Chaotic diffusion in randomly perturbed dynamical systems

or: A simple answer to a simple question

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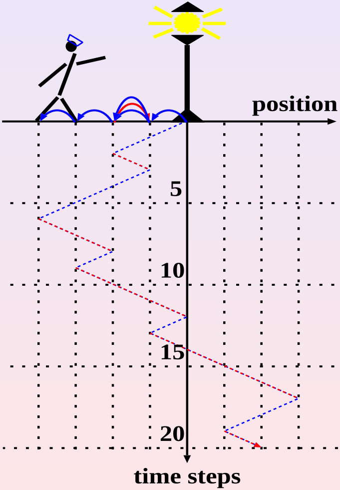
Outline

1. **Motivation:**
   random walk, deterministic diffusion and fractal diffusion coefficients
2. **Deterministic diffusion and random perturbations:**
   four different types of perturbations; computer simulations in comparison to simple analytical approximations
3. **Theoretical details:**
   derivation and discussion of the approximate formulas

The drunken sailor at a lamppost

random walk in one dimension (K. Pearson, 1905):



- steps of length s with probability $p(\pm s) = 1/2$ to the left/right
- single steps *uncorrelated*: Markov process
- define diffusion coefficient as

$$\boxed{D := \lim_{n\to\infty} \frac{1}{2n} < (x_n - x_0)^2 >_\varrho}$$

with discrete time step $n \in \mathbb{N}$ and average over the initial density $< \ldots >_\varrho := \int dx \ \varrho(x) \ldots$ of positions $x = x_0 \ , \ x \in \mathbb{R}$

- for *sailor:* $D = s^2/2$

A deterministic random walk

study **diffusion** on the basis of **dynamical systems theory**

piecewise linear deterministic map

$$M_a(x) = \begin{cases} ax \,, \; 0 < x \le \frac{1}{2} \\ ax + 1 - a \,, \; \frac{1}{2} < x \le 1 \end{cases}$$

lifted onto the real line by

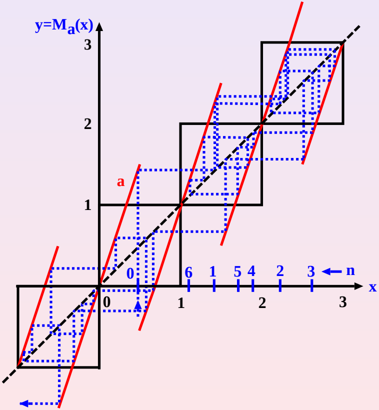
$$M_a(x+1) = M_a(x) + 1$$

with control parameter $a \ge 2$

equation of motion:

$$\boxed{x_{n+1} = M_a(x_n)}$$

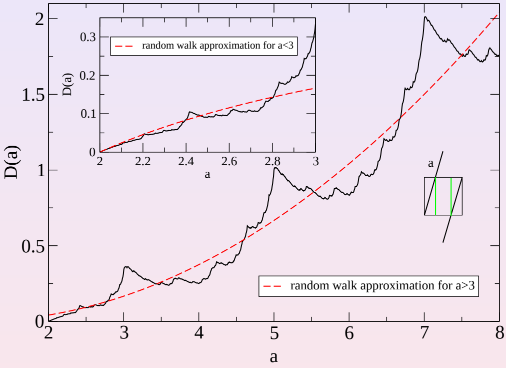
Lyapunov exponent $\lambda = \ln a > 0$:
map is chaotic



Grossmann/Geisel/Kapral (1982)

Fractal diffusion coefficient and random walks

$D(a)$ exists and is a **fractal function of the control parameter**



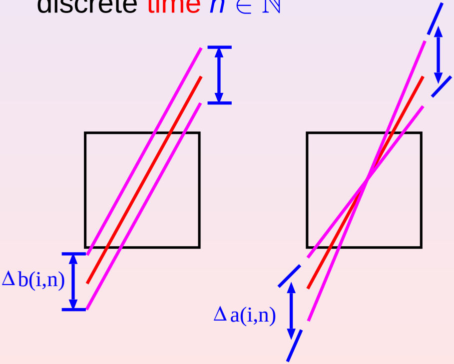
- exact results (R.K., Dorfman, 1995; Groeneveld, R.K., 2002)
- proof: $D(a)$ is Lipschitz continuous up to quadratic logarithmic corrections (Keller, Howard, R.K., 2008)
- cp. random walk with exact results (R.K., Dorfman, 1997)

Deterministic diffusion and random perturbations

What happens to $D(a)$ by imposing **random perturbations** onto the map? Four basic types:

$$\boxed{M_a(x) = (a + \Delta a(i,n))\,x + \Delta b(i,n)}$$

with random shifts or random slopes in discrete space $i \in \mathbb{Z}$ or discrete time $n \in \mathbb{N}$



(1) $\Delta b(i)$: quenched shifts
(2) $\Delta a(i)$: quenched slopes
(3) $\Delta a(n)$: noisy slopes
(4) $\Delta b(n)$: noisy shifts

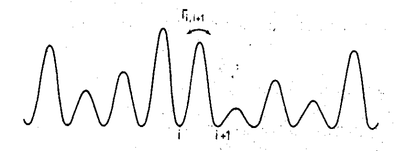
Deterministic diffusion and quenched disorder

(1): quenched random shifts $\Delta b(i)$; Radons (1996ff):

Golosov localization/Sinai diffusion: no normal diffusion

(2): quenched random slopes $\Delta a(i)$

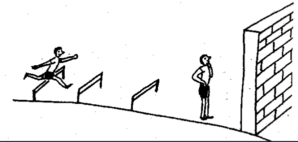
random walk in disordered lattices like random barrier model



transition rates $\Gamma_{i,i+1} = \Gamma_{i+1,i} \equiv \Gamma_i$; exact diffusion coefficient

$\boxed{d = < 1/\Gamma >_{\Gamma}^{-1} s^2}$ Alexander et al (1981), Derrida (1983), ...

disorder average $< 1/\Gamma >_{\Gamma} = 1/N \sum_{i=0}^{N} 1/\Gamma_i$, distance of sites s



Haus, Kehr (1987)

Quenched slopes: 'theory'

apply formula to **deterministic diffusion with quenched slopes**:

rewrite $d =< 1/\Gamma >_\Gamma^{-1} s^2 =< 1/d(s,\Gamma) >_\Gamma^{-1}$ with $d(s,\Gamma) = \Gamma s^2$

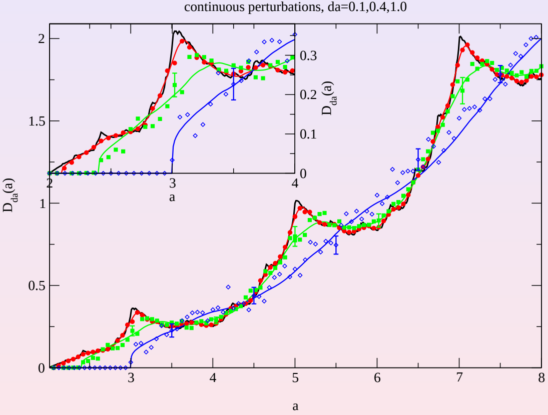
but for the map $M_a(x)$ we have $d(s,\Gamma) = d(s,\Gamma(s))$

approximation: identify $d(s,\Gamma(s))$ with $D(a+\Delta a)$, where $D(\cdot)$ is the unperturbed deterministic diffusion coefficient $D(a)$ with random variable Δa sampled from pdf $\chi_{da}(\Delta a)$ at perturbation strength $da \geq 0$, $-da \leq \Delta a \leq da$

$$\Rightarrow \boxed{D_{\mathrm{app}}(a,da) = \left[\int_{-da}^{da} d(\Delta a)\, \frac{\chi_{da}(\Delta a)}{D(a+\Delta a)}\right]^{-1}}$$

Quenched slopes: simulations $D_{da}(a)$

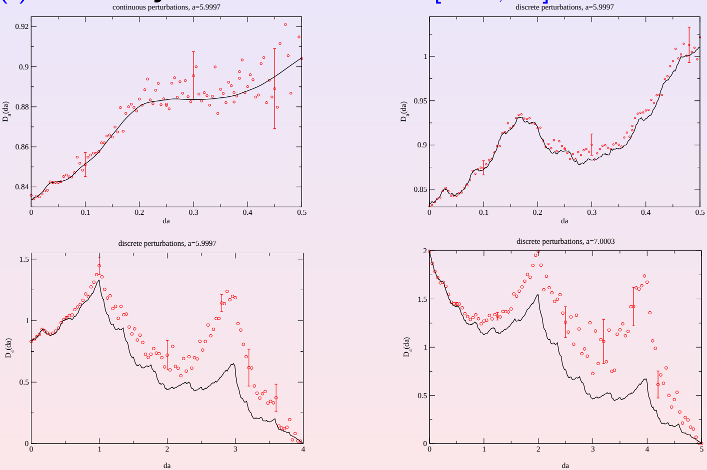
$\Delta a(i)$ **uniformly distributed** on $[-da, da]$:



- oscillatory structure persists under small perturbations
- dynamical phase transition for small parameters

Quenched slopes: simulations $D_a(da)$

$\Delta a(i)$ **uniformly** or δ**-distributed** on $[-da, da]$:

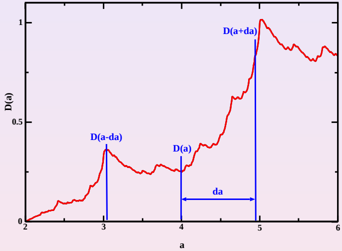


- excellent match theory and simulations for small da
- multiple suppression and enhancement with da

Deterministic diffusion and noise

approximate the diffusion coefficient by the unperturbed $D(a)$ (cp. to Geisel et al. (1982), Reimann (1994ff)):

basic idea for slopes with iid dichotomous noise $\pm da$:



$$\boxed{D_{\mathrm{app}}(a,da)=(D(a-da)+D(a+da))/2}$$

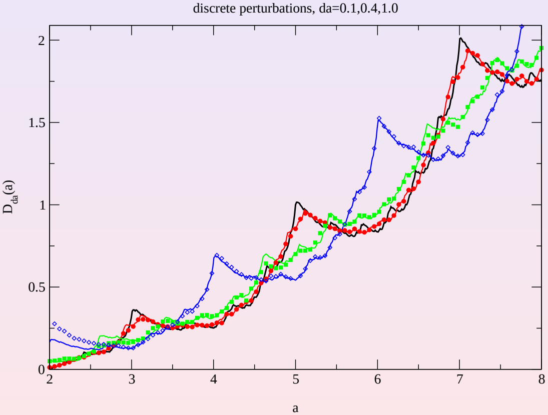
for a *general disorder distribution* $\chi_{da}(\Delta a)$

$$\boxed{D_{\mathrm{app}}(a,da)=\int_{-da}^{da} d(\Delta a)\,\chi_{da}(\Delta a)D(a+\Delta a)}$$

- cp. to quenched diffusion formula expanded for $da\to 0$

Noisy slopes: simulations $D_{da}(a)$

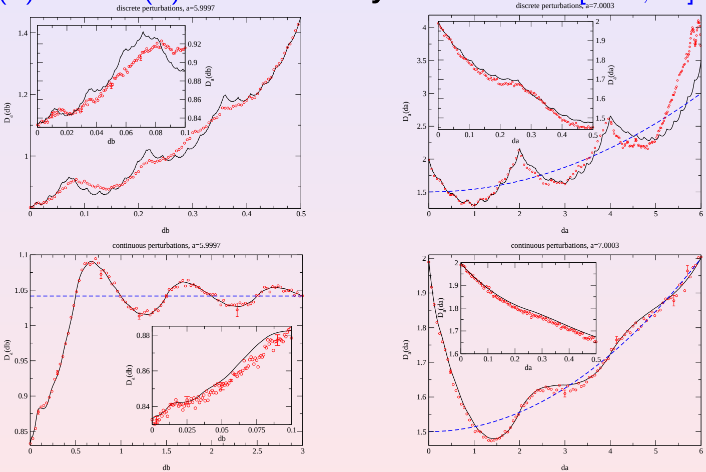
$\Delta a(n)$ δ**-distributed** with $\pm da$:



- shift of fractal structure under perturbations

Noisy slopes: simulations $D_a(da)$

$\Delta a(n)$ and $\Delta b(n)$ δ- or **uniformly** distributed on $[-da, da]$:



- transitions from deterministic to stochastic diffusion by suppression and enhancement

Theoretical aspects: precise definition of D(a,da)

Let Δ_n be iid random variables with pdf $\chi_d(\Delta_n)$ of perturbation strength $d \geq 0$. Let $\varrho(x)$ be the initial pdf of points x. Then

$$D(a,d) = \lim_{n\to\infty} \frac{1}{2n} \left(< x_n^2 >_{\varrho,\chi_d} - < x_n >^2_{\varrho,\chi_d} \right)$$

with

$$\begin{aligned} < x_n^k >_{\varrho,\chi_d} &= \int dx \int d(\Delta_0) d(\Delta_1) \ldots d(\Delta_{n-1}) \\ &\quad \varrho(x)\chi_d(\Delta_0)\chi_d(\Delta_1)\ldots\chi_d(\Delta_{n-1}) x_n^k \end{aligned}$$

special case: noisy slopes as an example (wlog),

$$d = da\,,\ \Delta_n = \Delta a_n \Rightarrow < x_n >_{\varrho,\chi_d} = 0$$

Deriving approximations in terms of the exact D(a)

Let Δa_n be uniformly distributed in $[-da, da]$ and $\Delta a = \Delta a_0$.

It holds: $\boxed{\Delta a_{n-1} = \Delta a + \epsilon\,,\ -2da \le \epsilon \le 2da}$

notation: $x_{n,a+\Delta a_{n-1}} = M_{a+\Delta a_{n-1}}(x_{n-1})$

step 1: $da \ll 1 \Rightarrow \epsilon \ll 1 \Rightarrow \Delta a_{n-1} \simeq \Delta a$

$$\begin{aligned} < x_n^2 >_{\varrho,\chi_{da}} &= \int dx \int d(\Delta a) d(\Delta a_1)\ldots d(\Delta a_{n-1}) \\ &\quad \rho_0(x)\chi_{da}(\Delta a)\chi_{da}(\Delta a_1)\ldots\chi_{da}(\Delta a_{n-1}) x^2_{n,a+\Delta a_{n-1}} \\ &= \int dx \int d(\Delta a)\varrho(x)\chi_{da}(\Delta a) x^2_{n,a+\Delta a} \end{aligned}$$

step 2: exchange time limit with integration

$$\begin{aligned} D_{app}(a,da) &= \lim_{n\to\infty} < x_n^2 >_{\varrho,\chi_{da}} /(2n) \\ &= \int d(\Delta a)\chi_{da}(\Delta a) \lim_{n\to\infty} \int dx\, \varrho(x) x^2_{n,a+\Delta a}/(2n) \\ &= \int d(\Delta a)\chi_{da}(\Delta a) D(a+\Delta a, 0) \end{aligned}$$

Validity of this approximation

note:

$\boxed{\text{This approximation needs to be handled with much care!}}$

- does not work for **quenched shifts**
- works for **noisy shifts**, but *only* in the limit of *very small* perturbations, because $\exists$ current; cp. to numerical results
- works well for **noisy slopes** in the limit of small perturbations

Random walk approximation

unperturbed diffusion coefficient

$$D(a) = \lim_{n\to\infty} \frac{1}{2n} \int_0^1 dx\, \varrho_a^*(x)(x_n - x)^2$$

with invariant density $\varrho_a^*(x)$ of map $M_a(x) \mod 1$

random walk approximation:

1. no memory in jumps: replace $\Delta x_n = x_n - x$ by single jump at time $n = 1$ over distance $\Delta x = \Delta x_1$
2. no memory in invariant density: assume $\rho_a^*(x) \simeq 1$

$$\Rightarrow D_{rw}(a) = \frac{1}{2} \int_0^1 dx\, \Delta x^2$$

Deriving random walk approximations

$$D_{rw}(a) = \frac{1}{2}\int_0^1 dx\, \Delta x^2$$

consider **two limiting cases of jumps:**

• random walk I: $a \ll 3$;
set $\Delta x = 0$ if $0 \le M_a(x) \le 1$ and $|\Delta x| = 1$ otherwise,
$D_{rwI}(a) = 1/2 \int_{0,\forall \Delta x=1}^1 dx = (a-2)/(2a) \simeq (a-2)/4 \; (a \to 2)$

• random walk II: $a \gg 3$; set $\Delta x = M_a(x) - x$ exactly,
$D_{rwII}(a) = 1/2 \int_0^1 dx (M_a(x) - x)^2 = (a-1)^2/24$

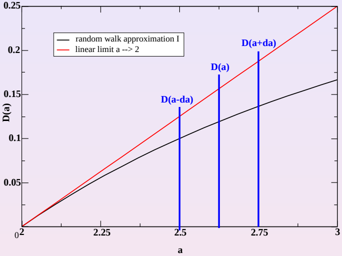
feed back results into **perturbed random walk** approximation

$$D_{app}(a, da) = \int d(\Delta a)\chi_{da}(\Delta a) D(a + \Delta a)$$

by replacing $D(a + \Delta a) \to D_{rw}(a + \Delta a)$

Perturbed random walk approximations

example: Let random slopes Δa be δ-distributed,



• for $2 \ll a \leq 3$ random walk I yields suppression of diffusion due to concavity (Reimann, 1994ff)

• for $a \to 2$ random walk I yields no change at all in the diffusion coefficient due to linearity

• for $3 \ll a$ random walk II yields enhancement of diffusion due to convexity, $\boxed{D_{rwII}(a, da) = D_{rwII}(a) + \Delta a^2/24}$

• in simulations random walk II well seen but not I

Summary

simple model with diffusion due to deterministic chaos under random perturbations:

1. in three of four cases of random perturbations the **diffusion coefficient exists**

2. the unperturbed **fractal diffusion coefficient** is **quite stable** against perturbations

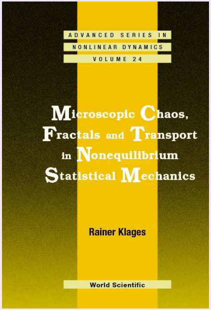
3. $\exists$ simple **approximations for the perturbed diffusion coefficient** in terms of the unperturbed diffusion coefficient

4. multiple (irregular) **suppression and enhancement of deterministic diffusion by stochastic perturbations**

5. transitions from deterministic to stochastic diffusion via suppresion and enhancement

Literature

- spatial disorder: R.K., PRE **65**, 055203(R) (2002)
- noise: R.K., EPL **57**, 796 (2002)
- all together:



see Part 1; file of this talk, details and taster sections on
www.maths.qmul.ac.uk/~klages