A simple non-chaotic map generating subdiffusive, diffusive and superdiffusive dynamics

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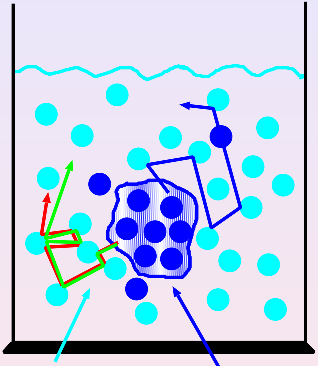
Outline

1. **Motivation:** chaos, diffusion and polygonal billiards
2. **Model:** a simple non-chaotic map with non-trivial diffusive properties
3. **Summary:** match results from the deterministic model to stochastic theory

Microscopic chaos in a glass of water?



- dispersion of a droplet of ink by diffusion
- chaotic collisions between billiard balls
- **chaotic hypothesis:**

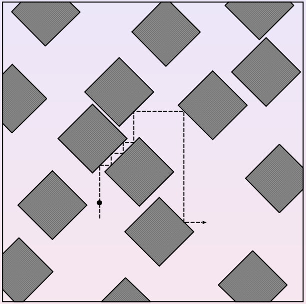
microscopic chaos
⇓
macroscopic diffusion

Gallavotti, Cohen (1995)

P.Gaspard et al. (1998): experiment on small colloidal particle in water; diffusion due to microscopic chaos based on positive *pattern entropy per unit time* $h(\epsilon, \tau) \leq h_{KS} = \sum_{\lambda_i > 0} \lambda_i$

The random wind tree model

counterexample:



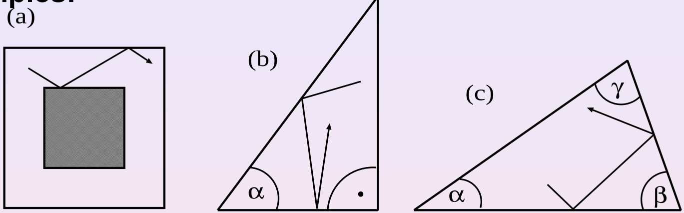
Ehrenfest, Ehrenfest (1959)

no positive Lyapunov exponent, hence non-chaotic dynamics

Dettmann et al. (1999): generates trajectories and $h(\epsilon, \tau)$ *indistinguishable from the colloidal particle*

Polygonal billiards

examples:



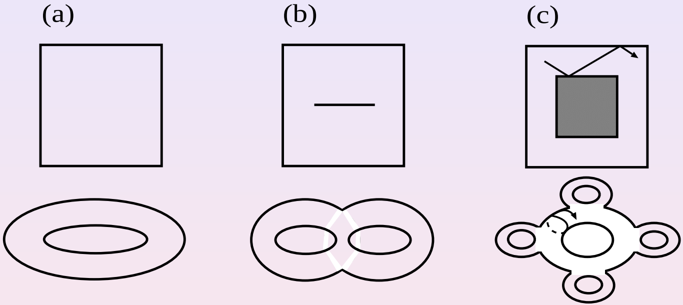
Artuso et al. (1997,2000); Casati et al. (1999)

rational billiards: all angles are rational multiples of π
irrational billiards: otherwise

non-trivial ergodic properties: rational billiards are not ergodic; phase space splits into invariant manifolds wrt initial angle of trajectory (e.g., Gutkin, 1996)

Pseudointegrability

joining all identical edges yields compact invariant surfaces:

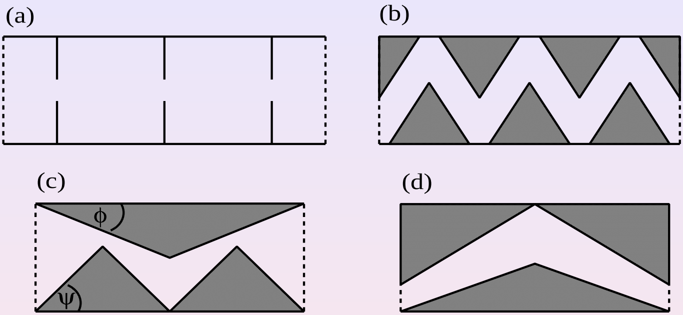


genus $g = 1$: billiard is *integrable*

$g > 1$: pseudointegrable (Richens, Berry, 1981); $\exists$ isolated saddles resembling hyperbolic fixed points imposing a 'chaotic character' onto the flow

asymptotic growth of displacement of two trajectories $\Delta(t) \sim t$

Diffusion in polygonal billiard channels



Zwanzig (1983), Zaslavsky et al. (2001), Li et al. (2002)

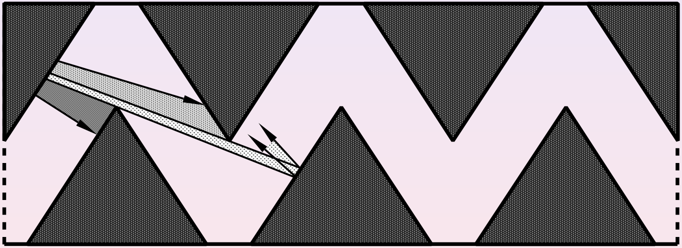
mean square displacement $< x^2 >:= \int dx\, x^2 \rho(x,t) \sim t^\gamma$
from simulations: sub- ($\gamma < 1$), super- ($\gamma > 1$) or normal ($\gamma = 1$) diffusion depending on parameters; partially conflicting results

Alonso et al. (2002), Jepps et al. (2006), Sanders et al. (2006)

Particle dispersion in polygonal billiards

simple picture:

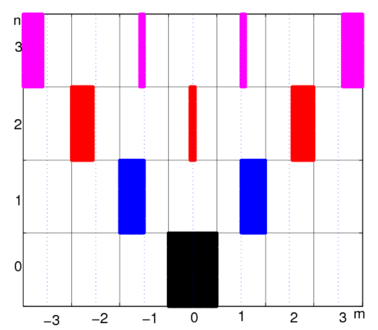
mechanism generating diffusion in these channels may be crucially determined by how scatterers slice a beam



this may in turn be captured by interval exchange transformations (Hannay, McCraw, 1990)

The slicer map I: basic idea

a simple one-dimensional *spatially dependent* interval exchange transformation:



zero Lyaponuv exponent: different points neither converge nor diverge from each other in time; slicer points are of Lebesgue measure zero; hence non-chaotic dynamics

The slicer map II: definition of slicers

consider a chain of intervals $\widehat{M} := M \times \mathbb{Z}$, $M := [0,1]$

product measure $\hat{\mu} := \lambda \times \delta_{\mathbb{Z}}$ on $\widehat{M}$ with Lebesgue measure λ on M and Dirac measure $\delta_{\mathbb{Z}}$ on integers

$\widehat{X} = (x, m)$ is a point in $\widehat{M}$ and $\widehat{M}_m := M \times \{m\}$ the m-th cell of $\widehat{M}$

subdivide each $\widehat{M}_m$ in 4 subintervals, separated by 3 points called **slicers**: $\{1/2\} \times \{m\}$, $\{\ell_m\} \times \{m\}$, $\{1 - \ell_m\} \times \{m\}$, where $0 < \ell_m < 1/2$ for every $m \in \mathbb{Z}$

The slicer map III: definition of model

slicer model:

the dynamical system $(\widehat{M}, \hat{\mu}, S)$ which, at each time step $n \in \mathbb{N}$, moves all subintervals from their cells to neighbouring cells by the rule $S : \widehat{M} \to \widehat{M}$,

$$S(x,m) = \begin{cases} (x, m-1) & \text{if } 0 \leq x < \ell_m \text{ or } \frac{1}{2} < x \leq 1 - \ell_m, \\ (x, m+1) & \text{if } \ell_m \leq x \leq \frac{1}{2} \text{ or } 1 - \ell_m < x \leq 1. \end{cases}$$

family of slicers:

$$L_\alpha = \left\{ \Big(\ell_j(\alpha), 1 - \ell_j(\alpha)\Big) : \ell_j(\alpha) = \frac{1}{\left(|j| + 2^{1/\alpha}\right)^\alpha}, j \in \mathbb{Z} \right\}, \ \alpha > 0$$

slicer map S_α: all slicers belong to L_α, $\ell_m = \ell_m(\alpha)$

Spreading of points in the slicer model

take an ensemble of points $\widehat{E}_0$ in the central cell $\widehat{M}_0 = M \times \{0\}$ and study how S_α spreads them in $\widehat{M}$

at time n they reach $\widehat{M}_n$ and $\widehat{M}_{-n}$; cells occupied at time n are odd/even if n odd/even

with

$$P_n = \{j \in \mathbb{Z} : j \text{ is even and } |j| \leq n\}$$

$$D_n = \{j \in \mathbb{Z} : j \text{ is odd and } |j| \leq n\}$$

we have

$$S_\alpha^n \widehat{M}_0 = \bigcup_{j \in P_n} (R_j \times \{j\}) \quad \text{if } n \text{ is even}$$

$$S_\alpha^n \widehat{M}_0 = \bigcup_{j \in D_n} (R_j \times \{j\}) \quad \text{if } n \text{ is odd}$$

with union of intervals $R_j \times \{j\} \subset \widehat{M}_j$; $R_j \subset M$, $R_i \cap R_j = \emptyset$ if $i \neq j$

Measure and density under slicer action

consider probability measure $d\nu_0 := \hat{\rho}_0(\widehat{X})d\hat{\mu}$ on $\widehat{M}$ with density

$$\hat{\rho}_0\left(\widehat{X}\right) = \begin{cases} 1, & \text{if } \widehat{X} \in \widehat{M}_0 \\ 0, & \text{otherwise} \end{cases}$$

which evolves under the action of S_α as

$$\hat{\rho}_n(\widehat{X}) = \begin{cases} 1 & \text{if } \widehat{X} \in S_\alpha^n \widehat{M}_0 \\ 0 & \text{otherwise} \end{cases}$$

the sets $\widehat{R}_j := S_\alpha^n \widehat{M_0} \cap \widehat{M}_j$, $j = -n, \ldots, n$, constitute the total phase space volume occupied at time n in cell $\widehat{M}_j$

the measure of $\widehat{R}_j$ equals the probability of cell j at time n, $A_j := \hat{\mu}(\widehat{R}_j) = \nu_n(\widehat{M}_j)$, yielding the coarse grained distribution

$$\rho_n^G(j) = \begin{cases} A_j & \text{if } j \in \{-n, \ldots, n\}, \\ 0 & \text{otherwise} \end{cases}$$

Diffusion in the slicer map

define the **mean square displacement** based on ρ_n^G:

$$\langle \Delta \hat{X}_n^2 \rangle := \sum_{j=-n}^{n} A_j j^2 \ ,$$

where j is the distance travelled by a point in $\widehat{M}_j$ at time n

for $\gamma \in [0,2]$ define

$$T_\alpha(\gamma) := \lim_{n\to\infty} \frac{\langle \Delta \hat{X}_n^2 \rangle}{n^\gamma}$$

if $T_\alpha(\gamma^t) \in (0,\infty)$ for $\gamma^t \in [0,2]$, γ^t is called the transport exponent of the slicer dynamics with generalized diffusion coefficient $T_\alpha(\gamma^t)$

The diffusing slicer density

For even $n > 2$ we get

$$\rho_n^G(j) = \begin{cases} 2(\ell_0 - \ell_1)\,, & j = 0 \\ \ell_{|2k-1|} - \ell_{|2k+1|}\,, & |j| = 2k,\ k = 1, \ldots, \frac{n-2}{2} \\ \ell_{|n-1|}\,, & |j| = n \\ 0\,, & \text{elsewhere} \end{cases}$$

and for odd $n > 3$

$$\rho_n^G(j) = \begin{cases} \ell_{|2k|} - \ell_{|2k+2|}\,, & |j| = 2k+1,\ k = 0, \ldots, \frac{n-3}{2} \\ \ell_{|n-1|}\,, & |j| = n \\ 0\,, & \text{elsewhere} \end{cases}$$

put in definition of slicer: for $\alpha \in [0, 2)$ and large n, j the tails correspond to **Lévy stable distributions**,

$$\rho_n^G(j) \sim 2\alpha/|j|^{\alpha+1}\mathbb{I}_{\{|j|<n\}}\,,$$

except in the *traveling regions* $j = \pm n$

Anomalous diffusion in the slicer

Proposition

Given $\alpha \in [0, 2)$ *and a uniform initial distribution in* $\widehat{M}_0$*, we have*

$$T_\alpha(\gamma) = \begin{cases} +\infty & \text{if } 0 \le \gamma < 2 - \alpha \\ \frac{4}{2-\alpha} & \text{if } \gamma = 2 - \alpha \\ 0 & \text{if } 2 - \alpha < \gamma \le 2 \end{cases},$$

hence the transport exponent γ^t *takes the value* $2 - \alpha$ *with* $\langle \Delta \hat{X}_n^2 \rangle \sim n^{2-\alpha}$. *For* $\alpha = 2$ *the transport regime is* logarithmically diffusive, *i.e.* $\langle \Delta \hat{X}_n^2 \rangle \sim \log n$ *asymptotically in* n.

for $\alpha > 2$ it is $\langle \Delta \hat{X}_n^2 \rangle \to const.$ $(n \to \infty)$, i.e., localisation sets in

The higher order moments in the slicer

Theorem

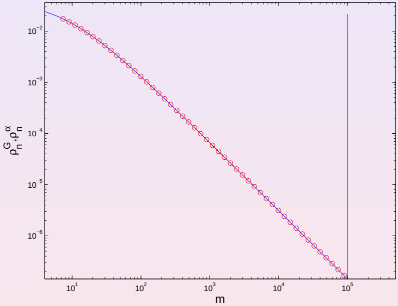
For $\alpha \in (0,2]$ *the moments* $\langle \Delta \hat{X}_n^p \rangle = \sum_{j=-n}^{n} A_j j^p$ *with* $p > 2$ *even and initial condition uniform in* $\widehat{M}_0$ *have the asymptotic behavior*

$$\langle \Delta \hat{X}_n^p \rangle \sim n^{p-\alpha}$$

while the odd moments $(p = 1, 3, \ldots)$ *vanish.*

Example: $\alpha = 1/3$

we have $\langle \Delta \widehat{X}_n^p \rangle \sim n^{p-1/3}$ and especially $\langle \Delta \widehat{X}_n^2 \rangle \sim n^{5/3}$:
superdiffusion; plot of analytic $\rho_n^G(m)$ (continuous line):



cp. with asymptotics: $\rho_n^\alpha(m) = \begin{cases} \dfrac{C_\alpha}{(m + 2^{1/\alpha})^{\alpha+1}} \ , & m < n \\ 0 \ , & m > n \end{cases}$

with normalisation C_α; note peak in the traveling area

Summary

slicer map generates subdiffusive, diffusive and superdiffusive dynamics:

1. $\alpha = 0$: ballistic motion with $\langle x_n^2 \rangle \sim n^2$
2. $0 < \alpha < 1$: superdiffusion with MSD $\langle x_n^2 \rangle \sim n^{2-\alpha}$
3. $\alpha = 1$: normal diffusion with linear MSD $\langle x_n^2 \rangle \sim n$
   note: non-chaotic normal diffusion with non-Gaussian density
4. $1 < \alpha < 2$: subdiffusion with MSD $\langle x_n^2 \rangle \sim n^{2-\alpha}$
   note: subdiffusion with ballistic peaks
5. $\alpha = 2$: logarithmic subdiffusion with MSD $\langle x_n^2 \rangle \sim \log n$
6. $\alpha > 2$: localisation in the MSD with $\langle x_n^2 \rangle \sim const.$

Matching to stochastic dynamics?

- one-dimensional stochastic Lévy Lorentz gas:

point particle moves ballistically between static point scatterers on a line from which it is transmitted / reflected with probability $1/2$

distance r between two scatterers is a random variable iid from Lévy distribution, $\lambda(r) \equiv \beta r_0^\beta \frac{1}{r^{\beta+1}}$, $r \in [r_0, +\infty)$ $\beta > 0$ and cutoff r_0

$\rightarrow$ model exhibits only superdiffusion

$\rightarrow$ all moments scale with the slicer moments for $\alpha \in (0, 1]$ (piecewise linearly depending on parameters)

Matching to stochastic dynamics?

- Lévy walk modeled by CTRW theory:

moments calculated to $\sim t^{p+1-\beta}$ for $p > \beta$, $1 < \beta < 2$
matches to slicer superdiffusion with $\beta = 1 + \alpha$
but conceptually a totally different process

- correlated Gaussian stochastic processes:

modeled by a generalized Langevin equation with a power law memory kernel
formal analogy in the subdiffusive regime
but Gaussian distribution and a conceptual mismatch

Summary

- **central theme:**
  diffusion generated by *non-chaotic dynamics*
- **main result:**
  slicer model generates 6 different types of diffusive dynamics under parameter variation covering the whole spectrum of diffusion
- this result might help to explain a **controversy about different stochastic models for diffusion in polygonal billiards**: sensitive dependence of diffusion on parameters matching to different stochastic processes

References

- slicer:

L.Salari, L.Rondoni, C.Giberti, RK, Chaos **25**, 073113 (2015)

- review about polygonal billiards: Section 17.4 in

R.Klages, *Microscopic Chaos, Fractals and Transport in Nonequilibrium Statistical Mechanics* (World Scientific, 2007)