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Spiral modes in the diffusion of a single granular particle on a vibrating surface

R. Klages [a,b,*], I.F. Barna [b,c], L. Mátyás [d]

[a] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*
[b] *Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany*
[c] *Institute for Theoretical Physics, TU Vienna, Wiedner Hauptstr. 8-10, A-1040 Vienna, Austria*
[d] *Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles,
Campus Plaine CP 231, Blvd. du Triomphe, B-1050 Brussels, Belgium*

Abstract

We consider a particle that is subject to a constant force and scatters inelastically on a vibrating periodically corrugated floor. At small friction and for small scatterers the dynamics is dominated by resonances forming spiral structures in phase space. These spiral modes lead to pronounced maxima and minima in the diffusion coefficient as a function of the vibration frequency, as is shown in computer simulations. Our theoretical predictions may be verified experimentally by studying transport of single granular particles on vibratory conveyors.
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1. Introduction

One might naively assume that a ball bouncing inelastically on an oscillating plate under the action of a gravitational field must be a very simple dynamical system. Surprisingly, such a ball exhibits an extremely rich dynamics providing a prominent example for complexity in a seemingly trivial nonlinear dynamical system. Experiments indicated period-doubling bifurcations into chaotic motion [1–4] while theoretical analyses classified different types of dynamical behavior in terms of phase locking, chattering, and strange attractors [5–8].

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* Corresponding author.
  E-mail addresses: r.klages@qmul.ac.uk (R. Klages),
barna@concord.itp.tuwien.ac.at (I.F. Barna), lmatyas@ulb.ac.be
(L. Mátyás).
  URLs: http://www.maths.qmul.ac.uk/~klages,
http://www.mpipks-dresden.mpg.de/~barna.

Oscillating surfaces, on the other hand, are often used in the field of granular material in order to drive a gas of granular particles into a nonequilibrium steady state [9–11]. A model that is somewhat intermediate between an interacting many-particle system and a single bouncing ball is the *bouncing ball billiard* (BBB) [12]. Here the surface is not flat but periodically corrugated mimicking a fixed periodic grid of steel balls [11]. A single particle bouncing inelastically on this vibrating surface thus performs diffusive motion. Studying the BBB at large friction and large radius of the scatterers, the diffusion coefficient turned out to be a highly irregular function of the driving frequency [12]. As the dynamical reason for the largest maxima some integer frequency locking between bouncing ball and vibrating plate could be identified, a phenomenon that is well known for a ball bouncing on a flat vibrating plate [2–4,6,7,13,14]. Additionally, we detected an irregular structure on fine scales that is supposedly due to some subtle effects like pruning of orbits under parameter variation [15].

In this Letter we show that yet there exists another microscopic mechanism creating nonmonotonicities in the frequency-dependent diffusion coefficient. Our starting point is to study diffusion in the BBB at smaller friction and smaller scatterer radius than in Ref. [12]. In this situation we find a *spiraling bouncing ball mode*, which changes under parameter variation. We show that this mode is again responsible for the emergence of local maxima and minima in the diffusion coefficient as a function of the vibration frequency. We conclude with a brief outlook towards consequences of our work for the standard bouncing ball as well as for transport on vibro-transporters.

2. The bouncing ball billiard

The model is depicted in Fig. 1: it consists of a floor oscillating with $y_f = -A\sin(2\pi f t)$, where $A = 0.01$ and f are the amplitude, respectively, the frequency of the vibration.[1] This floor is equipped with a periodic grid of circular scatterers of radius R whose
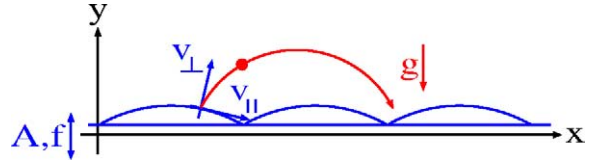

Fig. 1. Sketch of the bouncing ball billiard: a point particle collides inelastically with circular scatterers forming a periodic lattice on a line. Parallel to y there is an external field with constant acceleration g. The corrugated floor oscillates with amplitude A and frequency f.

centers are a distance of $S = 2$ apart from each other. We now consider a point particle[2] performing a free flight between two collisions in a gravitational field with acceleration $g = 9800 \parallel y$. The particle's coordinates (x_{n+1}^-, y_{n+1}^-) and velocities $(v_{x,n+1}^-, v_{y,n+1}^-)$ at time t_{n+1} immediately before the $(n + 1)$th collision and its coordinates (x_n^+, y_n^+) and velocities (v_{xn}^+, v_{yn}^+) at time t_n immediately after the nth collision are related by the equations [12]

$$x_{n+1}^- = x_n^+ + v_{x,n}^+(t_{n+1} - t_n), \tag{1}$$

$$y_{n+1}^- = y_n^+ + v_{y,n}^+(t_{n+1} - t_n) - g(t_{n+1} - t_n)^2/2, \tag{2}$$

$$v_{x,n+1}^- = v_{x,n}^+, \tag{3}$$

$$v_{y,n+1}^- = v_{y,n}^+ - g(t_{n+1} - t_n). \tag{4}$$

At a collision the velocities change according to

$$v_{\perp,n}^+ - v_{f\perp,n} = \alpha(v_{f\perp,n} - v_{\perp,n}^-), \tag{5}$$

$$v_{\parallel,n}^+ - v_{f\parallel,n} = \beta(v_{\parallel,n}^- - v_{f\parallel,n}), \tag{6}$$

where v_f is the velocity of the corrugated floor. The local coordinate system for the velocities $v_\perp$ and $v_\parallel$ is given in Fig. 1. We assume that the scattering process is inelastic by introducing the two restitution coefficients α and β perpendicular, respectively, tangential to the surface at the scattering point.

In Ref. [12] we studied the BBB for $R = 25$, $\alpha = 0.5$ and $\beta = 0.99$. Here we consider the case of $R = 15$ and $\alpha = 0.7$, which is closer to the experiments of Urbach et al. [11]. In contrast to naive expectations, this simple variation of parameters profoundly changes the diffusive dynamics of the BBB, as we will show in the following.

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[1] We work with reduced units, where all quantities are dimensionless in terms of the unit amplitude $A_0 = 1$ mm, respectively, the unit frequency $f_0 = 1$ Hz.

[2] Note that the radii of the moving particle and of a scatterer are additive, hence we only neglect any rotational energy.
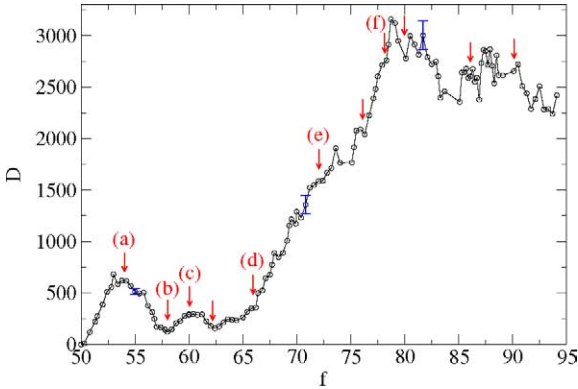
Fig. 2. The diffusion coefficient $D(f)$ of the bouncing ball billiard Fig. 1 at friction $\alpha = 0.7$ and scatterer radius $R = 15$ as a function of the vibration frequency f of the corrugated floor. The graph consists of 124 data points with error bars at $f = 55$, 70.8 and 82.1. The labels (a) to (e) refer to the respective phase space plots of Fig. 3.

3. Deterministic diffusion, frequency locking, and spiral modes

The diffusion coefficient D was computed from simulations according to the Einstein formula

$$D = \lim_{t \to \infty} \frac{\langle (x(t) - x(0))^2 \rangle}{2t}, \qquad (7)$$

where the brackets denote an ensemble average over moving particles; for further numerical details cf. Ref. [12]. Fig. 2 shows that D exhibits local maxima and minima as a function of the driving frequency f. However, already at a first view this curve looks very different from the corresponding one in Ref. [12], where the diffusion coefficient was computed at different parameter values.

The major irregularities of Fig. 2 can be understood by means of the phase space projections displayed in Fig. 3. Here we have plotted the positions y and the velocities v_y^+ of a moving particle immediately after its collisions with the plate. The onset of diffusion around $f = 50$ is characterized by the existence of many *creeping orbits*, where the particle performs long sequences of tiny little jumps along the surface hence moving parallel to y like the harmonically oscillating plate. In Fig. 3 this dynamics yields incomplete ellipses around $v_y^+ = 0$ thus representing a simple harmonic oscillator mode.

The first maximum of $D(f)$ around $f = 54$ reflects a 1/1 frequency locking between particle and plate, as is shown in Fig. 3(a) by the large number of points accumulating around $(y, v_y^+) \simeq (0.1, 100)$. In other words, the time of flight T_p of the particle is identical with the vibration period T_f of the floor. Additionally, the system may exhibit a dynamics where this behavior is interrupted by periods of creepy motion. Whether this happens depends on the initial conditions, hence the dynamics is nonergodic. However, only for the peak around $f = 55$ we could detect such difficulties, whereas for $f \geqslant 58$ our numerics indicated ergodic motion. Frequency locking has been widely discussed for the simple bouncing ball [2–7, 13,14] and has already been found to enhance diffusion in the BBB [12], where all the major peaks of the frequency-dependent diffusion coefficient could be identified in terms of integer frequency locking.

Around $f = 58$ this resonance is completely destroyed resulting in a local minimum of $D(f)$. In Fig. 3(b) this is represented by a "smearing out" of the 1/1 frequency locking spot together with a dominance of creeps. Surprisingly, the next local maximum at $f = 60$ corresponds to a new structure in phase space: similar to a harmonic oscillator, the granular particle locks onto a circle in the projected phase space. However, this circle is not situated around the origin but shifted up by $v_y^+ \simeq 70$ along the vertical axis, cf. Fig. 3(c), reflecting the dissipation due to the harmonic driving. We call this structure that obviously enhances diffusion a *virtual harmonic oscillator mode* (VHO). Around $f = 62$ this mode again becomes unstable leading to a local minimum. At $f = 66$ the VHO starts to "spiral out", see Fig. 3(d), eventually resulting in a two-loop spiral, Fig. 3(e). This causes a drastic increase of $D(f)$ on a global scale. Note that we could not detect any simple periodic motion on this spiral, hence this scenario may not be understood as a simple period-doubling bifurcation. We furthermore emphasize that parallel to x the motion is always highly irregular, as exemplified in the existence of a diffusion coefficient. At $f = 76$ another loop develops, see Fig. 3(f) with $f = 78$ for a slightly advanced stage, before this structure settles into a three-loop spiral that explains the washed-out local maximum in $D(f)$ around $f = 80$. A fourth loop emerges at $f = 86$ being completed around $f = 90$, followed by the onset of a fifth one at $f = 96$.
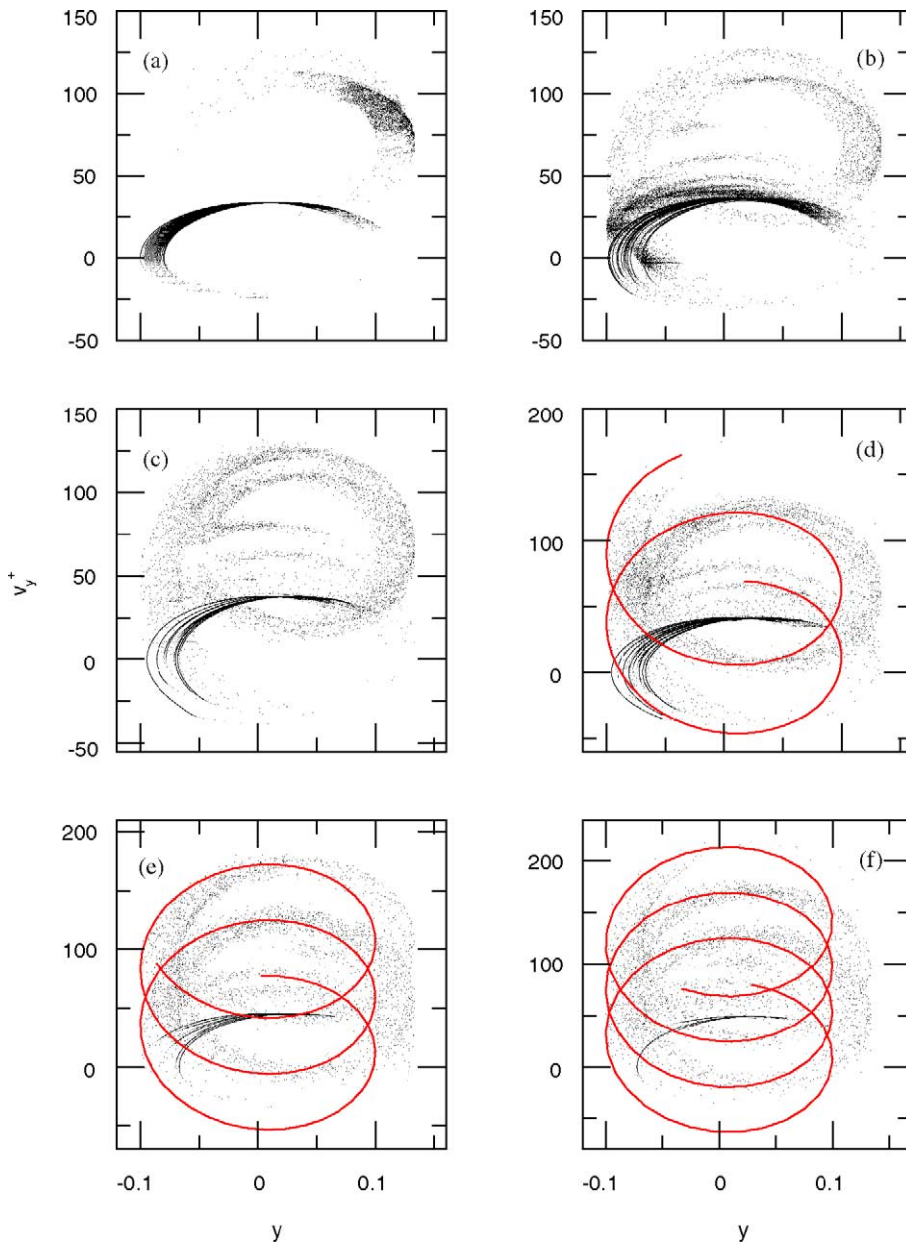
Fig. 3. Projections of the phase space of the bouncing ball billiard Eqs. (1) to (6) onto positions and velocities along the y-axis at the collisions. The frequencies of the vibrating plate are $f =$ (a) 54, (b) 58, (c) 60, (d) 66, (e) 72, (f) 78. The spiraling lines represent the analytical approximation Eqs. (9), (10).

In order to understand the frequency dependence of this spiral more quantitatively we analyze the data presented in Fig. 3 using a simple argument of Warr et al. [7]: the time of flight T_p between two collisions of the particle with the plate is determined by

$T_p = 2v_y^+/g$, hence one may calculate

$$k := \frac{T_p}{T_f} = \frac{2v_y^+ f}{g} \tag{8}$$

and check whether this fraction yields some rational number indicating some frequency locking. Extracting v_y^+ from Fig. 3 one can easily verify that the black spot in (a) indeed corresponds to $k \simeq 1$. Complete spiral loops at higher frequencies appear to be situated around $f_c + c\Delta f$, $c \in \mathbb{N}_0$, with $f_c \simeq 60$ and $\Delta f \simeq 10$. Let v_y^+ be the maximal values of the spirals in Fig. 3. We then find that $k \simeq 1.47$ around $f = 60$, $k \simeq 2.5$ around $f = 72$, $k \simeq 3.43$ around $f = 80$, and $k \simeq 4.5$ around $f = 90$. The spiral mode thus locks into "smeared-out" half-integer resonances starting around $f_i + i\Delta f$, $i \in \mathbb{N}_0$, $f_i \simeq 66$.

We remark that, for a ball bouncing on a flat plate, signs of such spiral modes were already observed experimentally [7] and described theoretically [3–6,8]. Particularly, Luck and Mehta [6] provided a simple analytical approximation for the coarse functional form of this mode. Their derivation can straightforwardly be repeated for Eqs. (1) to (6): let us approximate the surface to be flat, and let us assume that there are no correlations between the collisions. By using Eqs. (2) and (5) we then arrive at

$$y_1^+ = -A \sin(2\pi f t_1), \qquad (9)$$
$$v_{y,1}^+ = \alpha g/2(t_1 - t_0) - A 2\pi f(1+\alpha)\cos(2\pi f t_1), \qquad (10)$$

cp. Eq. (10) to Eq. (3.20) of Ref. [6], where t_0 is the initial time at which the particle was launched from the plate and t_1 is the time of the first collision. For t_0 we calculated the time at which the plate moves with maximum positive velocity, since here, for high enough frequency, a particle will be launched which previously is at rest. For t_1 we obtained the distributions of collision times t_c from simulations yielding a range of $0 \leqslant t_c \leqslant t_{\max}(f)$. In order to ensure that $t_1 - t_0 \geqslant 0$ we then defined $t_1 := t_c + t_0$. Consequently, Eqs. (9), (10) contain no fit parameter. Results from Eqs. (9), (10) are displayed in Fig. 3(d) to (f). Naturally, this approximation does not work well for small collision times, respectively, in the regime of creeping orbits. However, starting from the VHO it reproduces the whole structure of the spiral mode very well.

4. Outlook and conclusions

In this Letter we have studied the BBB for smaller friction and smaller scatterer radius than in our previous work Ref. [12]. Computer simulations demonstrated that, again, the diffusion coefficient is a highly irregular function of the vibration frequency. However, for the present parameters only the first local maximum could be identified in terms of simple frequency locking. We showed that there exist local extrema due to what we called a spiral mode in the BBB. Whenever the vertical velocity of the particle, respectively, the vibration frequency of the plate, is large enough such that the particle can lock into a multiple half-integer frequency of the plate, the spiral mode develops another loop. The coarse functional form of this spiral was extracted from the equations of motion by neglecting the radius of the scatterers and by assuming memory loss at the collisions.

In Ref. [8] these spiral structures have been denoted as "chattering bands" of the simple bouncing ball, and it has been argued that they are rather stable against random perturbations. In the BBB the scatterers are defocusing inducing an additional deterministic chaotic component into the dynamics [15]. Interpreting the geometry of the BBB as a "perturbation" thus leads to the conclusion that spiral modes form states of the bouncing ball dynamics which are much more stable than simple frequency locking resonances.

Interestingly, bouncing ball-type models have been used since quite some time in order to understand transport on vibro-transporters, which are common carriers of agricultural material and related matter. For a corrugated version that very much reminds of the BBB and is widely used in industrial applications see, e.g., Ref. [16]. In order to transport particles on such devices one generates an average drift velocity by means of some symmetry breaking. For tilted systems with a flat surface Hongler et al. [13,14] predicted an integer frequency locking leading to local extrema in the transport rate as a function of a control parameter, as was recently verified in experiments [17]. Grochowski et al. constructed a circular vibratory conveyor, where the trough was driven by asymmetric oscillations [18–20]. Both for a tracer particle in a layer of granular matter and for a single highly inelastic test particle they observed a nonmonotonic current reversal of the transport velocity under variation of control parameters. This ratchet-like effect was recently reproduced in simulations [21] again indicating some underlying frequency locking.

In contrast to single bouncing balls, the core of granular matter research deals with highly interacting many-particle systems, where phenomena like structure formation [19] arise because of statistical physical mechanisms rather than being of an intricate nonlinear deterministic origin. The main message we wish to convey with our Letter is that one might observe surprising transport properties for even a single, or just a few, granular particles moving on suitable surfaces. Looking for such counter-intuitive phenomena in experiments might thus open up new directions of research.

Our conclusion is that in vibro-transporters not only the current, but also other transport property quantifiers such as the diffusion coefficient may be irregular functions of control parameters. So far we have identified three basic mechanisms generating irregular parameter dependencies, which are (i) frequency locking [12], (ii) the spiral modes discussed here, and (iii) pruning effects [15] leading to fractal-like irregularities on finer and finer scales. Looking for irregular transport coefficients in experiments may thus pave the way for a very fine-tuned control of the dynamics of vibro-transporters. This may eventually provide a basis for practical applications such as, for example, sophisticated methods of particle separation [10].

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