\int_0^t K(t')P_u(x-ut',t-t')\,\mathrm{d}t' - \frac{1}{2}\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right]\int_0^t K(t')P_u(x+ut',t-t')\,\mathrm{d}t', \tag{S32}$$

with the memory kernel being defined as $K(\lambda) = \psi(\lambda)/\Psi(\lambda)$. It is easy to verify that Eq. **S32** yields Eq. **S30** in Fourier-Laplace space. This equation can be conveniently cast into the form

$$\left[\frac{\partial^2}{\partial t^2} - u^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\mathcal{D}_t^{(-u,u)} + \frac{1}{2}\mathcal{D}_t^{(u,-u)}\right]P_u(x,t) = 0, \tag{S33}$$

where $\mathcal{D}_t^{(v_1,v_2)}$ is the fractional operator

$$\mathcal{D}_t^{(v_1,v_2)}P_u(x,t) = \left[\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x}\right]\int_0^t K(t-t')P_u(x-v_2(t-t'),t')\mathrm{d}t', \tag{S34}$$

with Fourier-Laplace representation $\mathcal{D}_t^{(v_1,v_2)}P_u(x,t) \to (\lambda-ikv_1)K(\lambda-ikv_2)P_u(k,\lambda)$. For $v_1 = v_2 = -v_0$, $\mathcal{D}_t^{(v_1,v_2)}$ recovers the fractional substantial derivative Eq. **14**. Applying the GT Eq. **1** to $\mathcal{D}_t^{(v_1,v_2)}$ in Laplace space yields $(\lambda - ik(-v_0+v_1))K(\lambda - ik(-v_0+v_2))P_u(k,\lambda+ikv_0) \to \mathcal{D}_t^{(-v_0+v_1,-v_0+v_2)}\widetilde{P}_u(x,t)$. Therefore, we obtain the FP equation in $\widetilde{\mathcal{S}}$

$$\left[\frac{\partial^2}{\partial t^2} - 2v_0\frac{\partial}{\partial t}\frac{\partial}{\partial x} + (v_0^2-u^2)\frac{\partial^2}{\partial x^2} + \frac{1}{2}\mathcal{D}_t^{(-v_0-u,-v_0+u)} + \frac{1}{2}\mathcal{D}_t^{(-v_0+u,-v_0-u)}\right]\widetilde{P}_u(x,t) = 0, \tag{S35}$$

that can be written more neatly as

$$\left[\left(\frac{\partial}{\partial t}+u_+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+u_-\frac{\partial}{\partial x}\right)+\frac{1}{2}\mathcal{D}_t^{(u_-,u_+)}+\frac{1}{2}\mathcal{D}_t^{(u_+,u_-)}\right]\widetilde{P}_u(x,t)=0, \quad \text{[S36]}$$

which is the correct evolution equation for the Lévy walk dynamics in the comoving frame $\widetilde{\mathcal{S}}$.

3. Derivation of the propagator plotted in Fig. 2A

We consider the fractional equation

$$\frac{\partial}{\partial t}\widetilde{P}(x,t)=v_0\frac{\partial}{\partial x}\widetilde{P}(x,t)+\sigma\frac{\partial^2}{\partial x^2}\mathbb{D}_t\widetilde{P}(x,t), \quad \text{[S37]}$$

where $\mathbb{D}_t$ is the Riemann-Liouville operator with Fourier-Laplace representation $\mathbb{D}_t\widetilde{P}(x,t)\to\lambda^{1-\alpha}\widetilde{P}(k,\lambda)$ $(0<\alpha<1)$, that is plotted in Fig. 2A. Without loss of generality, we assume null initial condition. First, we solve Eq. **S37** in Fourier-Laplace space:

$$\widetilde{P}(k,\lambda)=\frac{1}{\lambda^{\alpha'}+b(k)\lambda^{\beta}+c(k)}, \quad \text{[S38]}$$

with the auxiliary parameters $\alpha'=1$, $\beta=1-\alpha$, $b(k)=\sigma k^2$ and $c(k)=ikv_0$. Note that $\alpha'>\beta$, $\forall\alpha\in(0,1)$. We then expand in series as (19)

$$\begin{aligned}\widetilde{P}(k,\lambda)&=\frac{1}{c(k)}\frac{1}{1+\frac{\lambda^{\alpha'}+b(k)\lambda^{\beta}}{c(k)}}=\frac{1}{c(k)}\frac{\lambda^{-\beta}c(k)}{\lambda^{\alpha'-\beta}+b(k)}\frac{1}{1+\frac{\lambda^{-\beta}c(k)}{\lambda^{\alpha'-\beta}+b(k)}}\\&=\frac{\lambda^{-\beta}}{\lambda^{\alpha'-\beta}+b(k)}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{-\beta\,n}[c(k)]^n}{[\lambda^{\alpha'-\beta}+b(k)]^n}=\sum_{n=0}^{\infty}[-c(k)]^n\frac{\lambda^{-\beta-\beta\,n}}{[\lambda^{\alpha'-\beta}+b(k)]^{n+1}}.\end{aligned} \quad \text{[S39]}$$

We can now make a term by term Laplace inverse transform of Eq. **S39** by recalling the formula for the Laplace transform of the three-parameter Mittag-Leffler function given in Eq. **S104**. Thus, $\widetilde{P}(k,t)$ is given as

$$\widetilde{P}(k,t)=\sum_{n=0}^{\infty}[-c(k)]^n t^n E_{\alpha,1+n}^{1+n}\left(-t^{\alpha}b(k)\right)=\sum_{n=0}^{\infty}(-iv_0t)^n k^n E_{\alpha,1+n}^{1+n}\left(-\sigma t^{\alpha}k^2\right). \quad \text{[S40]}$$

We now need to make a term by term inverse Fourier transform of Eq. **S40**. To this aim, we first rewrite it in terms of Fox H-functions by using the corresponding property given in Eq. **S105**. In our case, we obtain:

$$E_{\alpha,1+n}^{1+n}\left(-\sigma t^{\alpha}k^2\right)=\frac{1}{\Gamma(1+n)}H_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]. \quad \text{[S41]}$$

Using this formula, the Fourier inverse transform of $k^nE_{\alpha,1+n}^{1+n}\left(-\sigma t^{\alpha}k^2\right)$ is expressed by cosine and sine transforms of Fox H-functions, i.e., it is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(kx)k^nH_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\mathrm{d}k-\frac{i}{2\pi}\int_{-\infty}^{\infty}\sin(kx)k^nH_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\mathrm{d}k. \quad \text{[S42]}$$

Let us first assume $x>0$. We remark that (a) the first/second integral in Eq. **S42** is not null only for even/odd indices, i.e., for $n=2\nu/1+2\nu$, $\forall\nu\in\mathbb{N}_0$ respectively, due to the parity of the Fox H-function, and that (b) they are equal to twice the corresponding integral on the semi-half positive line, once not null. Thus, we can use the property of the H-function given in Eqs. **S100** to compute these integrals:

$$\int_0^{\infty}\cos(kx)k^{2\nu}H_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-2\nu,1)\\(0,1),(-2\nu,\alpha)\end{matrix}\right]\mathrm{d}k=\frac{\sqrt{\pi}2^{2\nu}}{|x|^{1+2\nu}}H_{3,2}^{1,2}\left[\frac{4\sigma t^{\alpha}}{x^2}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\nu,1\right),(-2\nu,1),(-\nu,1)\\(0,1),(-2\nu,\alpha)\end{matrix}\right], \quad \text{[S43a]}$$

$$\int_0^{\infty}\sin(kx)k^{1+2\nu}H_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-1-2\nu,1)\\(0,1),(-1-2\nu,\alpha)\end{matrix}\right]\mathrm{d}k=\frac{\sqrt{\pi}2^{1+2\nu}}{|x|^{2+2\nu}}H_{3,2}^{1,2}\left[\frac{4\sigma t^{\alpha}}{x^2}\,\middle|\,\begin{matrix}\left(-\frac{1}{2}-\nu,1\right),(-1-2\nu,1),(-\nu,1)\\(0,1),(-1-2\nu,\alpha)\end{matrix}\right]. \quad \text{[S43b]}$$

By using the further property in Eq. **S98** we obtain:

$$H_{1,2}^{1,1}\left[\sigma t^{\alpha}k^2\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\xrightarrow[\text{inverse}]{\text{Fourier}}\frac{1}{\sqrt{\pi}}\begin{cases}\frac{2^{2\nu}}{|x|^{1+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(1+2\nu,\alpha)\\\left(\frac{1}{2}+\nu,1\right),(1+2\nu,1),(1+\nu,1)\end{matrix}\right] & n=2\nu\\\frac{(-i)2^{1+2\nu}}{|x|^{2+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(2+2\nu,\alpha)\\\left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right] & n=1+2\nu\end{cases} \quad \text{[S44]}$$

These results enable us to write Eq. S38 explicitly in (x,t)-space in terms of two infinite series of Fox H-functions (corresponding to the original series over odd and even indices):

$$\widetilde{P}(x,t)=\frac{1}{\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(v_0t)^{2\nu}}{(2\nu)!}\frac{2^{2\nu}}{|x|^{1+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(1+2\nu,\alpha)\\ \left(\frac{1}{2}+\nu,1\right),(1+2\nu,1),(1+\nu,1)\end{matrix}\right]$$
$$+\frac{1}{\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{(-1)^{1+\nu}(v_0t)^{1+2\nu}}{(1+2\nu)!}\frac{2^{1+2\nu}}{|x|^{2+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(2+2\nu,\alpha)\\ \left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right].\quad\text{[S45]}$$

Finally, we can exploit Eq. S99 to absorb the x-dependent multiplicative factors into the Fox H-functions. For each term separately, we obtain:

$$\frac{2^{2\nu}}{|x|^{1+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(1+2\nu,\alpha)\\ \left(\frac{1}{2}+\nu,1\right),(1+2\nu,1),(1+\nu,1)\end{matrix}\right]=\frac{(\sigma t^{\alpha})^{-\nu}}{\sqrt{4\sigma t^{\alpha}}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\nu,1\right),\left(1+2\nu-\alpha\left(\frac{1}{2}+\nu\right),\alpha\right)\\ (0,1),\left(\frac{1}{2}+\nu,1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]\quad\text{[S46a]}$$

$$\frac{2^{1+2\nu}}{|x|^{2+2\nu}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(2+2\nu,\alpha)\\ \left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right]=\frac{(\sigma t^{\alpha})^{-\nu-1/2}}{\sqrt{4\sigma t^{\alpha}}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}(-\nu,1),(2+2\nu-\alpha(1+\nu),\alpha)\\ \left(\frac{1}{2},1\right),(1+\nu,1),(0,1)\end{matrix}\right].\quad\text{[S46b]}$$

In the opposite case $x<0$ the second term in the rhs of Eq. S42 changes sign, so that the sum over odd indices in Eq. S45 has an opposite sign as well. If we take this into account and substitute Eqs. S46a, S46b into Eq. S45, we obtain that $\widetilde{P}(x,t)$ is defined as an infinite series of Fox H-functions ($\forall x\neq 0$), i.e.,

$$\widetilde{P}(x,t)=\frac{1}{\sqrt{4\pi\sigma t^{\alpha}}}\left[\Theta(x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{v_0t}{\sqrt{\sigma t^{\alpha}}}\right)^n\overline{H}_{2,3}^{2,1}\left(\frac{x^2}{4\sigma t^{\alpha}};\alpha,n\right)+\Theta(-x)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{v_0t}{\sqrt{\sigma t^{\alpha}}}\right)^n\overline{H}_{2,3}^{2,1}\left(\frac{x^2}{4\sigma t^{\alpha}};\alpha,n\right)\right],\quad\text{[S47]}$$

where the auxiliary function $\overline{H}_{2,3}^{2,1}(x;\alpha,n)$ is defined as

$$\overline{H}_{2,3}^{2,1}(x;\alpha,n)=\begin{cases}(-1)^{\nu}H_{2,3}^{2,1}\left[x\,\middle|\,\begin{matrix}\left(\frac{1-2\nu}{2},1\right),\left(\frac{(2-\alpha)(1+2\nu)}{2},\alpha\right)\\ (0,1),\left(\frac{1+2\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right] & n=2\nu\\ (-1)^{\nu}H_{2,3}^{2,1}\left[x\,\middle|\,\begin{matrix}(-\nu,1),((2-\alpha)(1+\nu),\alpha)\\ \left(\frac{1}{2},1\right),(1+\nu,1),(0,1)\end{matrix}\right] & n=1+2\nu\end{cases}\quad\text{[S48]}$$

The previous formula is valid for $x\neq 0$. Therefore, we need to specify the value of the PDF in this point. In this case, only the sum over even indices contributes to the PDF in Eq. S40 (the sine transform in Eq. S42 is, in fact, null) with coefficients defined by solving the correspondent integral of Fox function with Eqs. S92, S101:

$$\int_0^{\infty}k^{2\nu}H_{1,1}^{1,2}\left[\sqrt{\sigma t^{\alpha}}|k|\,\middle|\,\begin{matrix}\left(-2\nu,\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right),\left(-2\nu,\frac{\alpha}{2}\right)\end{matrix}\right]\mathrm{d}k=\left(\frac{1}{\sqrt{\sigma t^{\alpha}}}\right)^{1+2\nu}\frac{\left[\Gamma\left(\frac{1}{2}+\nu\right)\right]^2}{\Gamma\left((1+2\nu)\left(1-\frac{\alpha}{2}\right)\right)}.\quad\text{[S49]}$$

By substituting such coefficients into the series over even indices, we obtain:

$$\widetilde{P}(0,t)=\frac{1}{\sqrt{4\sigma t^{\alpha}}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(2\nu)!}{4^{\nu}(\nu!)^2}\frac{1}{\Gamma\left(\left(1-\frac{\alpha}{2}\right)(1+2\nu)\right)}\left(\frac{v_0^2t^{2-\alpha}}{4\sigma}\right)^{\nu}$$
$$=\frac{1}{\sqrt{4\sigma t^{\alpha}}}E_{2-\alpha,(2-\alpha)/2}^{1/2}\left(-\frac{v_0^2t^{2-\alpha}}{4\sigma}\right).\quad\text{[S50]}$$

Note that Eq. S47 is expressed as an expansion in the constant force field v_0, i.e., the velocity of the frame $\widetilde{\mathcal{S}}$. As a sanity check, we compute the zero-th order term, which must be equal to the solution in the frame $\mathcal{S}$, i.e., the position PDF of a force-free CTRW (3). This is confirmed below (note that the corresponding terms in the two series in Eq. S47 are equal):

$$\widetilde{P}(x,t)=\frac{1}{\sqrt{4\pi\sigma t^{\alpha}}}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}\left(\frac{1}{2},1\right),\left(\frac{2-\alpha}{2},\alpha\right)\\ (0,1),\left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]=\frac{1}{\sqrt{4\pi\sigma t^{\alpha}}}H_{1,2}^{2,0}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}\left(\frac{2-\alpha}{2},\alpha\right)\\ (0,1),\left(\frac{1}{2},1\right)\end{matrix}\right].\quad\text{[S51]}$$

Here, we used the property of the Fox H-function given in Eq. S95.

At last, we check the normalisation of the derived formula for $\widetilde{P}$, which is expected as $\widetilde{P}(k=0,\lambda)=1/\lambda$. Due to the different sign of the sums over odd indices, only those over even ones contribute to the normalization of the PDF. Due to the parity of the Fox H-function, the integral can be restricted to the semi-half positive line:

$$\int_{-\infty}^{\infty}\widetilde{P}(x,t)\,\mathrm{d}x=\frac{1}{\sqrt{\pi\sigma t^{\alpha}}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu)!}\left(\frac{v_0t}{\sqrt{\sigma t^{\alpha}}}\right)^{2\nu}\int_0^{\infty}H_{2,3}^{2,1}\left[\frac{x^2}{4\sigma t^{\alpha}}\,\middle|\,\begin{matrix}\left(\frac{1-2\nu}{2},1\right),\left((2-\alpha)\left(\frac{1+2\nu}{2}\right),\alpha\right)\\ (0,1),\left(\frac{1+2\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]\mathrm{d}x.\quad\text{[S52]}$$

We compute the integral of the Fox H-function by recalling Eqs. **S97**, **S101**:

$$\int_0^\infty H^{2,1}_{2,3}\left[\frac{x^2}{4\sigma t^\alpha}\,\middle|\,\begin{matrix}\left(\frac{1-2\nu}{2},1\right),\left((2-\alpha)\left(\frac{1+2\nu}{2}\right),\alpha\right)\\(0,1),\left(\frac{1+2\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]\mathrm{d}x=\sqrt{\sigma t^\alpha}\,\Theta(-1), \tag{S53}$$

where the function Θ is defined in Eq. **S92**, which in this specific case is

$$\Theta(s)=\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\nu+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma\left((2-\alpha)\left(\frac{1+2\nu}{2}\right)+\frac{\alpha}{2}s\right)}=\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\nu+\frac{1}{2}s\right)}{\Gamma\left((2-\alpha)\left(\frac{1+2\nu}{2}\right)+\frac{\alpha}{2}s\right)}\prod_{i=0}^{\nu-1}\left(\frac{1}{2}-\frac{s}{2}+i\right). \tag{S54}$$

For $s=-1$ all terms, except that for $\nu=0$, which is equal to $\sqrt{\pi}$, cancel out. Eq. **S53** is then equal to $\sqrt{\pi\sigma t^\alpha}$, i.e., the PDF is correctly normalised.

4. Derivation of the characteristic functional of the noise $\bar{\xi}$

The noise $\bar{\xi}$ can be formally defined as (20)

$$\bar{\xi}(t)=\int_0^\infty \xi(s)\delta(t-T(s))\,\mathrm{d}s, \tag{S55}$$

where ξ is a white Gaussian noise with $\langle\xi(t)\rangle=0$ and $\langle\xi(t_1)\xi(t_2)\rangle=2\sigma\delta(t_1-t_2)$, and T is a strictly increasing Lévy process (21). Within the subordination description of CTRWs (13, 22–24), they specify respectively the stochastic process of jump lengths and that of waiting times of the underlying random walk. We recall the definition of the inverse subordinator $S(t)=\inf_{s>0}\{s:T(s)>t\}$, such that $\int_0^t \mathrm{d}t'\bar{\xi}(t')=B(S(t))$, where B is an ordinary Brownian motion.

Its characteristic functional is defined for a general test function $u(r)$ as

$$G[u(r)]=\left\langle \exp\left(i\int_0^\infty u(r)\,\bar{\xi}(r)\,\mathrm{d}r\right)\right\rangle. \tag{S56}$$

Note that the brackets denote an average over the realisations of both the stochastic processes ξ and T specifying Eq. **S55**. By substituting this definition into Eq. **S56**, we obtain

$$\begin{aligned}G[u(s_1)]&=\left\langle \exp\left[i\int_0^\infty u(s_1)\left(\int_0^\infty \xi(s_2)\delta(s_1-T(s_2))\,\mathrm{d}s_2\right)\mathrm{d}s_1\right]\right\rangle\\&=\left\langle \exp\left[i\int_0^\infty \xi(s_2)\left(\int_0^\infty u(s_1)\delta(s_1-T(s_2))\,\mathrm{d}s_1\right)\mathrm{d}s_2\right]\right\rangle\\&=\left\langle \exp\left[i\int_0^\infty \xi(s_1)\,f(s_1)\,\mathrm{d}s_1\right]\right\rangle.\end{aligned} \tag{S57}$$

In the previous expression, we changed the order of integration and defined the auxiliary function

$$f(s)=\int_0^\infty u(s')\delta(s'-T(s))\,\mathrm{d}s', \tag{S58}$$

which depends only on the different realisations of the process T. For each of them, f is completely determined and it can be used as a test function in the characteristic functional of ξ. Thus, Eq. **S57** can be simplified if we compute the average over ξ first. For a Gaussian noise of correlation function $\langle\xi(s_1)\,\xi(s_2)\rangle=C(|s_2-s_1|)$, we obtain (25)

$$\left\langle \exp\left(i\int_0^\infty \xi(s)\,f(s)\,\mathrm{d}s\right)\right\rangle=\left\langle \exp\left(-\frac{1}{2}\int_0^\infty\int_0^\infty f(s_1)f(s_2)C(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2\right)\right\rangle. \tag{S59}$$

The remaining average in its rhs is only on the realizations of the Lévy process T. Substituting Eq. **S58** into Eq. **S59** yields

$$G[u(r)]=\left\langle \exp\left(-\int_0^\infty\int_0^\infty u(r_1)u(r_2)\Lambda(r_1,\,r_2;\,T)\,\mathrm{d}r_1\,\mathrm{d}r_2\right)\right\rangle, \tag{S60a}$$

$$\Lambda(r_1,\,r_2;\,T)=\frac{1}{2}\int_0^\infty\int_0^\infty \delta(r_1-T(s_1))\delta(r_2-T(s_2))C(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{S60b}$$

For ξ white noise with correlation function $C(s_2-s_1)=2\sigma\delta(s_2-s_1)$, Eq. **S60b** reduces to

$$\begin{aligned}\Lambda(r_1,\,r_2;\,T)&=\sigma\int_0^\infty\int_0^\infty \delta(r_1-T(s_1))\delta(r_2-T(s_2))\delta(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2\\&=\sigma\int_0^\infty \delta(r_1-T(s))\delta(r_2-T(s))\,\mathrm{d}s\\&=\sigma\delta(r_2-r_1)\int_0^\infty \delta(r_1-T(s))\,\mathrm{d}s.\end{aligned} \tag{S61}$$

Substituting this result into Eq. **S60a**, we obtain the characteristic functional, i.e.,

$$\begin{aligned} G[u(r)] &= \left\langle \exp\left[-\sigma\int_0^\infty\int_0^\infty\int_0^\infty u(r_1)u(r_2)\delta(r_2-r_1)\delta(r_1-T(s))\,\mathrm{d}s\,\mathrm{d}r_1\,\mathrm{d}r_2\right]\right\rangle \\ &= \left\langle \exp\left[-\sigma\int_0^\infty\int_0^\infty [u(r)]^2\delta(r-T(s))\,\mathrm{d}s\,\mathrm{d}r\right]\right\rangle \\ &= \left\langle \exp\left[-\sigma\int_0^\infty [u(T(s))]^2\,\mathrm{d}s\right]\right\rangle. \end{aligned} \qquad [S62]$$

As a sanity check, we calculate the PDF of the process Y, satisfying the LE $\dot{Y}(t)=\overline{\xi}(t)$. If we set $u(r)=k\Theta(t-r)$ and employ the relation $\Theta(t-T(s))=1-\Theta(s-S(t))$ (26), we find

$$P(k,t)=\left\langle \exp\left(-\sigma k^2\int_0^\infty \Theta(t-T(s))\,\mathrm{d}s\right)\right\rangle = \left\langle \exp\left(-\sigma k^2 S(t)\right)\right\rangle, \qquad [S63]$$

which is the correct position PDF of a free diffusive CTRW (13).

Similarly, we can use this technique to prove Eq. **13** and find its propagator. For simplicity, we set the initial condition $Y_0=0$. Recalling that the PDF of the process $\widetilde{Y}$ satisfying the LE $\dot{\widetilde{Y}}(t)=-v_0+\overline{\xi}(t)$ is $\widetilde{P}(k,t)=\langle\exp[ik(-v_0t+\int_0^t\overline{\xi}(s)\,\mathrm{d}s)]\rangle$, we can write:

$$e^{ikv_0t}\widetilde{P}(k,t)=\left\langle e^{-\sigma k^2 S(t)}\right\rangle=\int_0^\infty h(s,t)e^{-\sigma k^2 s}\,\mathrm{d}s, \qquad [S64]$$

where $h(s,t)=\langle\delta(s-S(t))\rangle$ (22, 23) is the PDF of the inverse subordinator S. Then, we first take its time derivative, i.e.,

$$\left[ikv_0+\frac{\partial}{\partial t}\right]\widetilde{P}(k,t)=e^{-ikv_0t}\int_0^\infty e^{-\sigma k^2 s}\frac{\partial}{\partial t}h(s,t)\,\mathrm{d}s, \qquad [S65]$$

and secondly its Laplace transform. Recalling that $\widetilde{h}(s,\lambda)=[\Phi(\lambda)/\lambda]e^{-s\Phi(\lambda)}$, we obtain

$$\lambda\widetilde{P}(k,\lambda)-1=-ikv_0\widetilde{P}(k,\lambda)-\sigma k^2\frac{\lambda+iv_0k}{\Phi(\lambda+iv_0k)}\widetilde{P}(k,\lambda), \qquad [S66]$$

which is the Laplace transform of Eqs. **13**, **14**. Further solving it for $\widetilde{P}$, yields the propagator

$$\widetilde{P}(k,\lambda)=\frac{1}{\lambda+ikv_0}\left[1-\frac{\sigma k^2}{\Phi(\lambda+ikv_0)+\sigma k^2}\right]. \qquad [S67]$$

For the particular case of T being a Lévy stable process of order α, its inverse Laplace transform is

$$\widetilde{P}(x,t)=\frac{1}{\sqrt{4\sigma t^\alpha}}H^{1,0}_{1,1}\left[\frac{|x-v_0t|}{\sqrt{\sigma t^\alpha}}\,\middle|\,\begin{matrix}\left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right)\\(0,1)\end{matrix}\right], \qquad [S68]$$

which is the PDF plotted in Fig. 2B.

5. Derivation of the nonlocal advection-diffusion equation 13 via subordination

A CTRW is mathematically defined by a normal diffusive process X and a strictly increasing Lévy process T respectively specifying the stochastic process of jump lengths and that of waiting times of the random walk underlying its dynamics in the continuum limit (13, 22, 23). Their dynamics is described by the LEs

$$\dot{X}(s)=\xi(s), \qquad \dot{T}(s)=\eta(s), \qquad [S69]$$

where ξ is Gaussian white noise with $\langle\xi(s)\rangle=0$ and $\langle\xi(s_1)\xi(s_2)\rangle=2\sigma\delta(s_2-s_1)$, $\sigma>0$, and η is a one-sided positive Lévy process with characteristic functional (22–24)

$$G[u(r)]=\left\langle e^{-\int_0^\infty u(r)\eta(r)\,\mathrm{d}r}\right\rangle=e^{-\int_0^\infty \Phi(u(r))\,\mathrm{d}r}, \qquad [S70]$$

where u is an arbitrary test function. The function Φ is the Laplace exponent of η and is in general a Bernstein function (27). The anomalous CTRW process Y is defined by subordination of X with the inverse of T, i.e., $Y(t)=X(S(t))$, where S is the first passage time process $S(t)=\inf_{s>0}\{s:T(s)>t\}$. In the special case $\Phi(\lambda)=\lambda^\alpha$, $0<\alpha<1$, Eq. **S70** specifies a Lévy stable process that yields a subdiffusive CTRW with mean-square displacement that scales for long times as t^α.

A similar description can be defined for the process $\widetilde{Y}(t)$ satisfying Eq. **13**, i.e., we set $\widetilde{Y}(t)=\widetilde{X}(S(t))$, where $\widetilde{X}$ is described by the LE $\dot{\widetilde{X}}(s)=-v_0\eta(s)+\xi(s)$ instead of Eq. **S69**(left). As pointed out in the main text, weak GI requires a coupling between the LEs of the jump process X and that of the elapsed time process T. Here, we prove that its corresponding FP equation is Eq. **13**, following the technique of refs. (23, 24). The time-change S has continuous stochastic paths, such that $\widetilde{Y}$ is a continuous semi-martingale. Thus, its Itô formula for an arbitrary test function f is

$$f(\widetilde{Y}(t)) = f(Y_0) + \int_0^t \frac{\partial}{\partial y} f(\widetilde{Y}(t'))\,\mathrm{d}\widetilde{Y}(t') + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2} f(\widetilde{Y}(t'))\,\mathrm{d}[\widetilde{Y},\widetilde{Y}]_{t'}, \quad \text{[S71]}$$

where $\widetilde{Y}(0) = Y_0$ is the initial condition and $[\widetilde{Y},\widetilde{Y}]_t = 2\sigma\int_0^t \mathrm{d}S(t')$ is its quadratic variation. If we now evaluate Eq. **S71** for the specific choice $f(\widetilde{Y}(t)) = e^{ik\widetilde{Y}(t)}$, we obtain:

$$\begin{aligned} e^{ik\widetilde{Y}(t)} &= e^{ikx_0} + ik\int_0^t e^{ik\widetilde{Y}(t')}\,\mathrm{d}\widetilde{Y}(t') - \sigma k^2\int_0^t e^{ik\widetilde{Y}(t')}\,\mathrm{d}S(t') \\ &= e^{ikx_0} - ikv_0\int_0^t e^{ik\widetilde{Y}(t')}\,\mathrm{d}t' + ik\int_0^t e^{ik\widetilde{Y}(t')}\xi(S(t'))\,\mathrm{d}S(t') - \sigma k^2\int_0^t e^{ik\widetilde{Y}(t')}\,\mathrm{d}S(t'). \end{aligned} \quad \text{[S72]}$$

Here, we substituted the stochastic trajectory of $\widetilde{Y}$, obtained by exact integration of its LE. Thus, if we now (a) ensemble average Eq. **S72** (which cancels out the third term in its rhs because ξ is Gaussian noise with null first moment), (b) make its Fourier inverse transform and (c) take the time derivative of the resulting equation, we obtain:

$$\frac{\partial}{\partial t}\widetilde{P}(x,t) = v_0\frac{\partial}{\partial x}\widetilde{P}(x,t) + \sigma\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\left\langle\int_0^t \delta(x-\widetilde{Y}(t'))\,\mathrm{d}S(t')\right\rangle. \quad \text{[S73]}$$

Let us now compute the averaged stochastic integral in its rhs (23, 24). Employing the relation $1 = \int_0^\infty \delta(s-S(t))\,\mathrm{d}s$, we define an auxiliary quantity Q as

$$Q(x,t) = \left\langle\int_0^t \delta(x-\widetilde{Y}(t'))\,\mathrm{d}S(t')\right\rangle = \left\langle\int_0^t\left[\int_0^\infty \delta(x-\widetilde{X}(s))\delta(s-S(t'))\,\mathrm{d}s\right]\mathrm{d}S(t')\right\rangle, \quad \text{[S74]}$$

leading in Fourier transform to

$$\begin{aligned} Q(k,t) &= \left\langle\int_0^t\left[\int_0^\infty e^{ik\widetilde{X}(s)}\delta(s-S(t'))\,\mathrm{d}s\right]\mathrm{d}S(t')\right\rangle \\ &= \int_0^t\left[\int_0^\infty\left\langle e^{ik\int_0^s\xi(r)\,\mathrm{d}r}\right\rangle\left\langle e^{-ikv_0T(s)}\delta(t'-T(s))\right\rangle\mathrm{d}s\right]\mathrm{d}t'. \end{aligned} \quad \text{[S75]}$$

This equation is obtained by recalling that $\Theta(s-S(t)) = 1-\Theta(t-T(s))$ (26), which, together with the continuity of the paths of S, implies the relation: $\delta(t-T(s)) = \delta(s-S(t))\dot{S}(t)$ (23, 24). Here, $\dot{S}(t) = \lim_{\Delta t\to 0}[S(t+\Delta t)-S(t)]/\Delta t$ denotes an integration with respect to the time-change S. This is conveniently employed to express the stochastic integral in the left-hand side (lhs) of Eq. **S75** in terms of time increments. By introducing a partition of the interval $[0,t]$ of finite mesh Δt, we can write ($N = t/\Delta t$):

$$\begin{aligned} \int_0^t\left[\int_0^\infty e^{ik\widetilde{X}(s)}\delta(s-S(t'))\,\mathrm{d}s\right]\mathrm{d}S(t') &= \lim_{\substack{N\to\infty\\ \Delta t\to 0}}\sum_{i=0}^{N-1}\left[\int_0^\infty e^{ik\widetilde{X}(s)}\delta(s-S(t'_i))\,\mathrm{d}s\right][S(t'_{i+1})-S(t'_i)] \\ &= \lim_{\substack{N\to\infty\\ \Delta t\to 0}}\sum_{i=0}^{N-1}\left[\int_0^\infty e^{ik\widetilde{X}(s)}\delta(t'_i-T(s))\,\mathrm{d}s\right](t'_{i+1}-t'_i) \\ &= \int_0^t\left[\int_0^\infty e^{ik\widetilde{X}(s)}\delta(t'-T(s))\,\mathrm{d}s\right]\mathrm{d}t'. \end{aligned} \quad \text{[S76]}$$

Eq. **S75** then follows from Eq. **S76** by substituting the exact expression of $\widetilde{X}$ and by using the independence of ξ and η to factorise the ensemble average. Finally, we take the Laplace transform of Eq. **S75** to obtain:

$$Q(k,\lambda) = \frac{1}{\lambda}\int_0^\infty\left\langle e^{ik\int_0^s\xi(r)\,\mathrm{d}r}\right\rangle\left\langle e^{-(\lambda+ikv_0)\,T(s)}\right\rangle\mathrm{d}s = \frac{1}{\lambda}\int_0^\infty\left\langle e^{ik\int_0^s\xi(r)\,\mathrm{d}r}\right\rangle e^{-s\Phi(\lambda+ikv_0)}\,\mathrm{d}s, \quad \text{[S77]}$$

where the average over T is computed by employing its characteristic functional Eq. **S70**.

On the other hand, we can rewrite the position PDF of $\widetilde{Y}$ by using (a) the relation with which Eq. **S74** has been obtained, (b) the definition of $\widetilde{X}$ and (c) the independence of ξ and η. We then obtain in Fourier space:

$$\widetilde{P}(k,t) = \int_0^\infty\left\langle\delta(s-S(t))e^{ik\widetilde{X}(s)}\right\rangle\mathrm{d}s = \int_0^\infty\left\langle\delta(s-S(t))e^{-ikv_0T(s)}\right\rangle\left\langle e^{ik\int_0^s\xi(r)\,\mathrm{d}r}\right\rangle\mathrm{d}s, \quad \text{[S78]}$$

whose Laplace transform can be calculated by recalling that $\int_0^\infty \delta(s-S(t))e^{-\lambda t}\,\mathrm{d}t = \eta(s)e^{-\lambda T(s)}$ (23). We find:

$$\widetilde{P}(k,\lambda) = \int_0^\infty \left\langle \eta(s)e^{-(\lambda+ikv_0)T(s)}\right\rangle \left\langle e^{ik\int_0^s \xi(r)\,\mathrm{d}r}\right\rangle \mathrm{d}s. \quad \text{[S79]}$$

The η-dependent term can then be rewritten as

$$\begin{aligned}\left\langle \eta(s)e^{-(\lambda+ikv_0)T(s)}\right\rangle &= \frac{-1}{\lambda+ikv_0}\frac{\mathrm{d}}{\mathrm{d}s}\left\langle e^{-(\lambda+ikv_0)\int_0^s \eta(s')\,\mathrm{d}s'}\right\rangle \\ &= \frac{-1}{\lambda+ikv_0}\frac{\mathrm{d}}{\mathrm{d}s}e^{-s\Phi(\lambda+ikv_0)} = \frac{\Phi(\lambda+ikv_0)}{(\lambda+ikv_0)}e^{-s\Phi(\lambda+ikv_0)},\end{aligned} \quad \text{[S80]}$$

where we used again Eq. **S70** with $u(r) = \Theta(s-r)(\lambda+ikv_0)$. Substituting this result into Eq. **S79**, we obtain:

$$\int_0^\infty \left\langle e^{ik\int_0^s \xi(r)\,\mathrm{d}r}\right\rangle e^{-s\Phi(\lambda+ikv_0)}\,\mathrm{d}s = \frac{(\lambda+ikv_0)}{\Phi(\lambda+ikv_0)}\widetilde{P}(k,\lambda). \quad \text{[S81]}$$

The lhs of Eq. **S81** coincides with the integral at the rhs of Eq. **S77**. By eliminating it, we obtain

$$\lambda Q(k,\lambda) = \frac{(\lambda+ikv_0)}{\Phi(\lambda+ikv_0)}\widetilde{P}(k,\lambda), \quad \text{[S82]}$$

or equivalently in (k,t)-space (recalling that $Q(x,0)=0$ by definition):

$$\frac{\partial}{\partial t}Q(k,t) = \left[ikv_0 + \frac{\partial}{\partial t}\right]\int_0^t e^{-ikv_0(t-s)}K(t-s)\widetilde{P}(k,s)\,\mathrm{d}s. \quad \text{[S83]}$$

Finally, by taking its inverse Fourier transform and substituting it back into Eq. **S73**, we derive Eq. **13**.

6. Derivation of the nonlocal advection-diffusion equation 13 in the superdiffusive regime

We consider the stochastic process $\widetilde{Y}(t)$ in the comoving frame $\widetilde{\mathcal{S}}$, whose dynamics is described by the LE $\dot{\widetilde{Y}}(t) = -v_0 + \overline{\xi}(t)$, where the noise $\overline{\xi}$ is defined by its hierarchy of correlation functions; specifically, the odd ones are null, i.e., $\langle\prod_{j=1}^{1+2N}\overline{\xi}(t_j)\rangle = 0$, while the even ones are (20)

$$\left\langle \prod_{j=1}^{2N}\overline{\xi}(t_j)\right\rangle = \frac{\sigma^{N/2}}{N!2^N}\sum_{\varsigma\in\Sigma_{2N}}\prod_{m=1}^{N}\delta\big(t_{\varsigma(2N-m+1)}-t_{\varsigma(m)}\big)\sum_{\varsigma'\in\Sigma_N}\Theta\big(t_{\varsigma(\varsigma'(m))}-t_{\varsigma(\varsigma'(m-1))}\big)K\big(t_{\varsigma(\varsigma'(m))}-t_{\varsigma(\varsigma'(m-1))}\big). \quad \text{[S84]}$$

Here, $\varsigma(\varsigma')$ is a permutation of $2N(N)$ elements, which keeps the initial time fixed, $\Sigma_{2N}(\Sigma_N)$ denotes the set of all such operations, Θ is an Heaviside function and K an arbitrary function of time. Eq. **S84** represents an equivalent characterisation of the noise obtained by time derivative of a subordinated Brownian motion (20) (section 4), in which case K is related to the Laplace exponent Φ of a strictly increasing Lévy process T by the formula $K(\lambda) = \Phi(\lambda)^{-1}$ (22, 23). This generally yields subdiffusive MSD behaviour. However, Eq. **S84** still characterises a well-defined noise, even if a corresponding process T cannot be defined. Thus, Y may exhibit even super-diffusive behaviour, e.g., by setting $K(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $1<\alpha<2$.

Recalling Eq. **S10**, we need to compute the averaged quantity $\langle\overline{\xi}(t)h(k,t)\rangle$, where we set $h(k,t) = e^{-ikv_0t+\int_0^t\overline{\xi}(s)\,\mathrm{d}s}$. In the expression of h, the second exponential is a functional of the noise path, that can be Taylor expanded as (14, 15)

$$\begin{aligned}e^{ikv_0t}h(k,t)-1 &= \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^\infty \mathrm{d}s_1\ldots\int_0^\infty \mathrm{d}s_n H^{(n)}(k,s_1,\ldots,s_n)\overline{\xi}(s_1)\ldots\overline{\xi}(s_n) \\ &= \sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\int_0^t \mathrm{d}s_1\ldots\int_0^t \mathrm{d}s_n\overline{\xi}(s_1)\ldots\overline{\xi}(s_n),\end{aligned} \quad \text{[S85]}$$

where the variational derivatives are $H^{(n)}(k,s_1,\ldots,s_n) = \frac{\delta^{(n)}e^{ik\int_0^t\overline{\xi}(s)\,\mathrm{d}s}}{\delta\overline{\xi}(s_1)\ldots\delta\overline{\xi}(s_n)}\bigg|_{\overline{\xi}=0} = (ik)^n\Theta(t-s_1)\ldots\Theta(t-s_n)$.

Let us take the ensemble average of Eq. **S85** and then its time derivative. As the odd correlation functions of $\overline{\xi}$ are null, only the terms with even indices survive, so that we obtain:

$$\begin{aligned}\left[ikv_0+\frac{\partial}{\partial t}\right]\widetilde{P}(k,t) = e^{-ikv_0t}\Bigg[&\frac{\sigma(ik)^2}{2}K(t) + \frac{\sigma^2(ik)^4}{4}\int_0^t K(t-s)K(s)\,\mathrm{d}s \\ &+\sum_{n=3}^{\infty}\frac{\sigma^n(ik)^{2n}}{2^n}\int_0^t \mathrm{d}s_{n-1}K(t-s_{n-1})\ldots\int_0^{s_2}K(s_2-s_1)K(s_1)\,\mathrm{d}s_1\Bigg].\end{aligned} \quad \text{[S86]}$$

This result is understood by recalling that the $(2n)$-th order correlation function of $\bar{\xi}$ contains $(2n)!/(2^n n!)$ terms, each corresponding to a different structure of the delta functions. In addition, for each of the sequences of the n distinct times, set by the product of delta functions, there are $n!$ different orderings. However, once we integrate over time, all of them give the same contribution, so that we obtain $(2n)!/2^n$ integrals of the same type, thus leading to the final result Eq. **S86**.

We then multiply Eq. **S85** by $\bar{\xi}(t)$ and take its ensemble average. By eliminating the null terms, we obtain:

$$\left\langle \bar{\xi}(t) h(k,t) \right\rangle = \sum_{n=0}^{\infty} \frac{\sigma^{1+n}(ik)^{1+2n}}{(1+2n)!} e^{-ikv_0 t} \int_0^t \mathrm{d}s_1 \ldots \int_0^t \mathrm{d}s_{1+2n} \left\langle \bar{\xi}(t)\bar{\xi}(s_1)\ldots\bar{\xi}(s_{1+2n}) \right\rangle . \quad \text{[S87]}$$

We then find (a)$\int_0^t \mathrm{d}s_1 \left\langle \bar{\xi}(t)\bar{\xi}(s_1) \right\rangle = \sigma K(t)$ for $n = 0$, (b) $\int_0^t \mathrm{d}s_1 \int_0^t \mathrm{d}s_2 \int_0^t \mathrm{d}s_3 \left\langle \bar{\xi}(t)\bar{\xi}(s_1)\bar{\xi}(s_2)\bar{\xi}(s_3) \right\rangle = 3\sigma^2 \int_0^t \mathrm{d}s_1 K(t - s_1)K(s_1)\,\mathrm{d}s_1$ for $n = 1$ and for general $n > 1$:

$$\int_0^t \mathrm{d}s_1 \ldots \int_0^t \mathrm{d}s_{1+2n} \left\langle \bar{\xi}(t)\bar{\xi}(s_1)\ldots\bar{\xi}(s_{1+2n}) \right\rangle = \frac{\sigma^{1+n}(1+2n)!}{2^n} \int_0^t \mathrm{d}s_n K(t-s_n) \prod_{m=2}^{n} \int_0^{s_m} \mathrm{d}s_m K(s_m - s_{m-1}) K(s_1). \quad \text{[S88]}$$

Substituting these results into Eq. **S87**, we find $(s_n = s)$:

$$\left\langle \bar{\xi}(t) h(k,t) \right\rangle = ik\sigma e^{-ikv_0 t} K(t) + ik\sigma \times$$
$$\times \int_0^t \mathrm{d}s K(t-s) e^{-ikv_0 t} \left[\frac{\sigma (ik)^2}{2} K(s) + \sum_{n=2}^{\infty} \frac{\sigma^n (ik)^{2n}}{2^n} \prod_{m=2}^{n} \int_0^{s_m} \mathrm{d}s_{m-1} K(s_m - s_{m-1}) K(s_1) \right]. \quad \text{[S89]}$$

Comparing Eqs. **S86**, **S89**, we obtain the equation:

$$\begin{aligned} \left\langle \bar{\xi}(t) h(k,t) \right\rangle &= ik\sigma e^{-ikv_0 t} K(t) + ik\sigma \int_0^t \mathrm{d}s K(t-s) e^{-ikv_0(t-s)} \left[ikv_0 + \frac{\partial}{\partial s} \right] \widetilde{P}(k,s) \\ &= ik\sigma \left[ikv_0 + \frac{\partial}{\partial t} \right] \int_0^t \mathrm{d}s K(t-s) e^{-ikv_0(t-s)} \widetilde{P}(k,s). \end{aligned} \quad \text{[S90]}$$

The equivalence of the two expressions at the rhs of Eq. **S90** is proved by taking their Laplace transforms. Substituting this formula into Eq. **S10** yields the Fourier transform of Eq. **13**.

7. Special Functions: Definitions and Useful Relations

Here, we review definitions and useful properties of the three parameter Mittag-Leffler function and the Fox H-function. For further details on these special functions and derivations of the relations presented below we refer to (28).

A. The Fox H-Function. The Fox H-function is formally defined in terms of the following Mellin-Barnes type integral:

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\Omega} \Theta(s)\, z^{-s}\, \mathrm{d}s, \quad \text{[S91]}$$

where $i = (-1)^{-1/2}$, $z \neq 0$ and $z^{-s} = \exp\left[-s\left(\ln|z| + i \arg z\right)\right]$. Here, $\ln|z|$ stands for the natural logarithm of $|z|$, whereas $\arg z$ is not necessarily its principal value. The function $\Theta(s)$ is defined in terms of Gamma functions as

$$\Theta(s) = \frac{\left\{\prod_{j=1}^{m} \Gamma(b_j + B_j s)\right\} \left\{\prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)\right\}}{\left\{\prod_{j=1+m}^{q} \Gamma(1 - b_j - B_j s)\right\} \left\{\prod_{j=1+n}^{p} \Gamma(a_j + A_j s)\right\}}, \quad \text{[S92]}$$

where $m, n, p, q \in \mathbb{N}_0$ with $0 \leq n \leq p$ and $1 \leq m \leq q$; $A_i, B_j \in \mathbb{R}_+$; $a_i, b_j \in \mathbb{R}$ (or alternatively $\mathbb{C}$) with $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Any empty product in Eq. **S92** is to be interpreted as unity. The contour Ω in Eq. **S91** is suitably chosen to separate the poles $\xi_{j\nu} = -(\nu + b_j)/B_j$, with $j = 1, \ldots, m$ and $\nu \in \mathbb{N}_0$, of $\Gamma(b_j + B_j s)$ from the poles $\chi_{i\nu} = (1 - a_i + \nu)/A_i$, with $i = 1, \ldots, n$ and same ν, of $\Gamma(1 - a_j - A_j s)$. Thus, the condition $A_i(b_j + \nu) \neq B_j(a_i - 1 - \nu)$ ensures the existence of the contour Ω and consequently the convergence of the integral in Eq. **S91**. A popular choice for the contour Ω consists in a path running parallel to the imaginary axis from $\gamma - i\infty$ to $\gamma + i\infty$, where $\gamma \in \mathbb{R} = (-\infty, +\infty)$ is chosen arbitrarily such that it separates all the poles $\xi_{j\nu}$ from all the poles $\chi_{i\nu}$. If we choose such a contour, the convergence of the Mellin-Barnes integral in Eq. **S91** is obtained if $a^* > 0$ and $|\arg z| < (\pi/2)a^*$, $z \neq 0$, with a^* being the following parameter:

$$a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \quad \text{[S93]}$$

The integral also converges if $a^* = 0$, $\gamma\mu + \mathrm{Re}(\delta) < -1$, $\arg z = 0$ and $z \neq 0$, where

$$\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \tag{S94}$$

Other equivalent choices of Ω, with the corresponding convergence conditions for the integral of Eq. **S91**, are available. A first useful property of the H-function is its symmetry under exchange of the pairs of parameters (a_p, A_p) and/or (b_p, B_p). Specifically, the H-function is symmetric under permutations of the pairs (a_i, A_i) for $i = 1, \ldots, n$ or separately for $i = n+1, \ldots, p$; likewise it is symmetric if we make a permutation of the pairs (b_j, B_j) for $j = m+1, \ldots, q$ or separately for $j = 1, \ldots, m$. A second property enables us to reduce the order of the function if one of the pairs (a_i, A_i) for $i = 1, \ldots, n$ is equal to one of the pairs (b_j, B_j) for $j = 1+m, \ldots, q$ or alternatively for $i = 1+n, \ldots, p$ and $j = 1, \ldots, m$. In these different cases, the H-function reduces to one of lower order with p, q and n (or m respectively) decreased by one. In formulas, we have:

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_{q-1}, B_{q-1}), (a_1, A_1) \end{array} \right. \right] = H_{p-1,q-1}^{m,n-1}\left[z \left| \begin{array}{l} (a_2, A_2), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_{q-1}, B_{q-1}) \end{array} \right. \right], \tag{S95}$$

provided $n \geq 1$ and $q > m$; and alternatively:

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_1, A_1), \ldots, (a_{p-1}, A_{p-1}), (b_1, B_1) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right. \right] = H_{p-1,q-1}^{m-1,n}\left[z \left| \begin{array}{l} (a_1, A_1), \ldots, (a_{p-1}, A_{p-1}) \\ (b_2, B_2), \ldots, (b_q, B_q) \end{array} \right. \right], \tag{S96}$$

provided $m \geq 1$ and $p > n$. The Fox H-function satisfies the following scaling relation:

$$H_{p,q}^{m,n}\left[z^r \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] = \frac{1}{r} H_{q,p}^{n,m}\left[z \left| \begin{array}{l} (a_p, A_p/r) \\ (b_q, B_q/r) \end{array} \right. \right], \quad \forall r \in \mathbb{R}_+/\{0\}. \tag{S97}$$

Two further properties enable us either to invert the independent variable inside the H-function:

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] = H_{q,p}^{n,m}\left[\frac{1}{z} \left| \begin{array}{l} (1-b_q, B_q) \\ (1-a_p, A_p) \end{array} \right. \right] \tag{S98}$$

or to absorb powers of the independent variable of general exponent $\sigma \in \mathbb{C}$ inside the H-function:

$$z^\sigma H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] = H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_p + \sigma A_p, A_p) \\ (b_q + \sigma B_q, B_q) \end{array} \right. \right]. \tag{S99}$$

On the one hand, the Mellin-cosine(sine) transform of the Fox H-function is given by (29):

$$\int_0^\infty z^{\rho-1} \begin{Bmatrix} \sin(\kappa z) \\ \cos(\kappa z) \end{Bmatrix} H_{p,q}^{m,n}\left[a z^r \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] \mathrm{d}z = \frac{2^{\rho-1}\sqrt{\pi}}{\kappa^\rho} H_{p+2,q}^{m,n+1}\left[a \left(\frac{2}{\kappa}\right)^r \left| \begin{array}{l} \left(\left(\frac{3\mp1-2\rho}{4}\right), \frac{r}{2}\right), (a_p, A_p), \left(\left(\frac{3\pm1-2\rho}{4}\right), \frac{r}{2}\right) \\ (b_q, B_q) \end{array} \right. \right] \tag{S100}$$

where the following conditions must be satisfied: (i) $a^*, r, \kappa > 0$, (ii) $|\arg(a)| < a^*\pi/2$, (iii) $\mathrm{Re}(\rho) + r \min_{1\leq j\leq m} \mathrm{Re}\left(\frac{b_j}{B_j}\right) > \frac{(-1\mp1)}{2}$, (iv) $\mathrm{Re}(\rho) + r \max_{1\leq j\leq n} \mathrm{Re}\left(\frac{a_j - 1}{A_j}\right) < 1$. On the other hand, the Mellin transform of a general H-function is

$$\int_0^\infty z^{\xi-1} H_{p,q}^{m,n}\left[a z \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] \mathrm{d}z = a^{-\xi}\Theta(\xi), \tag{S101}$$

with Θ defined as in Eq. **S92**. In conclusion, we provide a formula for the general n-th order derivative of the H-function, i.e.,

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r} H_{p,q}^{m,n}\left[(c\,x+d)^h \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] = \left(\frac{c}{c\,x+d}\right)^r H_{1+p,1+q}^{m,1+n}\left[(c\,x+d)^h \left| \begin{array}{l} (0,h), (a_p, A_p) \\ (b_q, B_q), (r,h) \end{array} \right. \right]. \tag{S102}$$

B. The Three Parameter Mittag-Leffler Function. The three parameter Mittag-Leffler function is defined by the following power-series:

$$E_{\alpha,\beta}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\beta + \alpha\,n)} \frac{z^n}{n!}, \tag{S103}$$

where $(\delta)_n = \Gamma(\delta+n)/\Gamma(\delta)$ is the Pochhammer symbol. The two and one parameter Mittag-Leffler functions $E_{\alpha,\beta}(z)$ and $E_\alpha(z)$ are obtained as special cases of Eq. **S103** by setting $\delta = 1$, and also $\beta = 1$ for the latter one. Its Laplace transform is

$$\mathcal{L}\left\{z^{\beta-1} E_{\alpha,\beta}^{\delta}(\pm c\,z^\alpha)\right\}(\lambda) = \frac{\lambda^{\alpha\,\delta-\beta}}{(\lambda^\alpha \mp c)^\delta} \tag{S104}$$

with $\mathrm{Re}(\lambda) > |c|^{1/\alpha}$. The three parameter Mittag-Leffler function can be expressed as a Fox H-function as

$$E^{\delta}_{\alpha,\beta}\left(\pm z\right) = \frac{1}{\Gamma(\delta)} H^{11}_{12}\left[\mp z \,\middle|\, \begin{matrix}(1-\delta,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right]. \tag{S105}$$

This formula is derived by solving the corresponding integral of Eq. **S91** with the residue theorem. In several anomalous diffusive systems, this function plays a major role, as it typically describes their mean square displacement (in this case then z is the time variable). It is then important to study its asymptotic scaling for both small and large values of z. In the former case, the function $E^{\delta}_{\alpha,\beta}\left(-z^{\alpha}\right)$ behaves as a stretched exponential. In fact, by looking at Eq. **S103**, we can write:

$$E^{\delta}_{\alpha,\beta}\left(-z^{\alpha}\right) \sim \frac{1}{\Gamma(\beta)} - \delta\,\frac{z^{\alpha}}{\Gamma(\alpha+\beta)} \sim \frac{1}{\Gamma(\beta)}\exp\left(-\delta\,\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} z^{\alpha}\right). \tag{S106}$$

In the latter case, it is convenient to look at the equivalent definition (valid for $|z| > 1$) (30)

$$E^{\delta}_{\alpha,\beta}\left(-z\right) = \frac{z^{-\delta}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta+n)}{\Gamma\left(\beta-\alpha(\delta+n)\right)} \frac{z^{-n}}{n!}, \tag{S107}$$

which then predicts a asymptotic power-law behaviour for $|z| \gg 1$, i.e.,

$$E^{\delta}_{\alpha,\beta}\left(-z^{\alpha}\right) \sim \frac{z^{-\alpha\,\delta}}{\Gamma(\beta-\alpha\,\delta)}. \tag{S108}$$

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Table S1. Overview of generic stochastic models for normal and anomalous diffusion. For simplicity, we show their representations in terms of generalized Fokker-Planck or Klein-Kramers equations and neglect the explicit dependencies of the distributions P, $\widetilde{P}$ on the sample variables. For all models, except the Continuous time random walk, property ii holds, i.e., their evolution equations in different inertial frames are related by a Galilean transformation of their independent variables. We define the diffusion operator $\mathcal{L} = \sigma \frac{\partial^2}{\partial x^2}$.

Stochastic model	Fokker-Planck/Klein-Kramers eq. in $\mathcal{S}$	Fokker-Planck/Klein-Kramers eq. in $\widetilde{\mathcal{S}}$
Normal diffusion (overdamped)	$\left[\frac{\partial}{\partial t} - \mathcal{L}\right] P = 0$	$\left[\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} - \mathcal{L}\right] \widetilde{P} = 0$
Normal diffusion (underdamped) *	$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v - \frac{\partial}{\partial v}\gamma v - \gamma\sigma\frac{\partial^2}{\partial v^2}\right] P = 0$	$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v - \frac{\partial}{\partial v}\gamma(v + v_0) - \gamma\sigma\frac{\partial^2}{\partial v^2}\right] \widetilde{P} = 0$
Fractional/Scaled Brownian motion †	$\left[\frac{\partial}{\partial t} - \beta t^{\beta-1}\mathcal{L}\right] P = 0$	$\left[\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} - \beta t^{\beta-1}\mathcal{L}\right] \widetilde{P} = 0$
Generalized Langevin equation ‡	$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v - \frac{\partial}{\partial v}\Gamma(t) v\right] P$ $= \left[\frac{\partial^2}{\partial v^2}\sigma\Gamma(t) + \frac{\partial^2}{\partial x \partial v} D_{xv}(t)\right] P$	$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v - \frac{\partial}{\partial v}\Gamma(t)(v + v_0)\right] \widetilde{P}$ $= \left[\frac{\partial^2}{\partial v^2}\sigma\Gamma(t) + \frac{\partial^2}{\partial x \partial v} D_{xv}(t)\right] \widetilde{P}$
Lévy flight §	$\left[\frac{\partial}{\partial t} - \nabla^\beta\right] P = 0$	$\left[\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} - \nabla^\beta\right] \widetilde{P} = 0$
Lévy walk ¶	$\left[\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - u\frac{\partial}{\partial x}\right)\right] P_u$ $= -\left[\frac{1}{2}\mathcal{D}_t^{(-u,u)} + \frac{1}{2}\mathcal{D}_t^{(u,-u)}\right] P_u$	$\left[\left(\frac{\partial}{\partial t} + u_+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + u_-\frac{\partial}{\partial x}\right)\right] \widetilde{P}_u$ $= -\left[\frac{1}{2}\mathcal{D}_t^{(u_-,u_+)} + \frac{1}{2}\mathcal{D}_t^{(u_+,u_-)}\right] \widetilde{P}_u$
Continuous time random walk	$\left[\frac{\partial}{\partial t} - \mathcal{L}\mathbb{D}_t\right] P = 0$	?

* $\gamma > 0$ is the friction coefficient.

† $0 < \beta < 2$ is the exponent of the characteristic power-law dependence of the noise correlations.

‡ Γ, D_{xv} are time dependent friction and diffusion coefficients, respectively, given in Eq. **S21**.

§ ∇^β $(0 < \beta < 2)$ denotes the fractional Laplacian, defined in Fourier space as $\nabla^\beta \to -|k|^\beta$.

¶ u is the absolute value of the velocity in the frame $\mathcal{S}$, while in $\widetilde{\mathcal{S}}$ the forward/backward velocities are $u_\pm = -v_0 \pm u$. The operator $\mathcal{D}_t^{(v_1,v_2)}$ has the representation $\mathcal{D}_t^{(v_1,v_2)} P(x,t) \to (\lambda - ikv_1)K(\lambda - ikv_2)P(k,\lambda)$ (see Eq. **S34**). For $v_1 = v_2 = -v_0$, $\mathcal{D}_t^{(v_1,v_2)}$ recovers the fractional substantial derivative Eq. **14**.