Anomalous diffusion generated by randomly perturbed deterministic dynamics

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Bad Wildbad, 05 October 2015

Outline

huge progress by stochastic theory of anomalous transport

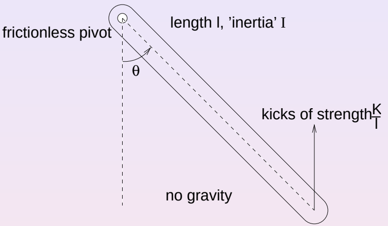
but: *microscopic origin* of anomalous dynamics from deterministic equations of motion?

here in-between: randomly perturbed deterministic dynamics

1. **Motivation:** standard map, diffusion, and dissipation
2. **Randomly perturbed** dynamical systems
3. **Noise-induced diffusion** in the dissipative standard map
4. Compare with **stochastic theory:** CTRW

The kicked rotor and the standard map

rotating bar kicked periodically:



equations of motion:

$$\dot{\theta} = \omega \quad , \quad \dot{\omega} = k \sin\theta \sum_{m=0}^{\infty} \delta(t - m\tau) \quad ; \quad k = K/I$$

integration $\int_{n+0^+}^{(n+1)+0^+} dt \ldots, \tau = 1$ yields the (Chirikov-Taylor) standard map

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega_n \\ \omega_{n+1} &= \omega_n + k \sin\theta_{n+1} \end{aligned}$$

The standard map and diffusion

paradigmatic Hamiltonian dynamical system

in the following:

$$\begin{aligned} x_{n+1} &= x_n + y_n \mod 2\pi \\ y_{n+1} &= y_n + K \sin x_{n+1} \end{aligned}$$

define (momentum) diffusion coefficient as

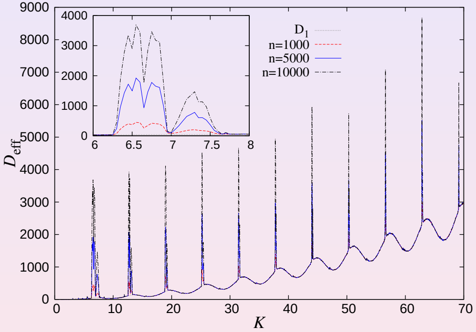
$$D(K) = \lim_{n\to\infty} \frac{1}{n} < (y_n - y_0)^2 >$$

with ensemble average over the initial density

$$< \ldots > = \int dx\, dy\, \varrho(x,y) \ldots \;, x \in [0,2\pi)\;,\; y = y_0 \in [0,2\pi)$$

Diffusion in the standard map

analytical (Rechester, White, 1980) and numerical studies of parameter-dependent diffusion $D_{eff}(K)$:



Manos, Robnik, PRE (2014)

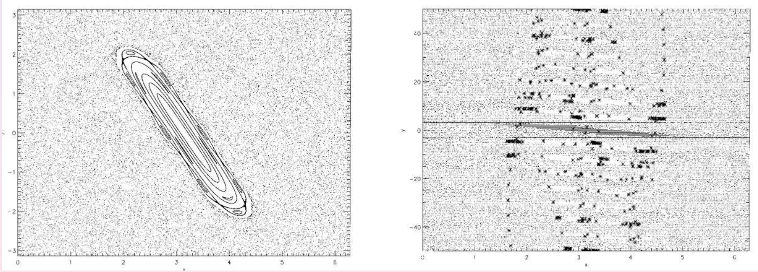
- $D(K)$ is highly irregular
- for some K **superdiffusion** with mean square displacement $< y_n^2 > \sim n^\gamma \;,\; \gamma > 1$ due to accelerator modes

The dissipative standard map

kicked rotor / standard map with damping:

$$x_{n+1} = x_n + y_n \mod 2\pi$$
$$y_{n+1} = (1-\nu)y_n + f_0 \sin x_{n+1}$$

with $\nu \in [0,1]$:



Feudel, Grebogi, Hunt, Yorke, PRE (1996)

- islands in phase space for $\nu = 0$ (left) become coexisting periodic attractors (right): 150 found for $\nu = 0.02$, $f_0 = 4$
- simple argument yields $|y_n| < y_{max}$: quick trapping

Randomly perturbed deterministic dynamics

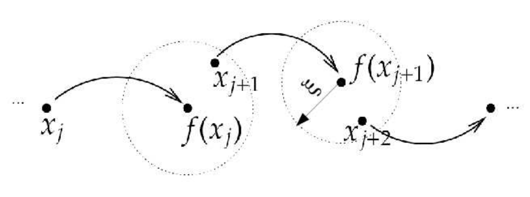
Question: What happens to deterministic dynamics $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ under random perturbations?

Consider the dissipative standard map with additive noise:

$$
\begin{aligned}
x_{n+1} &= x_n + y_n + \epsilon_{x,n} \mod 2\pi \\
y_{n+1} &= (1-\nu)y_n + f_0 \sin x_{n+1} + \epsilon_{y,n}
\end{aligned}
$$

with iid random variables $\boldsymbol{\epsilon}_n = (\epsilon_{x,n}, \epsilon_{y,n})$ drawn from uniform distribution bounded by $||\boldsymbol{\epsilon}_n|| < \xi$ of noise amplitude ξ

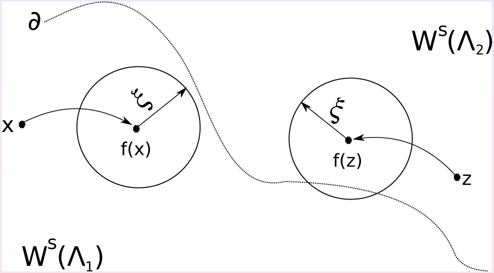
perturbed dynamics $\mathbf{F}(\mathbf{x}_j) = \mathbf{f}(\mathbf{x}_j) + \boldsymbol{\epsilon}_j$:



From attractors to hopping on pseudo attractors

Consequences of the random perturbations:

• beyond a noise threshold $\xi \geq \xi_0$ the attracting sets $W^s(\Lambda_i)$ lose their stability due to holes

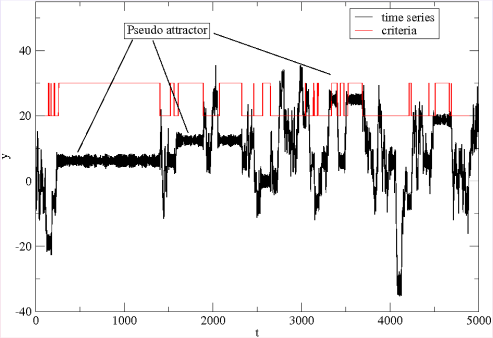


• the (invariant) attractors become (quasi-invariant) pseudo attractors from which there is noise-induced escape

• the noise induces a hopping process between all coexisting pseudo attractors

Intermittency and stickiness

the resulting perturbed dissipative dynamics is intermittent:

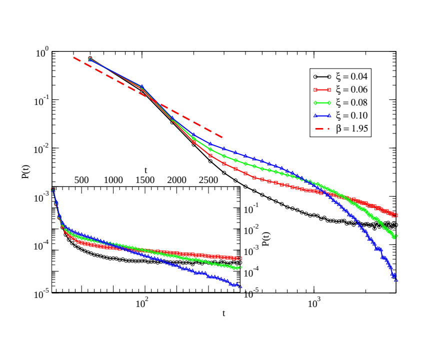


$$f_0 = 4\ ,\ \xi = 0.06\ ,\ \nu = 0.002$$

- stickiness to pseudo attractors measured by criterion that maximal eigenvalue of the Jacobian matrix along orbit < 1

Escape time distribution

probability distributions $P(t)$ of escape times t from pseudo attractors computed by using eigenvalue criterion (plus a Markov assumption and averaging over all non-uniform pseudo attractors):

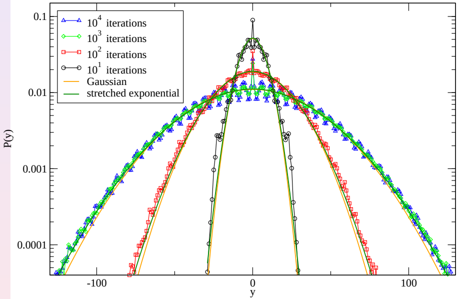


dissipation $\nu = 0.002$ with different noise strength ξ

- transition from power law (stickiness) to exponential
- transition takes longer when $\xi \to 0$

Diffusion

probability distribution function $P_n(y)$ for position y at different time steps n:

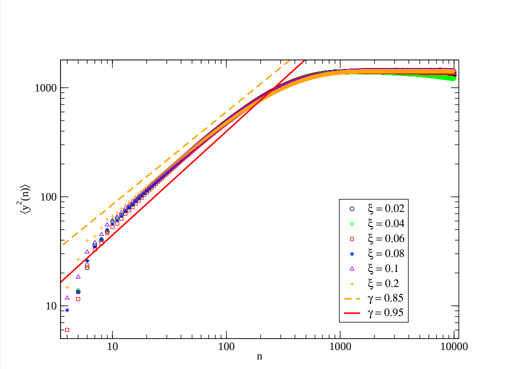


$\xi = 0.06\ ,\ \nu = 0.002$

- there is Gaussian-like diffusive spreading up to $n < 1000$
- localization trivially due to boundedness of pseudo attractors

Mean square displacement

mean square displacement $< y^2(n) >$ for position y and different noise amplitudes ξ at $\nu = 0.002$:



- transient subdiffusion $< y^2(n) > \sim n^\gamma$ up to $n < 1000$
- only small variation of the subdiffusive exponent $0.85 < \gamma < 0.95$ for different ξ

Continuous time random walk theory

reproduce simulation results by **CTRW theory** (Montroll, Weiss, Scher, 1973): define stochastic process by master equation with *waiting time distribution* $w(t)$ and *jump distribution* $\lambda(x)$

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt' \, w(t-t') \, \varrho(x',t') + \\ +(1-\textstyle\int_0^t dt' w(t'))\delta(x)$$

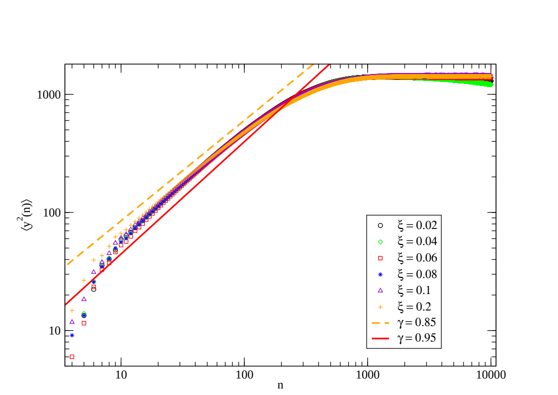
structure: jump + no jump for points starting at $(x,t)=(0,0)$
$\hat{\text{F}}$ourier-$\tilde{\text{L}}$aplace transform yields **Montroll-Weiss eqn (1965)**

$$\boxed{\hat{\tilde{\varrho}}(k,s) = \frac{1-\tilde{w}(s)}{s} \frac{1}{1-\hat{\lambda}(k)\tilde{w}(s)}}$$

with mean square displacement $\tilde{\langle x^2 \rangle}(s) = -\left.\frac{\partial^2 \hat{\tilde{\varrho}}(k,s)}{\partial k^2}\right|_{k=0}$

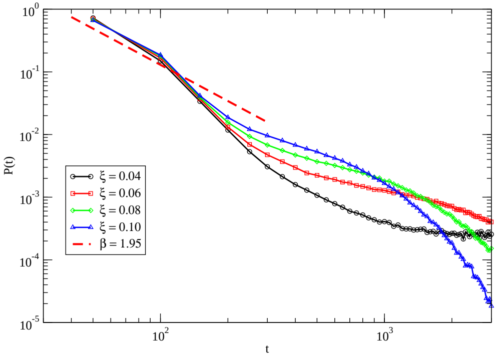
CTRW theory and mean square displacement

CTRW theory predicts that solving the MW eqn. for a power law waiting time distribution $w(t) \sim t^{-(\gamma+1)}$ with jump distribution $\lambda(x) = \delta(|x| - const.)$ yields $< x^2(t) >\sim t^\gamma$



for $\nu = 0.002$, $\xi = 0.06$ we have $< y_n^2 >\sim n^\gamma$ with $\gamma \simeq 0.95$

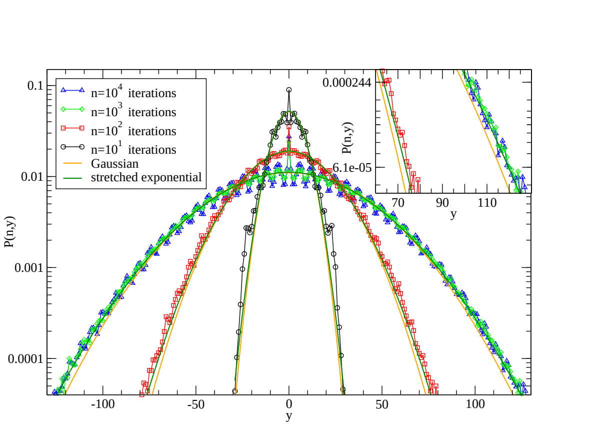
CTRW theory and escape time distribution



the dashed red line represents the CTRW theory prediction of $P(t) \sim t^{-1.95}$ corresponding to $< y^2(n) > \sim n^{0.95}$

CTRW theory and position pdf

CTRW theory also predicts a stretched exponential position pdf, here: $P_n(y) \sim \exp\left(-cx^{2/(2-\gamma)}\right)$



green lines represent the CTRW theory pdf for $\gamma = 0.95$: corrects the mismatch to Gaussian in the tails

Summary

- **central theme:** *diffusion* generated by *randomly perturbed deterministic dynamics*
- **main result:** for the dissipative standard map stickiness to pseudo attractors under random perturbations generates
  - *power law escape time distributions* and
  - *stretched exponential position distributions* leading to
  - *subdiffusion*

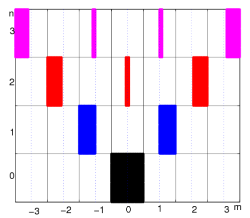
  simulation results consistently explained by *CTRW theory*

reference:
C.S.Rodrigues A.V.Chechkin, A.P.S. de Moura, C.Grebogi, RK, Europhys.Lett. **108**, 40002 (2014)

outlook: ∃ *generic mechanism* generating novel types of anomalous diffusion in randomly perturbed dynamical systems (Sato, RK, in prep.)

Wild and bad I

the **slicer map**:

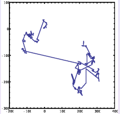


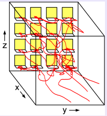
non-chaotic interval exchange transformation generating subdiffusive, diffusive and superdiffusive dynamics:

1. $\alpha = 0$: ballistic motion with $\langle x_n^2 \rangle \sim n^2$
2. $0 < \alpha < 1$: superdiffusion with MSD $\langle x_n^2 \rangle \sim n^{2-\alpha}$
3. $\alpha = 1$: normal diffusion with linear MSD $\langle x_n^2 \rangle \sim n$
4. $1 < \alpha < 2$: subdiffusion with MSD $\langle x_n^2 \rangle \sim n^{2-\alpha}$
5. $\alpha = 2$: logarithmic subdiffusion with MSD $\langle x_n^2 \rangle \sim \log n$
6. $\alpha > 2$: localisation in the MSD with $\langle x_n^2 \rangle \sim const.$

L. Salari, L. Rondoni, C. Giberti, R. Klages, Chaos **25**, 073113 (2015)

Wild and bad II





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