Diffusion in randomly perturbed dissipative dynamics

C.S. Rodrigues[1]   A.V. Chechkin[2]   A.P.S. de Moura[3]
C. Grebogi[3]   R. Klages[4]

[1]Max Planck Institute for Mathematics in the Sciences, Leipzig

[2]Institute for Theoretical Physics NSC KIPT, Kharkov, Ukraine

[3]Institute for Complex Systems and Mathematical Biology, University of Aberdeen

[4]Queen Mary University of London, School of Mathematical Sciences

Open Statistical Physics Annual Meeting
The Open University, 25 March 2015

Queen Mary
University of London

Outline

1. **Motivation: standard map, diffusion, and dissipation**

2. **Randomly perturbed dissipative dynamics**

3. **Randomly perturbed dissipative standard map**

4. **Continuous Time Random Walk theory**

The standard map and diffusion

- paradigmatic Hamiltonian dynamical system:

standard map

$$x_{n+1} = x_n + y_n \mod 2\pi$$
$$y_{n+1} = y_n + K \sin x_{n+1}$$

derived from kicked rot(at)or where $x_n \in \mathbb{R}$ is an angle, $y_n \in \mathbb{R}$ the angular velocity with $n \in \mathbb{N}$ and $K > 0$ the kick strength
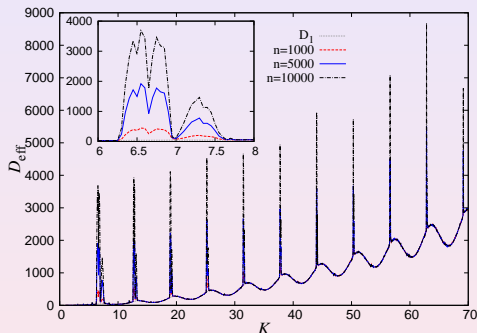
- define **diffusion coefficient** as

$$D(K) = \lim_{n \to \infty} \frac{1}{n} < (y_n - y_0)^2 >$$

with ensemble average over the initial density
$< \ldots > = \int dx\, dy\, \varrho(x, y) \ldots \, , \ x \in [0, 2\pi) \, , \ y = y_0 \in [0, 2\pi)$

Diffusion in the standard map

analytical (Rechester, White, 1980) and numerical studies of parameter-dependent diffusion $D_{eff}(K)$:

Manos, Robnik, PRE (2014)

- $D(K)$ is highly irregular
- for some K there is superdiffusion with mean square displacement $< y_n^2 > \sim n^\gamma$, $\gamma > 1$ due to accelerator modes
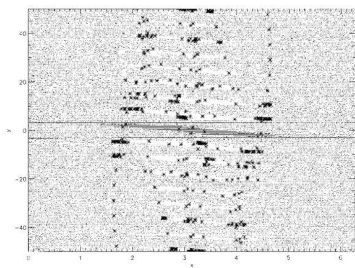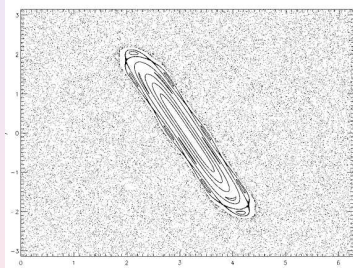
The dissipative standard map

model damping in the standard map by

$$x_{n+1} = x_n + y_n \mod 2\pi$$
$$y_{n+1} = (1 - \nu)y_n + f_0 \sin x_{n+1}$$

with $\nu \in [0, 1]$:

Feudel, Grebogi, Hunt, Yorke, PRE (1996)

• islands in phase space for $\nu = 0$ (left) become coexisting periodic attractors (right): 150 found for $\nu = 0.02$, $f_0 = 4$
• simple argument yields $|y_n| < y_{max}$: quick trapping

Dissipative dynamics and random perturbations

Question: What happens to dissipative deterministic dynamics $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ under random perturbations?
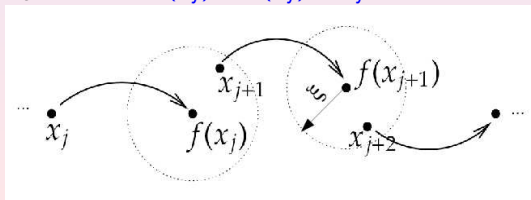
Consider the dissipative standard map with additive noise:

$$
\begin{aligned}
x_{n+1} &= x_n + y_n + \epsilon_{x,n} \mod 2\pi \\
y_{n+1} &= (1-\nu)y_n + f_0 \sin x_{n+1} + \epsilon_{y,n}
\end{aligned}
$$

with iid random variables $\epsilon_n = (\epsilon_{x,n}, \epsilon_{y,n})$ drawn from uniform distribution bounded by $||\epsilon_n|| < \xi$ of noise amplitude ξ
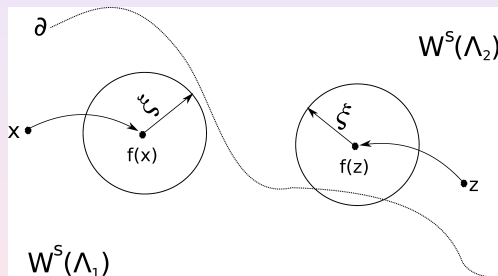
perturbed dynamics $\mathbf{F}(\mathbf{x}_j) = \mathbf{f}(\mathbf{x}_j) + \epsilon_j$:

From attractors to hopping on pseudo attractors

Consequences of the random perturbations:

• beyond a noise threshold $\xi \geq \xi_0$ the attracting sets $W^S(\Lambda_i)$ lose their stability due to holes

• the (invariant) attractors become (quasi-invariant) pseudo attractors from which there is noise-induced escape

• the noise induces a hopping process between all coexisting pseudo attractors

Intermittency and stickiness

the resulting perturbed dissipative dynamics is intermittent:
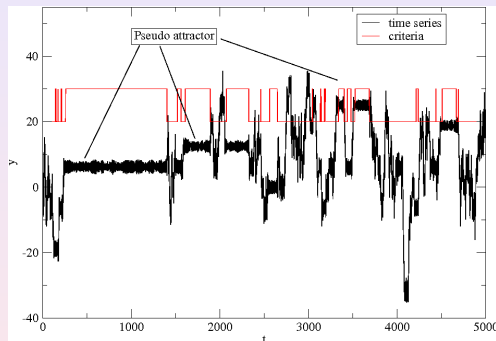
$$f_0 = 4 \,,\; \xi = 0.06 \,,\; \nu = 0.002$$

• stickiness to pseudo attractors measured by criterion that maximal eigenvalue of the Jacobian matrix along orbit < 1

Escape time distribution

probability distributions $P(t)$ of escape times t from pseudo attractors computed by using eigenvalue criterion (plus a Markov assumption and averaging over all non-uniform pseudo attractors):
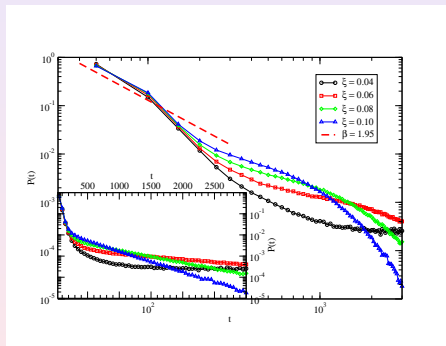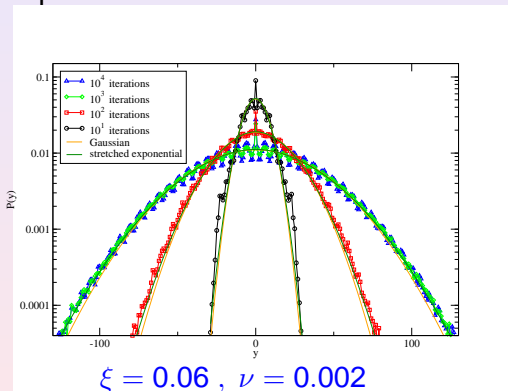

dissipation $\nu = 0.002$ with different noise strength ξ

- transition from power law (stickiness) to exponential
- transition takes longer when $\xi \to 0$

Diffusion

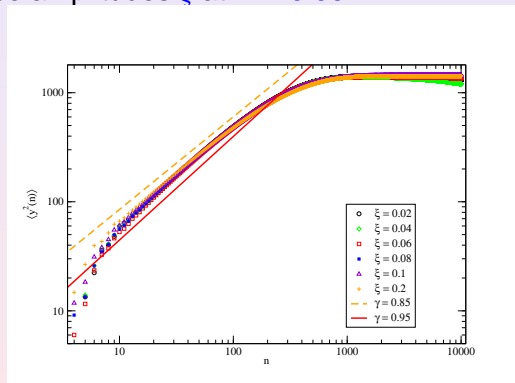**probability distribution function** $P_n(y)$ for position y at different time steps n:

$$\xi = 0.06 \, , \, \nu = 0.002$$

- there is Gaussian-like diffusive spreading up to $n < 1000$
- localization trivially due to boundedness of pseudo attractors

Mean square displacement

mean square displacement $<y^2(n)>$ for position y and different noise amplitudes ξ at $\nu = 0.002$:

- transient subdiffusion $<y^2(n)> \sim n^\gamma$ up to $n < 1000$
- only small variation of the subdiffusive exponent $0.85 < \gamma < 0.95$ for different ξ

Continuous time random walk theory

reproduce simulation results by **CTRW theory** (Montroll, Weiss, Scher, 1973): define stochastic process by master equation with *waiting time distribution $w(t)$* and *jump distribution $\lambda(x)$*

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt' \, w(t-t') \, \varrho(x',t') +$$
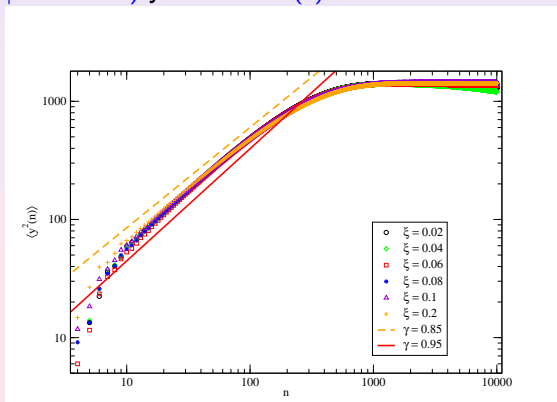$$+ (1 - \int_0^t dt' w(t')) \delta(x)$$

structure: jump + no jump for points starting at $(x,t) = (0,0)$
$\hat{\text{F}}$ourier-$\tilde{\text{L}}$aplace transform yields **Montroll-Weiss eqn (1965)**

$$\hat{\tilde{\varrho}}(k,s) = \frac{1 - \tilde{w}(s)}{s} \frac{1}{1 - \hat{\lambda}(k)\tilde{w}(s)}$$

with mean square displacement $\langle x^2(\tilde{s}) \rangle = -\frac{\partial^2 \hat{\tilde{\varrho}}(k,s)}{\partial k^2}\bigg|_{k=0}$
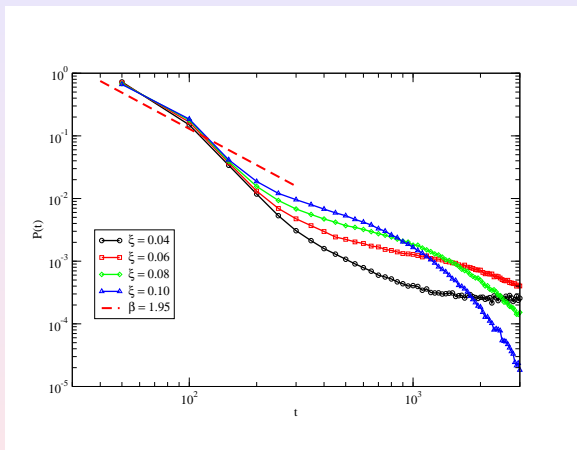
CTRW theory and mean square displacement

CTRW theory predicts that solving the MW eqn. for a power law waiting time distribution $w(t) \sim t^{-(\gamma+1)}$ with jump distribution $\lambda(x) = \delta(|x| - const.)$ yields $< x^2(t) > \sim t^\gamma$

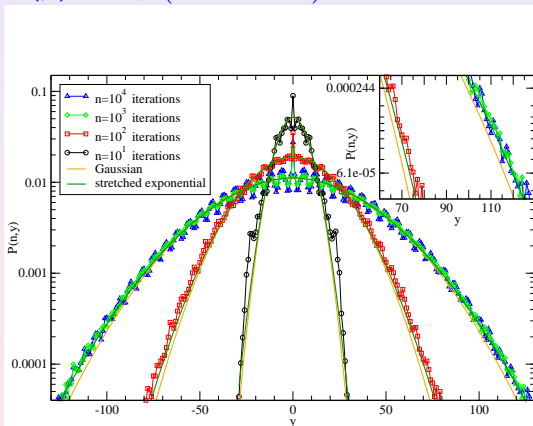for $\nu = 0.002$, $\xi = 0.06$ we have $< y_n^2 > \sim n^\gamma$ with $\gamma \simeq 0.95$

CTRW theory and escape time distribution

the dashed red line represents the CTRW theory prediction of $P(t) \sim t^{-1.95}$ corresponding to $< y^2(n) > \sim n^{0.95}$

CTRW theory and position pdf

CTRW theory also predicts a stretched exponential position pdf, here: $P_n(y) \sim \exp\left(-cx^{2/(2-\gamma)}\right)$

green lines represent the CTRW theory pdf for $\gamma = 0.95$: corrects the mismatch to Gaussian in the tails

Summary

- **central theme:** study of *diffusion generated by randomly perturbing dissipative deterministic dynamics*

- **main result:** for the dissipative standard map non-hyperbolic stickiness to pseudo attractors under random perturbations generates
  - *power law escape time distributions* and
  - *stretched exponential position distributions* leading to
  - *subdiffusion*

  simulation results consistently explained by *CTRW theory*

- **outlook:** similar phenomena in other randomly perturbed deterministic dynamical systems?

reference:
C.S.Rodrigues A.V.Chechkin, A.P.S. de Moura, C.Grebogi, R.Klages, Europhys.Lett. **108**, 40002 (2014)