Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions

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Abstract. We consider transfer operators acting on spaces of holomorphic functions, and provide explicit bounds for their eigenvalues. More precisely, if Ω is any open set in $\mathbb{C}^d$, and $\mathcal{L}$ is a suitable transfer operator acting on Bergman space $A^2(\Omega)$, its eigenvalue sequence $\{\lambda_n(\mathcal{L})\}$ is bounded by $|\lambda_n(\mathcal{L})| \leq A \exp(-an^{1/d})$, where $a, A > 0$ are explicitly given.

1. Introduction

The study of transfer operators acting on spaces of holomorphic functions was initiated by Ruelle [**Rue3**] in 1976. He showed that certain dynamical zeta functions, including those of Artin-Mazur [**AM**] and Smale [**Sma**], could be expressed in terms of the determinant of such operators. The setting for Ruelle's theory is (the complexification of) a real analytic expanding map. If $(\phi_i)_{i\in\mathcal{I}}$ are the local inverse branches of this map, and $(w_i)_{i\in\mathcal{I}}$ is a suitable collection of holomorphic functions, then the associated *transfer operator* $\mathcal{L}$, defined by

$$(\mathcal{L}f)(z) = \sum_{i\in\mathcal{I}} w_i(z) f(\phi_i(z))\,, \tag{1}$$

preserves the space of functions holomorphic on some appropriate open subset Ω of d-dimensional complex Euclidean space.

Transfer operators of this form arise in statistical mechanics (see [**Rue2**]), and have been applied to hyperbolic dynamical systems, notably by Ruelle [**Rue1**], Sinai [**Sin**], and Bowen [**Bow**], as part of their program of thermodynamic formalism (cf. [**Rue4**]). Up until 1976 the setting for this formalism was symbolic dynamics: a hyperbolic system can be coded by a subshift of finite type Σ, and the transfer operator $\mathcal{L}$ preserves the space of Lipschitz functions on Σ. If the functions w_i are positive then $\mathcal{L}$ inherits a positivity property, and an infinite dimensional analogue of the Perron-Frobenius theorem can be established (cf. [**Rue1**]): the leading eigenvalue of $\mathcal{L}$ is simple, positive, and isolated. This leads to important ergodic-theoretic information (e.g. exponential decay of correlations) about a wide class of invariant measures (equilibrium states). Variations on this result have continued to be a fruitful area of active development (see [**Bal**] for a comprehensive overview), with transfer operators studied on various other spaces, notably C^k spaces [**Rue7, Rue8**], and the space of functions of bounded variation [**LY, HK, BG**]. In each of these cases $\mathcal{L}$, although not a compact operator, does enjoy the Perron-Frobenius property of having an isolated and positive dominant eigenvalue. In the case where $\mathcal{L}$ acts on certain *holomorphic* function spaces, however, Ruelle [**Rue3**] showed that it enjoys much stronger properties. In particular $\mathcal{L}$ is compact, so that its spectrum is a sequence $\{\lambda_n(\mathcal{L})\}$ converging to zero, together with zero itself.

The present article is concerned with obtaining completely explicit upper bounds on the eigenvalue moduli $|\lambda_n(\mathcal{L})|$, ordered by decreasing modulus and counting algebraic multiplicities. Spectral estimates of this kind have a long history (see e.g. [**Pie3**, Ch. 7]), and the theory is particularly well developed in the case where $\mathcal{L}$ is the Laplacian, or more generally a selfadjoint differential operator. Relatively little is known in the non-selfadjoint case, however, and existing explicit bounds on the eigenvalues of transfer operators are mainly restricted to the first two eigenvalues, where positivity arguments can be employed.

Explicit information on the spectrum of transfer operators is desirable for a variety of reasons. For example any explicit estimate on the second eigenvalue $\lambda_2(\mathcal{L})$ yields an explicit bound on the exponential rate of mixing for the underlying dynamical system. There are several such a priori bounds in the literature, notably the one due to Liverani [**Liv**]. Although $|\lambda_2(\mathcal{L})|$ is the optimal bound on the exponential rate of mixing which holds for *all* correlation functions with holomorphic observables, faster exponential decay can occur for observables in certain subspaces of finite codimension. More precisely, $|\lambda_n(\mathcal{L})|$ bounds the exponential rate of mixing on the subspace of observables with vanishing spectral projections corresponding to $\lambda_2(\mathcal{L}), \ldots, \lambda_{n-1}(\mathcal{L})$. Therefore the set of possible exponential rates of mixing (the *correlation spectrum*, cf. [**CPR**]) is determined by the full eigenvalue sequence $\{\lambda_n(\mathcal{L})\}$. Any a priori bounds on these eigenvalues thus yields information on the finer mixing properties of the underlying system. The correlation spectrum is also closely related to the *resonances* of the underlying dynamical system (see [**Rue5, Rue6**]).

Explicit *a priori* bounds on $\lambda_n(\mathcal{L})$ also yield explicit bounds on the Taylor coefficients of the determinant $\det(I - \zeta\mathcal{L})$, which in turn facilitate a rigorous *a posteriori* error analysis of any computed approximations to the $\lambda_n(\mathcal{L})$ (see §6 for details). This rigorous justification of accurate numerical bounds has applications to a number of topics in dynamical systems (e.g. the correlation spectrum [**CPR**], the linearised Feigenbaum renormalisation operator [**AAC, CCR, Pol**], Hausdorff dimension estimates [**JP3**], the Selberg zeta function for hyperbolic surfaces [**GLZ, May**], zeta functions for more general Anosov flows [**Fri**]), as well as to other areas of mathematics (e.g. regularity estimates for refinable functions [**Dau**], and the determinant of the Laplacian on surfaces of negative curvature [**PR**]).

Our approach to explicitly bounding the eigenvalues of $\mathcal{L}$ is to consider completely general non-empty open subsets $\Omega \subset \mathbb{C}^d$ in arbitrary complex dimension d, and systematically work with Bergman space $A^2(\Omega)$, consisting of those holomorphic functions in $L^2(\Omega, dV)$, where V denotes $2d$-dimensional Lebesgue measure on Ω. For $\mathcal{I}$ a finite or countably infinite set, consider a collection $(\phi_i)_{i\in\mathcal{I}}$ of holomorphic maps[1] $\phi_i : \Omega \to \Omega$ such that the closure of $\cup_{i\in\mathcal{I}}\phi_i(\Omega)$ is a compact subset of Ω, and a collection $(w_i)_{i\in\mathcal{I}}$ of functions $w_i \in A^2(\Omega)$ with $\sum_{i\in\mathcal{I}} |w_i| \in L^2(\Omega, dV)$ (this condition obviously holds whenever $\mathcal{I}$ is finite). We then call $(\Omega, \phi_i, w_i)_{i\in\mathcal{I}}$ a *holomorphic map-weight system* on Ω and associate with it the transfer operator $\mathcal{L}$ defined as in (1). Our main result is:

Theorem. *If $\mathcal{L} : A^2(\Omega) \to A^2(\Omega)$ is the transfer operator corresponding to a holomorphic map-weight system $(\Omega, \phi_i, w_i)_{i\in\mathcal{I}}$ on a non-empty open set $\Omega \subset \mathbb{C}^d$, then*

$$|\lambda_n(\mathcal{L})| \le A \exp(-a n^{1/d}) \quad \text{for all } n \in \mathbb{N}, \tag{2}$$

where the constants $a, A > 0$ can be determined explicitly in terms of computable properties of $(\Omega, \phi_i, w_i)_{i\in\mathcal{I}}$.

The above theorem is proved as Theorem 5.13, where the coefficients $a, A > 0$ are given explicitly. This theorem is something of a folklore result. Ruelle [**Rue3**, p. 236] had originally asserted that the eigenvalues of $\mathcal{L}$ tend to zero exponentially fast, following a claim of Grothendieck [**Gro**, II, Remarque 9, pp. 62–4]. In 1986 Fried [**Fri**] noted that in fact this assertion is false: in dimension greater than one the eigenvalue decay rate can be slower than exponential. More precisely, for each dimension d he exhibited a transfer operator $\mathcal{L}$ whose eigenvalue sequence satisfies (2) for some $a, A > 0$, but is not $O(\exp(-an^\gamma))$ for any $\gamma > 1/d$.

[1] The ϕ_i here need not be complexified local inverses of some expanding map; in particular they need not be contractions with respect to the Euclidean metric.

Recently, the bound (2) has appeared [**FR**, Thm. 4] in the setting of dynamical systems on the torus, and also in [**GLZ**, (3.6), p. 157])[2], although in these papers the constants a, A are not given explicitly. The bound (2) is proved in [**BJ1, BJ2**], with explicit formulae for a and A, in the special case where Ω is a Euclidean ball.

The main contribution of the present paper is a rigorous proof of (2) for arbitrary non-empty open sets $\Omega \subset \mathbb{C}^d$, including explicit upper bounds on the positive constants a and A. The principal step towards proving (2) consists of establishing the estimate

$$s_n(\mathcal{L}) \leq B\exp(-bn^{1/d}) \quad \text{for all } n \in \mathbb{N}\,, \tag{3}$$

for explicit $b, B > 0$, where $s_n(\mathcal{L})$ denotes the n-th singular value of $\mathcal{L}$. The proof of (3) consists of the following three stages. In §3 the analogous singular value estimate is first derived for canonical embedding operators between Bergman spaces on *strictly circled* domains. In §4 the result is established for canonical identification operators J between Bergman spaces on arbitrary non-empty open subsets $\Omega_2 \subset \Omega_1 \subset \mathbb{C}^d$, subject to the condition that the closure of Ω_2 is a compact subset of Ω_1. For this we introduce the notion of a *relative cover* of the pair (Ω_1, Ω_2) by strictly circled domains. To each relative cover is associated its *efficiency*, a quantity which is readily computable, and which can be used to explicitly bound the singular values of J. In §5, by factorising $\mathcal{L}$ as the product of a bounded operator and a canonical identification operator, we arrive at an explicit version of (3).

Having established (3), there are two possible routes to deducing the eigenvalue bound (2). The first, suggested by Grothendieck [**Gro**], and sketched in more detail by Fried [**Fri**, pp. 505–7], is based on growth estimates for the determinant $\det(I - \zeta\mathcal{L})$. We instead take a more direct approach by applying Weyl's multiplicative inequality, relating eigenvalues to singular values (see §5). For completeness we develop the Grothendieck-Fried strategy as Appendix B. Section 6 contains explicit bounds on the Taylor coefficients of the determinant $\det(I - \zeta\mathcal{L})$, derived from the singular value estimates of §5, together with an outline of how these bounds can be used to obtain explicit a *a posteriori* error bounds for spectral approximation procedures applied to transfer operators. Finally, in Appendix A we show how our Theorem 4.7 can be used to provide a short proof of the correct statement of Grothendieck's Remarque 9, which does not seem to have appeared in the literature yet: if L is any bounded linear operator on the Fréchet space $\mathcal{H}(\Omega)$ of holomorphic functions on an open set $\Omega \subset \mathbb{C}^d$, then its eigenvalues are $O(\exp(-an^{1/d}))$ as $n \to \infty$, for some $a > 0$.

The methods of this paper can be extended to prove an analogue of the main result for more general transfer operators arising in the study of limit sets of iterated function schemes [**MU1, MU2**] or of certain Kleinian groups [**JP3**], and whenever the underlying dynamical system is a real analytic expanding Markov map. We do not pursue this generalisation here, however, preferring to present the main ideas in the simplest possible combinatorial setting.

Notation 1.1. Let $\mathbb{N}$ denote the set of strictly positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, let $\mathcal{O}_d$ denote the collection of non-empty open subsets of $\mathbb{C}^d$.

For Hilbert spaces H_1, H_2, let $L(H_1, H_2)$ denote the Banach space of bounded linear operators from H_1 to H_2, equipped with the usual norm, and let $S_\infty(H_1, H_2) \subset L(H_1, H_2)$ denote the closed subspace of compact operators from H_1 to H_2. We write L or S_∞ whenever the Hilbert spaces H_1 and H_2 are understood.

For $A \in S_\infty(H, H)$ let $\lambda(A) = \{\lambda_n(A)\}_{n=1}^\infty$ denote the sequence of eigenvalues of A, each eigenvalue repeated according to its algebraic multiplicity, and ordered by magnitude (where distinct eigenvalues of the same modulus can be written in any order), so that $|\lambda_1(A)| \geq$

[2] The focus in [**GLZ**] is on the asymptotics of the determinant with respect to a complex parameter s, rather than on completely explicit eigenvalue bounds. In fact the derivation of the eigenvalue bound (3.6) in [**GLZ**] is not quite complete: no argument is given for the bound on the norm of the Bergman space operator $\mathcal{L}^\rho_{ij}(s)$ [**GLZ**, p. 159], and simple examples (see [**CM**, §3.5], [**KS**]) show that in general the operator is not bounded.

$|\lambda_2(A)| \geq \ldots$. For $A \in S_\infty(H_1, H_2)$, we define the n-th *singular value* of A by $s_n(A) := \sqrt{\lambda_n(A^*A)}$, $n \geq 1$, where A^* denotes the adjoint of A.

2. Preliminaries

2.1. Exponential classes. Much modern work on eigenvalue distributions has been carried out within the framework of *operator ideals* (cf. [**GK, Pie2, Pie3, Sim**]). This framework, however, is not well adapted to our setting: as we shall see, the transfer operators considered here are always trace class (see Theorem 5.9), and hence belong to any symmetrically normed ideal (see e.g. [**GK**, Chap. 3.2]), so that the results from this theory are too conservative. We instead use the theory of exponential classes developed in [**Ban**].

DEFINITION 2.1. Let H_1, H_2 be infinite dimensional Hilbert spaces. For $a, \alpha > 0$, define

$$E(a,\alpha) := \left\{ A \in S_\infty(H_1,H_2) \,:\, |A|_{a,\alpha} := \sup_{n\in\mathbb{N}} s_n(A)\exp(an^\alpha) < \infty \right\},$$

the *exponential class of operators of type* (a, α). Define $E(\alpha) := \cup_{a>0} E(a, \alpha)$.

Exponential classes enjoy the following closure properties (see [**Ban**, Propositions 2.5 and 2.8]):

LEMMA 2.2. *Let* $\alpha, a, a_1, \ldots, a_N > 0$.

(i) *If* $A, C \in L$ *and* $B \in E(a, \alpha)$, *then* $ABC \in E(a, \alpha)$, *and* $|ABC|_{a,\alpha} \leq \|A\| \; |B|_{a,\alpha} \; \|C\|$. *In particular,* $LE(a, \alpha)L \subset E(a, \alpha)$.

(ii) *Let* $A_n \in E(a_n, \alpha)$ *for* $1 \leq n \leq N$ *and let* $A = \sum_{n=1}^N A_n$. *Then*

$$A \in E(a',\alpha) \text{ with } |A|_{a',\alpha} \leq N \max_{1\leq n\leq N} |A_n|_{a_n,\alpha}$$

where $a' := (\sum_{n=1}^N a_n^{-1/\alpha})^{-\alpha}$. *In particular,* $E(a_1, \alpha) + \cdots + E(a_N, \alpha) \subset E(a', \alpha)$, *and this inclusion is sharp in the sense that* $E(a_1, \alpha) + \cdots + E(a_N, \alpha) \not\subset E(b, \alpha)$ *whenever* $b > a'$.

2.2. Bergman spaces. Bergman spaces, originally introduced by Stefan Bergman in his 1921 PhD thesis [**Ber**], are among the simplest examples of Hilbert spaces of holomorphic functions. Less delicate in their definition than Hardy spaces, they provide a convenient setting for our analysis of transfer operators.

DEFINITION 2.3. For $\Omega \in \mathcal{O}_d$, let $\mathcal{H}(\Omega)$ denote the Fréchet space of holomorphic functions $f : \Omega \to \mathbb{C}$, equipped with the topology of uniform convergence on compact subsets of Ω. Let $A^\infty(\Omega)$ be the Banach space of bounded $f \in \mathcal{H}(\Omega)$, equipped with the norm $\|f\|_{A^\infty(\Omega)} := \sup_{z\in\Omega} |f(z)|$. If V denotes $2d$-dimensional Lebesgue measure on $\mathbb{C}^d$, normalised so that the $2d$-dimensional Euclidean unit ball has unit mass,

$$A^2(\Omega) := \left\{ f \in \mathcal{H}(\Omega) \,:\, \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\}$$

is called *Bergman space over* Ω.

This definition of Bergman space is slightly more general then the usual one, in that we allow arbitrary non-empty open sets rather than just domains. However most of their familiar properties (see e.g. [**Kra**, Chapter 1.4]) are easily seen to carry over to the more general setting. In particular, $A^2(\Omega)$ is a separable Hilbert space with inner product

$$(f,g)_{A^2(\Omega)} = \int_\Omega f(z)\overline{g(z)}\, dV(z) \quad (f,g \in A^2(\Omega)).$$

The following quantitative refinement of a well known lemma (see [**Kra**, Lemma 1.4.1]) will be used in Lemma 5.3.

LEMMA 2.4. *If $\Omega \in \mathcal{O}_d$, and $K \subset \Omega$ is compact, there is a constant $C_K > 0$ such that $\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}$ for all $f \in A^2(\Omega)$. Moreover, it is possible to choose $C_K = r^{-d}$, where $r = \operatorname{dist}(\partial K, \partial\Omega) = \operatorname{dist}(K, \partial\Omega)$.*

PROOF. By hypothesis, $r > 0$ and $B(z,r) \subset \Omega$ for every $z \in K$, where $B(z,r)$ denotes the Euclidean ball of radius r centred at z. If $f \in A^2(\Omega)$ then, just as in the standard case (see [**Kra**, Lemma 1.4.1]), $f(z) = (\int_{B(z,r)} f\, dV)/V(B(z,r))$, so by the Cauchy-Schwarz inequality,

$$|f(z)| \leq \frac{1}{V(B(z,r))} \int_{B(z,r)} |f|\, dV \leq V(B(z,r))^{-1/2} \|f\|_{L^2(B(z,r))} \leq r^{-d} \|f\|_{A^2(\Omega)} .$$

□

3. Canonical embeddings for simple geometries

Suppose that $\Omega_1, \Omega_2 \in \mathcal{O}_d$, and that $\Omega_2 \subset \Omega_1$. By restriction to Ω_2 every element in $A^2(\Omega_1)$ can also be considered as an element of $A^2(\Omega_2)$. This restriction yields a linear transformation $J : A^2(\Omega_1) \to A^2(\Omega_2)$ defined by $Jf = f|_{\Omega_2}$, which will be referred to as *canonical identification* (we use J throughout to denote canonical identifications; the spaces involved will always be clear from the context). If Ω_1 is connected, then the canonical identification is a proper embedding of $A^2(\Omega_1)$ in $A^2(\Omega_2)$. Clearly J is continuous, with norm at most one.

DEFINITION 3.1. For $\Omega_1, \Omega_2 \in \mathcal{O}_d$, if $\overline{\Omega}_2$ is a compact subset of Ω_1 then we say that Ω_2 is *compactly contained* in Ω_1, and write $\Omega_2 \Subset \Omega_1$.

It turns out that if $\Omega_2 \Subset \Omega_1$ then $J : A^2(\Omega_1) \to A^2(\Omega_2)$ is a compact operator; to see this note that $J(A^2(\Omega_1))$ is contained in the Banach space $C^b(\Omega_2)$ of bounded continuous functions on Ω_2 and $J : A^2(\Omega_1) \to C^b(\Omega_2)$ has closed graph, hence $\{Jf : \|f\|_{A^2(\Omega_1)} \leq 1\}$ is uniformly bounded on Ω_2 and therefore a normal family in $A^2(\Omega_2)$. In fact rather more is true: $J \in E(c, 1/d)$ for some $c > 0$. The proof of this result for general open sets $\Omega_2 \Subset \Omega_1$ requires a certain amount of preparation and will be presented in §4. In this section we shall be content with proving the result for certain subclasses of open sets Ω_1, Ω_2 for which the decay rate c can be identified precisely; these subclasses are defined as follows.

DEFINITION 3.2. Let $D \subset \mathbb{C}^d$ and $\zeta \in \mathbb{C}^d$. We call D *strictly circled, with centre ζ*, if

$$\mu(D - \zeta) \Subset D - \zeta \qquad \text{for all } \mu \in \mathbb{C} \text{ with } |\mu| < 1.$$

For $r > 0$ we define $D(r) := r(D - \zeta) + \zeta$.

Note that a strictly circled set is necessarily bounded. Moreover, the boundary of a strictly circled open set has zero Lebesgue measure, a fact which will be used in §4.

LEMMA 3.3. *If $D \in \mathcal{O}_d$ is strictly circled then $V(\partial D) = 0$.*

PROOF. By translation invariance of V, it suffices to prove the assertion for D with centre 0. Since D is open, $D = \cup_{0<r<1} r\overline{D}$, where $\overline{D}$ denotes the closure of D. Thus $V(D) = \sup_{0<r<1} V(r\overline{D}) = \sup_{0<r<1} r^{2d} V(\overline{D}) = V(\overline{D})$. □

We now consider canonical embeddings of Bergman spaces over strictly circled open sets.

PROPOSITION 3.4. *If $D \in \mathcal{O}_d$ is strictly circled then:*

(i) *There is a set consisting of homogeneous polynomials which is a complete orthogonal system for every $A^2(D(r))$, $r > 0$.*
(ii) *If $\gamma > 1$, then the singular values of the canonical embedding $J : A^2(D(\gamma)) \hookrightarrow A^2(D)$ are given by $s_n(J) = \gamma^{-(k+d)}$ for $\binom{k+d-1}{d} < n \leq \binom{k+d}{d}$ and $k \in \mathbb{N}_0$.*

PROOF. (i) Assume for the moment that D is centred at the origin. Any function holomorphic on the strictly circled set $D(r)$ has a unique expansion in terms of homogeneous polynomials, which is convergent uniformly on compact subsets of $D(r)$ (see [**And**, Chapter I, §10.3, Thm. 2] or [**Mal**, Chapter II, Thm. 3]), hence the collection of homogeneous polynomials is total (i.e. its linear span is dense) in $A^2(D(r))$. It remains to show that this collection can be orthogonalised so as to yield a system that is simultaneously orthogonal for all $A^2(D(r))$, $r > 0$. To do this we introduce the short-hand $(f,g)_r := (f,g)_{A^2(D(r))}$. Let f and g be monomials of degree n and m respectively. Since D is bounded, $f, g \in A^2(D(r))$ for all $r > 0$. Moreover, since D is strictly circled, each $D(r)$ is invariant under the transformation $z \mapsto e^{it}z$. Thus

$$(f,g)_r = \int_{D(r)} f(e^{it}z)\overline{g(e^{it}z)}\, dV(z) = e^{it(n-m)}(f,g)_r\,,$$

which implies $(f,g)_r = 0$ for $n \neq m$, and for each $r > 0$.

For any $r_1 > 0$, an application of the Gram-Schmidt orthogonalisation procedure now yields an orthonormal basis of $A^2(D(r_1))$ consisting of homogeneous polynomials. We shall show that this basis is also orthogonal with respect to all other scalar products $(\cdot,\cdot)_r$, for $r > 0$. To see this fix $r > 0$ and let f and g be homogeneous polynomials of degree n and m respectively. Then

$$\begin{aligned}(f,g)_r &= \int_{D(r)} f(z)\overline{g(z)}\, dV(z)\\ &= \int_{D(r_1)} f((r/r_1)z)\overline{g((r/r_1)z)}(r/r_1)^{2d}\, dV(z)\\ &= (r/r_1)^{n+m+2d}(f,g)_{r_1}\,.\end{aligned} \tag{4}$$

Thus, if $(f,g)_{r_1} = 0$ then $(f,g)_r = 0$, and (i) is proved.

(ii) If $J^* : A^2(D) \to A^2(D(\gamma))$ denotes the adjoint of $J : A^2(D(\gamma)) \hookrightarrow A^2(D)$, then setting $r = 1$, $r_1 = \gamma$ in (4) gives $(f, J^*Jg)_\gamma = (Jf, Jg)_1 = \gamma^{-(n+m+2d)}(f,g)_\gamma$. Thus $J^*J : A^2(D(\gamma)) \to A^2(D(\gamma))$ is diagonal with respect to the orthogonal basis of homogeneous polynomials. Its eigenvalues therefore belong to the set $\{\, \gamma^{-(2k+2d)} : k \in \mathbb{N}_0 \,\}$. Therefore the singular values of $J : A^2(D(\gamma)) \hookrightarrow A^2(D)$ belong to the set $\{\, \gamma^{-(k+d)} : k \in \mathbb{N}_0 \,\}$. As there are $\binom{k+d-1}{d-1}$ linearly independent homogeneous polynomials of degree k, the value $\gamma^{-(k+d)}$ occurs with multiplicity $\binom{k+d-1}{d-1}$. Thus the largest n for which $s_n(J) = \gamma^{-(k+d)}$ is equal to $\sum_{l=0}^{k}\binom{l+d-1}{d-1} = \binom{k+d}{d}$. This completes the proof in the case of D centred at 0. The general case can be reduced to this case by shifting the origin and using translation invariance of Lebesgue measure. $\square$

The precise location of J in the scale of exponential classes $\{E(a,\alpha)\}$ is as follows:

PROPOSITION 3.5. *If $D \in \mathcal{O}_d$ is strictly circled, and $\gamma > 1$, then the canonical embedding $J : A^2(D(\gamma)) \hookrightarrow A^2(D)$ satisfies*

$$J \in E(c, 1/d), \quad \textit{where } c = (d!)^{1/d}\log\gamma\,, \tag{5}$$

and

$$|J|_{c,1/d} = \gamma^{(1-d)/2} \tag{6}$$

That is, its singular value sequence has the following asymptotics:

$$\lim_{n\to\infty} \frac{\log|\log s_n(J)|}{\log n} = \frac{1}{d}\,; \tag{7}$$

$$\lim_{n\to\infty} \frac{\log s_n(J)}{n^{1/d}} = -(d!)^{1/d}\log\gamma\,; \tag{8}$$

$$\sup_{n\in\mathbb{N}}\Big(\log s_n(J)+(nd!)^{1/d}\log\gamma\Big)=\frac{1-d}{2}\log\gamma\,. \tag{9}$$

Proof. If $h_d(k):=\binom{k+d-1}{d}$, and $h_d(k)<n\le h_d(k+1)$, Proposition 3.4 gives

$$\frac{\log\left|\log\gamma^{-1}\right|+\log(k+d)}{\log h_d(k+1)}\le\frac{\log|\log s_n(J)|}{\log n}\le\frac{\log\left|\log\gamma^{-1}\right|+\log(k+d)}{\log h_d(k)}\,.$$

It is easily seen that $\lim_{k\to\infty}\frac{\log(k+d)}{\log h_d(k+1)}=\lim_{k\to\infty}\frac{\log(k+d)}{\log h_d(k)}=\frac{1}{d}$, so (7) follows. Similarly,

$$\frac{(k+d)\log\gamma^{-1}}{h_d(k+1)^{1/d}}\le\frac{\log s_n(J)}{n^{1/d}}\le\frac{(k+d)\log\gamma^{-1}}{h_d(k)^{1/d}}\,,$$

and $\lim_{k\to\infty}\frac{(k+d)}{h_d(k+1)^{1/d}}=\lim_{k\to\infty}\frac{(k+d)}{h_d(k)^{1/d}}=(d!)^{1/d}$, so (8) follows.

To prove (9), we first establish that for all $d\in\mathbb{N}$,

$$\sup_{x\ge 0}\prod_{j=1}^{d}(x+j)^{1/d}-(x+d)=\lim_{j\to\infty}\prod_{j=1}^{d}(x+j)^{1/d}-(x+d)=-\frac{d-1}{2}\,. \tag{10}$$

The case $d=1$ of (10) is obvious, so suppose $d\ge 2$. If $h(x):=\prod_{j=1}^{d}(x+j)^{1/d}-(x+d)$ then

$$h'(x)=\frac{1}{d}\prod_{j=1}^{d}(x+j)^{1/d-1}\sum_{j=1}^{d}\prod_{\substack{l=1\\ l\ne j}}(x+l)-1.$$

Now $\frac{1}{d}\sum_{j=1}^{d}(x+j)^{-1}\ge\prod_{j=1}^{d}(x+j)^{-1/d}$ by the arithmetic-geometric mean inequality, so

$$\frac{1}{d}\sum_{j=1}^{d}\prod_{\substack{l=1\\ l\ne j}}(x+l)\ge\prod_{j=1}^{d}(x+j)^{1-1/d}\,,$$

and therefore $h'(x)\ge 0$ for $x\ge 0$. If $t:=(x+d)^{-1}$ then

$$h(x)=(x+d)\left(\prod_{j=1}^{d}\left(\frac{x+j}{x+d}\right)^{1/d}-1\right)=t^{-1}\left(\prod_{j=0}^{d-1}(1-jt)^{1/d}-1\right)\,,$$

so $\sup_{x\ge 0}h(x)=\lim_{x\to\infty}h(x)=\lim_{t\to 0}t^{-1}\left(\prod_{j=0}^{d-1}(1-jt)^{1/d}-1\right)=-\frac{1}{d}\sum_{j=0}^{d-1}j=-\frac{d-1}{2}$ by l'Hôpital's rule, and (10) is proved.

Now $\log s_n(J)+(nd!)^{1/d}\log\gamma\le\big((h_d(k+1)d!)^{1/d}-(k+d)\big)\log\gamma\le\big(-\frac{d-1}{2}\big)\log\gamma$, by (10), so $\sup_{n\in\mathbb{N}}\big(\log s_n(J)+(nd!)^{1/d}\log\gamma\big)\le\frac{1-d}{2}\log\gamma$. To obtain equality we consider $s_{h_d(k+1)}(J)$ and again apply (10). Finally, note that (9) is a restatement of (5) and (6). □

Remark 3.6. Proposition 3.5 is optimal: as a consequence of (7) and (8), membership of J in (5) is sharp, in the sense that neither $1/d$ nor c can be replaced by anything larger; moreover, $|J|_{c,1/d}$ is known exactly.

4. Canonical identifications and relative covers

We shall now show how identifications of Bergman spaces over more general sets can be obtained from identifications of Bergman spaces over strictly circled sets. The main tool is the following construction:

Lemma 4.1. *If $\Omega_1,\Omega_2,\Omega_3\in\mathcal{O}_d$, with $\Omega_1\subset\Omega_2\subset\Omega_3$, the operator $T_{\Omega_1}:A^2(\Omega_2)\to A^2(\Omega_3)$, defined implicitly by $(T_{\Omega_1}f,g)_{A^2(\Omega_3)}=\int_{\Omega_1}f\,\overline{g}\,dV$, is bounded with norm at most 1.*

Proof. Notice that for any $f \in A^2(\Omega_2)$ and any $g \in A^2(\Omega_3)$

$$\left|\int_{\Omega_1} f\overline{g}\, dV\right|^2 \leq \int_{\Omega_1} |f|^2 \, dV \int_{\Omega_1} |g|^2 \, dV \leq \|f\|^2_{A^2(\Omega_2)} \, \|g\|^2_{A^2(\Omega_3)} .$$

Thus T_{Ω_1} is well-defined and continuous with norm at most 1. □

Definition 4.2. Let $\{\Omega_n\}_{1\leq n\leq N}$ be a finite collection of open subsets of $\mathbb{C}^d$. If $\{\widetilde{\Omega}_n\}_{1\leq n\leq N}$ is a partition (modulo sets of zero Lebesgue measure) of $\bigcup_{n=1}^N \Omega_n$, where each $\widetilde{\Omega}_n$ is open, and $\widetilde{\Omega}_n \subset \Omega_n$ for each n, then we say that $\{\widetilde{\Omega}_n\}_{1\leq n\leq N}$ is a *disjointification* of $\{\Omega_n\}_{1\leq n\leq N}$.

Remark 4.3. If a collection $\{\Omega_n\}_{1\leq n\leq N}$ has the property that the boundary of each Ω_n is a Lebesgue null set, then a disjointification exists and can, for example, be obtained by defining $\widetilde{\Omega}_n$ as the interior of $\big(\Omega_n \setminus (\cup_{i=1}^{n-1}\Omega_i)\big)$ for $n = 1, \ldots, N$.

The usefulness of the operator defined in Lemma 4.1 is due to the following key result.

Proposition 4.4. *For $1 \leq n \leq N$, let $\Omega_n, \Omega \in \mathcal{O}_d$, with $\Omega_n \subset \Omega$, and let $J_n : A^2(\Omega) \to A^2(\Omega_n)$ denote the canonical identification. If $\{\widetilde{\Omega}_n\}_{1\leq n\leq N}$ is a disjointification of $\{\Omega_n\}_{1\leq n\leq N}$, then the canonical identification $J : A^2(\Omega) \to A^2(\bigcup_{n=1}^N \Omega_n)$ can be written as*

$$J = \sum_{n=1}^{N} T_{\widetilde{\Omega}_n} J_n \,,$$

where $T_{\widetilde{\Omega}_n} : A^2(\Omega_n) \to A^2(\bigcup_{n=1}^N \Omega_n)$ is the operator defined in Lemma 4.1.

Proof. Let $f \in A^2(\Omega)$ and $g \in A^2(\bigcup_{n=1}^N \Omega_n)$. Then

$$\begin{aligned}
(\sum_{n=1}^{N} T_{\widetilde{\Omega}_n} J_n f, g)_{A^2(\bigcup_n \Omega_n)} &= \sum_{n=1}^{N} \int_{\widetilde{\Omega}_n} f(z)\overline{g(z)}\, dV(z) \\
&= \int_{\bigcup_n \Omega_n} f(z)\overline{g(z)}\, dV(z) \\
&= (Jf, g)_{A^2(\bigcup_n \Omega_n)},
\end{aligned}$$

and the assertion follows. □

Definition 4.5. Let $\Omega_1, \Omega_2 \in \mathcal{O}_d$, with $\Omega_2 \subset\subset \Omega_1$, and $N \in \mathbb{N}$. A finite collection $D_1, \ldots, D_N$ of strictly circled open subsets of $\mathbb{C}^d$ is a *relative cover* of the pair (Ω_1, Ω_2) if

(a) $\Omega_2 \subset \bigcup_{n=1}^N D_n$, and
(b) for each $1 \leq n \leq N$ there exists $\gamma_n > 1$ such that $\bigcup_{n=1}^N D_n(\gamma_n) \subset \Omega_1$.

We call N the *size*, $(\gamma_1, \ldots, \gamma_N)$ a *scaling*, and $\Gamma = (\log\gamma_1, \ldots, \log\gamma_N)$ the *efficiency*, of the relative cover.

Remark 4.6. Since $\Omega_2 \subset\subset \Omega_1$, there always exists a relative cover for (Ω_1, Ω_2).

Theorem 4.7. *If $\Omega_1, \Omega_2 \in \mathcal{O}_d$, with $\Omega_2 \subset\subset \Omega_1$, then the canonical identification $J : A^2(\Omega_1) \to A^2(\Omega_2)$ belongs to $E(1/d)$.*
More precisely, if $\{D_n\}_{1\leq n\leq N}$ is a relative cover of (Ω_1, Ω_2) of size N with efficiency Γ then

$$J \in E(c, 1/d), \quad \textit{where} \quad c = \|\Gamma\|_d \,,$$

and

$$|J|_{c,1/d} \leq N e^{-\frac{d-1}{2}\min(\Gamma)} \,, \tag{11}$$

where $\min(\Gamma)$ denotes the smallest entry in Γ and $\|(x_1, \ldots, x_N)\|_d := \Big(\sum_{j=1}^N |x_j|^{-d}\Big)^{-1/d}$.

Proof. For $1 \le n \le N$, let $T_{\widetilde\Omega_n} : A^2(D_n) \to A^2(\bigcup_{n=1}^N D_n)$ denote the operator defined in Lemma 4.1, where $\{\widetilde\Omega_n\}_{1\le n\le N}$ is a disjointification of $\{D_n\}_{1\le n\le N}$ (which exists by Lemma 3.3 and Remark 4.3). For $(\gamma_1,\ldots,\gamma_N)$ a scaling of $\{D_n\}_{1\le n\le N}$, consider the canonical identifications $\widetilde J_n : A^2(\Omega_1) \to A^2(D_n(\gamma_n))$, $J_n : A^2(D_n(\gamma_n)) \to A^2(D_n)$, and $\widetilde J : A^2(\bigcup_{n=1}^N D_n) \to A^2(\Omega_2)$. By Proposition 4.4,

$$J = \sum_{n=1}^N \widetilde J T_{\widetilde\Omega_n} J_n \widetilde J_n\,.$$

Trivially $\|\widetilde J\| \le 1$ and $\|\widetilde J_n\| \le 1$, while $\|T_{\widetilde\Omega_n}\| \le 1$ by Lemma 4.1, so Lemma 2.2 (i) and Proposition 3.5 imply that each $\widetilde J T_{\widetilde\Omega_n} J_n \widetilde J_n \in E(c_n, 1/d)$, where $c_n = (d!)^{1/d}\log\gamma_n$, and $|\widetilde J T_{\widetilde\Omega_n} J_n \widetilde J_n|_{c_n,1/d} \le \gamma_n^{-\frac{d-1}{2}}$. The assertion now follows from Lemma 2.2 (ii). □

5. Singular values and eigenvalues of transfer operators

Our previous analysis of the singular values of identification operators can now be applied to the study of the singular values of transfer operators. Since a transfer operator can be expressed in terms of multiplication operators and composition operators, we begin by considering such operators.

Definition 5.1. Let $\Omega, \widetilde\Omega \in \mathcal{O}_d$.

(a) If $\phi : \Omega \to \widetilde\Omega$ is holomorphic, the linear transformation $C_\phi : \mathcal{H}(\widetilde\Omega) \to \mathcal{H}(\Omega)$ defined by $C_\phi f := f \circ \phi$ is called a *composition operator* (with *symbol* ϕ).
(b) If $w \in \mathcal{H}(\Omega)$, the linear transformation $M_w : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ defined by $(M_w f)(z) := w(z)f(z)$ is called a *multiplication operator* (with *symbol* w).
(c) An operator of the form $M_w C_\phi$, where C_ϕ is a composition operator and M_w is a multiplication operator, is called a *weighted composition operator*.

Notation 5.2. If F, G are Banach spaces, and $A : F \to G$ is a bounded linear operator, the norm of A will sometimes be denoted by $\|A\|_{F\to G}$.

Lemma 5.3. *If $\Omega, \widetilde\Omega \in \mathcal{O}_d$, $\phi : \Omega \to \widetilde\Omega$ is holomorphic, and $r := \operatorname{dist}(\phi(\Omega), \partial\widetilde\Omega) > 0$, then $C_\phi : A^2(\widetilde\Omega) \to A^\infty(\Omega)$ is bounded, with norm*

$$\|C_\phi\|_{A^2(\widetilde\Omega)\to A^\infty(\Omega)} \le r^{-d}\,. \tag{12}$$

If, in addition, Ω has finite volume, then $C_\phi : A^2(\widetilde\Omega) \to A^2(\Omega)$ is bounded, with norm

$$\|C_\phi\|_{A^2(\widetilde\Omega)\to A^2(\Omega)} \le \sqrt{V(\Omega)}\, r^{-d}\,.$$

Proof. By Lemma 2.4, $\|C_\phi f\|_{A^\infty(\Omega)} = \sup_{z\in\Omega} |f(\phi(z))| = \sup_{z\in\phi(\Omega)} |f(z)| \le r^{-d}\,\|f\|_{A^2(\widetilde\Omega)}$ for $f \in A^2(\widetilde\Omega)$, thus C_ϕ maps $A^2(\widetilde\Omega)$ continuously to $A^\infty(\Omega)$, with norm as in (12). The remaining assertions follow from the fact that if Ω has finite volume then the canonical identification $J : A^\infty(\Omega) \to A^2(\Omega)$ is continuous with norm $\|J\| = \sqrt{V(\Omega)}$. □

Remark 5.4.
(i) There is a sizable literature on criteria for continuity of composition operators between Bergman spaces, beginning with Littlewood's subordination principle [**Lit**], guaranteeing that if $\Omega = \widetilde\Omega$ is the open unit disc then C_ϕ is *always* bounded (see [**MS**, Prop. 3.4]). This need not be the case for more general simply connected domains in $\mathbb{C}$ (see [**KP, SS**]), or indeed when $\Omega = \widetilde\Omega$ is the open unit ball in $\mathbb{C}^d$, $d > 1$ (see e.g. [**CM**, §3.5]). A novelty of our approach is that we consider Bergman spaces in arbitrary dimension, and over arbitrary open sets Ω.

(ii) There is no known general formula (in terms of the symbol ϕ) for the norm of the composition operator C_ϕ; see [**Sha**, p. 195] for a discussion of this problem.

Next we consider weighted composition operators. Again we may ask under what conditions $M_w C_\phi$ maps $A^2(\widetilde{\Omega})$ continuously into $A^2(\Omega)$. A necessary condition is that $w \in A^2(\Omega)$, since the image of the constant function 1 is w. In general this is not enough to guarantee the continuity of M_w itself (in one complex dimension, necessary and sufficient conditions for the continuity of multiplication operators are given in [**KP**]), but in our context it *is* sufficient for the boundedness of $M_w C_\phi$:

LEMMA 5.5. *Suppose $\Omega, \widetilde{\Omega} \in \mathcal{O}_d$, $\phi : \Omega \to \widetilde{\Omega}$ is holomorphic, and $r := \operatorname{dist}(\phi(\Omega), \partial\widetilde{\Omega}) > 0$. If $w \in A^2(\Omega)$, the weighted composition operator $M_w C_\phi : A^2(\widetilde{\Omega}) \to A^2(\Omega)$ is bounded, with*

$$\|M_w C_\phi\|_{A^2(\widetilde{\Omega}) \to A^2(\Omega)} \le r^{-d} \|w\|_{A^2(\Omega)} .$$

PROOF. If $f \in A^2(\widetilde{\Omega})$ then $w \cdot (f \circ \phi) \in \mathcal{H}(\Omega)$. Now $\sup_{z \in \Omega} |f(\phi(z))|^2 \le r^{-2d} \|f\|^2_{A^2(\widetilde{\Omega})}$ by Lemma 5.3, so $\|M_w C_\phi f\|^2_{A^2(\Omega)} = \int_\Omega |w(z)|^2 |f(\phi(z))|^2 \, dV(z) \le r^{-2d} \|f\|^2_{A^2(\widetilde{\Omega})} \|w\|^2_{A^2(\Omega)}$. □

DEFINITION 5.6. Let $\Omega, \widetilde{\Omega} \in \mathcal{O}_d$, and let $\mathcal{I}$ be either a finite or countably infinite set. Suppose we are given the following data:

(a) a collection $(\phi_i)_{i \in I}$ of holomorphic maps $\phi_i : \Omega \to \widetilde{\Omega}$ with $\cup_{i \in \mathcal{I}} \phi_i(\Omega) \Subset \widetilde{\Omega}$;
(b) a collection $(w_i)_{i \in \mathcal{I}}$ of functions $w_i \in A^2(\Omega)$ with $\sum_{i \in \mathcal{I}} |w_i| \in L^2(\Omega, dV)$, i.e. the series of the moduli of the w_i converges in $L^2(\Omega, dV)$.

We then call $((\Omega, \widetilde{\Omega}), \phi_i, w_i)_{i \in \mathcal{I}}$ a *holomorphic map-weight system (on $(\Omega, \widetilde{\Omega})$)*. If $\widetilde{\Omega} = \Omega$ then we simply refer to a *holomorphic map-weight system on* Ω, denoted by $(\Omega, \phi_i, w_i)_{i \in \mathcal{I}}$.

To each holomorphic map-weight system we associate a transfer operator as follows (note that when $\widetilde{\Omega} = \Omega$, the definition coincides with the one given in §1):

DEFINITION 5.7. Let $((\Omega, \widetilde{\Omega}), \phi_i, w_i)_{i \in \mathcal{I}}$ be a holomorphic map-weight system. Then the linear operator $\mathcal{L} : A^2(\widetilde{\Omega}) \to A^2(\Omega)$ defined as the sum of weighted composition operators

$$\mathcal{L} = \sum_{i \in \mathcal{I}} M_{w_i} C_{\phi_i} ,$$

is called the associated *transfer operator (on $(\Omega, \widetilde{\Omega})$)*.

If $\mathcal{I}$ is infinite, it is not obvious that this definition of $\mathcal{L}$ produces a well-defined continuous operator from $A^2(\widetilde{\Omega})$ to $A^2(\Omega)$. We now prove that this is indeed the case.

PROPOSITION 5.8. *Let $((\Omega, \widetilde{\Omega}), \phi_i, w_i)_{i \in \mathcal{I}}$ be a holomorphic map-weight system, with $r_i := \operatorname{dist}(\phi_i(\Omega), \partial\widetilde{\Omega})$. The associated transfer operator $\mathcal{L} : A^2(\widetilde{\Omega}) \to A^2(\Omega)$ is bounded, with norm*

$$\|\mathcal{L}\|_{A^2(\widetilde{\Omega}) \to A^2(\Omega)} \le \left\| \sum_{i \in \mathcal{I}} |w_i| \, r_i^{-d} \right\|_{L^2(\Omega)} . \tag{13}$$

PROOF. Let $f \in A^2(\widetilde{\Omega})$. If $\mathcal{J} \subset \mathcal{I}$ is finite, $\sum_{i \in \mathcal{J}} M_{w_i} C_{\phi_i} f \in A^2(\Omega)$ by Lemma 5.5. Now

$$\left\| \sum_{i \in \mathcal{J}} M_{w_i} C_{\phi_i} f \right\|^2_{A^2(\Omega)} \le \int_\Omega \left(\sum_{i \in \mathcal{J}} |w_i(z)| \, |f(\phi_i(z))| \right)^2 dV(z) ,$$

and $\sup_{z \in \Omega} |f(\phi_i(z))| \le r_i^{-d} \|f\|_{A^2(\widetilde{\Omega})}$ by Lemma 5.3, so

$$\left\| \sum_{i \in \mathcal{J}} M_{w_i} C_{\phi_i} f \right\|^2_{A^2(\Omega)} \le \|f\|^2_{A^2(\widetilde{\Omega})} \int_\Omega \left(\sum_{i \in \mathcal{J}} |w_i(z)| r_i^{-d} \right)^2 dV(z) . \tag{14}$$

Since each $r_i \geq \operatorname{dist}(\cup_{i\in\mathcal{I}}\phi_i(\Omega), \partial\widetilde{\Omega}) =: r > 0$,

$$\int_\Omega \left(\sum_{i\in\mathcal{J}} |w_i(z)| r_i^{-d}\right)^2 dV(z) \leq r^{-2d} \int_\Omega \left(\sum_{i\in\mathcal{J}} |w_i(z)|\right)^2 dV(z)\,.$$

So (14) implies that $\sum_{i\in\mathcal{I}} M_{w_i} C_{\phi_i} f$ is Cauchy in $A^2(\Omega)$, hence converges to an element in $A^2(\Omega)$. Thus $\mathcal{L}$ defines a bounded operator from $A^2(\widetilde{\Omega})$ to $A^2(\Omega)$, by the uniform boundedness principle. Choosing $\mathcal{J} = \mathcal{I}$ in (14) yields the desired upper bound on the norm of $\mathcal{L}$. □

We now prove that for any holomorphic map-weight system, the corresponding transfer operator lies in an exponential class $E(c, 1/d)$, with explicit estimates on both c and $|\mathcal{L}|_{c,1/d}$:

Theorem 5.9. *Suppose that $((\Omega, \Omega'), \phi_i, w_i)_{i\in\mathcal{I}}$ is a holomorphic map-weight system with $\Omega, \Omega' \in \mathcal{O}_d$, and $r_i := \operatorname{dist}(\phi_i(\Omega), \partial\widetilde{\Omega})$. Let $\widetilde{\Omega} \subset\subset \Omega'$ be such that*

$$\cup_{i\in\mathcal{I}}\phi_i(\Omega) \subset\subset \widetilde{\Omega} \subset\subset \Omega'\,, \tag{15}$$

and such that $(\Omega', \widetilde{\Omega})$ has a relative cover of size N with efficiency Γ.

Then the corresponding transfer operator $\mathcal{L} : A^2(\Omega') \to A^2(\Omega)$ belongs to the exponential class $E(c, 1/d)$, where

$$c = \|\Gamma\|_d\,, \tag{16}$$

and

$$|\mathcal{L}|_{c,1/d} \leq N e^{-\frac{d-1}{2}\min(\Gamma)} \left\| \sum_{i\in\mathcal{I}} |w_i| r_i^{-d} \right\|_{L^2(\Omega)}\,. \tag{17}$$

Proof. By Proposition 5.8 the transfer operator $\mathcal{L} : A^2(\Omega') \to A^2(\Omega)$ can be lifted to a continuous operator $\widetilde{\mathcal{L}} : A^2(\widetilde{\Omega}) \to A^2(\Omega)$. If $J : A^2(\Omega') \to A^2(\widetilde{\Omega})$ denotes the canonical identification, $\mathcal{L}$ factorises as $\mathcal{L} = \widetilde{\mathcal{L}} J$. By Theorem 4.7, $J \in E(c, 1/d)$, where c is as in (16), and (11) gives $|J|_{c,1/d} \leq N e^{\frac{1-d}{2}\min(\Gamma)}$. Lemma 2.2 now shows that $\mathcal{L} = \widetilde{\mathcal{L}} J \in E(c, 1/d)$, with $|\mathcal{L}|_{c,1/d} \leq \|\widetilde{\mathcal{L}}\|_{A^2(\widetilde{\Omega})\to A^2(\Omega)} |J|_{c,1/d}$, and (13) yields the desired bound for $|\mathcal{L}|_{c,1/d}$. □

Remark 5.10. In Theorem 5.9 there is some freedom in the choice of $\widetilde{\Omega}$. The condition $\cup_{i\in\mathcal{I}}\phi_i(\Omega) \subset\subset \widetilde{\Omega}$ ensures that $\widetilde{\mathcal{L}} : A^2(\widetilde{\Omega}) \to A^2(\Omega)$ is bounded, while $\widetilde{\Omega} \subset\subset \Omega'$ is required so that $J : A^2(\Omega') \to A^2(\widetilde{\Omega})$ lies in some exponential class $E(c, 1/d)$. In practice the choice of $\widetilde{\Omega}$ subject to (15) would be made according to the relative importance of a sharp bound on c or on $|\mathcal{L}|_{c,1/d}$; for the former it is preferable to choose $\widetilde{\Omega}$ only slightly larger than $\cup_{i\in\mathcal{I}}\phi_i(\Omega)$, whereas the latter is achieved by taking $\widetilde{\Omega}$ only slightly smaller than Ω'.

We now wish to consider the transfer operator $\mathcal{L}$ as an endomorphism of a space $A^2(\Omega)$, and derive explicit bounds on its eigenvalues. For this it is convenient to define, for $a, \alpha > 0$,

$$\mathcal{E}(a, \alpha) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \,:\, |x|_{a,\alpha} := \sup_{n\in\mathbb{N}} |x_n| \exp(a n^\alpha) < \infty \right\}\,, \quad \mathcal{E}(\alpha) := \bigcup_{a>0} \mathcal{E}(a, \alpha)\,.$$

The following result is from [**Ban**]; for completeness we give the short proof here.

Lemma 5.11. *Let $\alpha > 0$. If $A \in E(\alpha)$ then $\lambda(A) \in \mathcal{E}(\alpha)$. More precisely, if $A \in E(c, \alpha)$ then $\lambda(A) \in \mathcal{E}(c/(1+\alpha), \alpha)$, with $|\lambda(A)|_{c/(1+\alpha),\alpha} \leq |A|_{c,\alpha}$.*

Proof. If $A \in E(c, \alpha)$ then $s_k(A) \leq |A|_{c,\alpha} \exp(-ck^\alpha)$. The multiplicative Weyl inequality [**Pie3**, 3.5.1] gives

$$|\lambda_k(A)|^k \leq \prod_{l=1}^k |\lambda_l(A)| \leq \prod_{l=1}^k s_l(A) \leq \prod_{l=1}^k |A|_{c,\alpha} \exp(-cl^\alpha) = |A|_{a,\alpha}^k \exp(-c \sum_{l=1}^k l^\alpha)\,,$$

and $\sum_{l=1}^k l^\alpha \geq \int_0^k x^\alpha\, dx = \frac{1}{1+\alpha} k^{\alpha+1}$, so $|\lambda_k(A)| \leq |A|_{c,\alpha} \exp(-ck^\alpha/(1+\alpha))$. □

Remark 5.12. Lemma 5.11 is sharp, in the sense that there exists an operator $A \in E(c,\alpha)$ such that $\lambda(A) \notin \mathcal{E}(b,\alpha)$ whenever $b > c/(1+\alpha)$ (see [**Ban**, Proposition 2.10]).

The following result is a detailed version of the theorem stated in §1.

Theorem 5.13. *Let $(\Omega, \phi_i, w_i)_{i\in\mathcal{I}}$ be a holomorphic map-weight system on $\Omega \in \mathcal{O}_d$. Let $\widetilde{\Omega} \subset\subset \Omega$ be such that*

$$\cup_{i\in\mathcal{I}}\phi_i(\Omega) \subset\subset \widetilde{\Omega} \subset\subset \Omega\,,$$

and such that $(\Omega, \widetilde{\Omega})$ has a relative cover of size N with efficiency Γ. Then the eigenvalue sequence $\lambda(\mathcal{L})$ of the corresponding transfer operator $\mathcal{L} : A^2(\Omega) \to A^2(\Omega)$ satisfies

$$\lambda(\mathcal{L}) \in \mathcal{E}(dc/(1+d), 1/d) \quad \textit{with } |\lambda(\mathcal{L})|_{dc/(1+d),1/d} \le |\mathcal{L}|_{c,1/d},$$

where $c = \|\Gamma\|_d$, and $|\mathcal{L}|_{c,1/d}$ can be bounded as in (17).

In particular,

$$|\lambda_n(\mathcal{L})| \le |\mathcal{L}|_{c,1/d} \exp\left(-\left(\frac{dc}{1+d}\right) n^{1/d}\right) \quad \textit{for all } n \in \mathbb{N}\,.$$

Proof. This follows from Theorem 5.9 and the case $\alpha = 1/d$ in Lemma 5.11. □

6. An application: Taylor coefficients of the determinant

By Theorem 5.9, the transfer operator $\mathcal{L} : A^2(\Omega) \to A^2(\Omega)$ for a holomorphic map-weight system on Ω is trace class, so we may consider the corresponding spectral determinant $\det(I - \zeta\mathcal{L})$, given for small $\zeta \in \mathbb{C}$ by (see e.g. [**Sim**, Chapter 3])

$$\det(I - \zeta\mathcal{L}) = \exp(-\sum_{n=1}^{\infty} a_n(\mathcal{L})\zeta^n)\,, \tag{18}$$

where $a_n(\mathcal{L}) = \frac{1}{n}\operatorname{tr}(\mathcal{L}^n)$. This formula admits a holomorphic extension to the whole complex plane, so that $\zeta \mapsto \det(I - \zeta\mathcal{L})$ becomes an entire function. Writing

$$\det(I - \zeta\mathcal{L}) = 1 + \sum_{n=1}^{\infty} \alpha_n(\mathcal{L})\,\zeta^n\,, \tag{19}$$

the Taylor coefficients $\alpha_n(\mathcal{L})$ can be bounded as follows:

Theorem 6.1. *Let $\mathcal{L}$ be the transfer operator associated to the holomorphic map-weight system $(\Omega, \phi_i, w_i)_{i\in\mathcal{I}}$ on $\Omega \in \mathcal{O}_d$. If $\det(I - \zeta\mathcal{L}) = \sum_{n=0}^{\infty} \alpha_n(\mathcal{L})\zeta^n$ then*

$$|\alpha_n(\mathcal{L})| \le |\mathcal{L}|^n_{c,1/d} \exp\left(-\frac{d}{d+1} c n^{1+1/d} + \sum_{i=0}^{d} \frac{d!}{(d-i)!}\frac{n^{1-i/d}}{c^i}\right) \tag{20}$$

for all $n \in \mathbb{N}$, where c and $|\mathcal{L}|_{c,1/d}$ can be chosen as in Theorem 5.9.

Proof. By [**Sim**, Lemma 3.3],

$$\alpha_n(\mathcal{L}) = \sum_{i_1<\ldots<i_n} \prod_{j=1}^{n} \lambda_{i_j}(\mathcal{L})\,,$$

the summation being over n-tuples of positive integers $(i_1, \ldots, i_n)$ with $i_1 < \ldots < i_n$. Now

$$\sum_{i_1<\ldots<i_n} \left|\prod_{j=1}^{n} \lambda_{i_j}(\mathcal{L})\right| \le \sum_{i_1<\ldots<i_n} \prod_{j=1}^{n} s_{i_j}(\mathcal{L})\,,$$

by [**GGK**, Cor. VI.2.6]. But $s_n(\mathcal{L}) \le |\mathcal{L}|_{c,1/d} \exp(-cn^{1/d})$ for all $n \in \mathbb{N}$, so

$$|\alpha_n(\mathcal{L})| \le |\mathcal{L}|^n_{c,1/d}\,\beta_n(c,d)\,, \tag{21}$$

where $\beta_n = \beta_n(c, d)$ are the Taylor coefficients of the function $f_{c,1/d}$ defined by

$$f_{c,1/d}(\zeta) = \prod_{n=1}^{\infty}(1 + \zeta \exp(-cn^{1/d})) = \sum_{n=0}^{\infty} \beta_n(c, d)\zeta^n .$$

Fried [**Fri**, p. 507] estimates $\log 1/\beta_n \geq n \log r - c^{-d}P(\log r)$, where $P(x) := \sum_{i=0}^{d+1} \frac{d!}{i!}x^i$. Setting $\log r = cn^{1/d}$ gives

$$\beta_n \leq \exp\left(-cn^{1+1/d} + c^{-d}P(cn^{1/d})\right) = \exp\left(-\frac{d}{d+1}cn^{1+1/d} + \sum_{i=0}^{d} \frac{d!}{(d-i)!}\frac{n^{1-i/d}}{c^i}\right) ,$$

and combining with (21) gives the required bound on $\alpha_n(\mathcal{L})$. □

One motivation for Theorem 6.1 is the possibility of obtaining *a posteriori* bounds on the eigenvalues of transfer operators $\mathcal{L} : A^2(\Omega) \to A^2(\Omega)$. In other words, for a particular $\mathcal{L}$, we wish to rigorously bound the quality of computed approximations to the eigenvalues $\lambda_i(\mathcal{L})$. In particular cases these bounds may be sharper than the a priori estimates of §5. In dimension $d = 1$ such rigorous a posteriori analysis has been performed in [**JP1, JP3**]. The bounds on $\alpha_n(\mathcal{L})$ in Theorem 6.1 are sharper than those of [**JP1, JP3**], and valid for arbitrary Ω in arbitrary dimension d.

We now outline the method of a posteriori analysis based on Theorem 6.1. Comparison of the two expressions (18) and (19) for $\det(I - \zeta\mathcal{L})$ yields the identity

$$\alpha_n(\mathcal{L}) = \sum_{\substack{(n_1,\dots,n_j) \\ n_1+\cdots+n_j=n}} \frac{(-1)^j}{j!} \prod_{l=1}^{j} a_{n_l}(\mathcal{L}) . \tag{22}$$

In particular, each $\alpha_n(\mathcal{L})$ is expressible in terms of $a_1(\mathcal{L}), \dots, a_n(\mathcal{L})$. The importance of this is underscored by Ruelle's observation [**Rue3**] that each $a_i(\mathcal{L})$ can itself be expressed in terms of fixed points (which are numerically computable) of compositions of the maps $(\phi_i)_{i\in\mathcal{I}}$. More precisely, if $\underline{i} := (i_1, \dots, i_n) \in \mathcal{I}^n$ then $\phi_{\underline{i}} := \phi_{i_n} \circ \cdots \circ \phi_{i_1}$ has a unique fixed point $z_{\underline{i}}$ [**Rue3**, Lem. 1]. If $w_{\underline{i}} := \prod_{j=0}^{n-1} w(z_{\sigma^j \underline{i}})$, where $\sigma^j\underline{i} := (i_{j+1}, \dots i_n, i_1, \dots, i_j)$, Ruelle's formula is

$$a_n(\mathcal{L}) = \frac{1}{n}\mathrm{tr}(\mathcal{L}^n) = \frac{1}{n}\sum_{\underline{i}\in\mathcal{I}^n} \frac{w_{\underline{i}}}{\det(I - \phi'_{\underline{i}}(z_{\underline{i}}))} , \tag{23}$$

where $\phi'_{\underline{i}}$ denotes the derivative of $\phi_{\underline{i}}$.

Now fix $N \in \mathbb{N}$ such that for all $\underline{i} \in \cup_{1\leq n\leq N}\mathcal{I}^n$, the fixed point $z_{\underline{i}}$ can be determined computationally to a given numerical precision. The Taylor coefficients $\alpha_1(\mathcal{L}), \dots, \alpha_N(\mathcal{L})$ may then be computed via (22), (23), and used to define the polynomial function $\Delta_N(\zeta) := 1 + \sum_{n=1}^{N} \alpha_n(\mathcal{L})\zeta^n$, an approximation to $\Delta(\zeta) := \det(I - \zeta\mathcal{L})$. If $\zeta_1, \zeta_2, \dots$ are the zeros of Δ, ordered by increasing modulus and listed with multiplicity, then each $\zeta_i = \lambda_i(\mathcal{L})^{-1}$. Let $\zeta_{N,1}, \dots, \zeta_{N,N}$ denote the zeros of Δ_N, ordered by increasing modulus and listed with multiplicity; these zeros can be computed to a given precision, and their reciprocals will approximate the corresponding eigenvalues of $\mathcal{L}$. In this way any eigenvalue $\lambda_i(\mathcal{L})$ may be approximated by the numerically computable values $\zeta_{N,i}^{-1}$. A practical issue concerns the quality of this approximation, and it is here that the a priori bounds on the $\alpha_n(\mathcal{L})$ can be used. The error $|\zeta_i - \zeta_{N,i}|$ may be bounded using Rouché's theorem: if C is a circle of radius $\varepsilon > 0$, centred at $\zeta_{N,i}$ and enclosing no other zero of Δ_N, and if it can be shown that

$$|\Delta_N(\zeta) - \Delta(\zeta)| < |\Delta_N(\zeta)| \quad \text{for } \zeta \in C , \tag{24}$$

then ζ_i lies inside C, so $|\zeta_i - \zeta_{N,i}| < \varepsilon$. As $\Delta(\zeta) - \Delta_N(\zeta) = \sum_{n=N+1}^{\infty} \alpha_n(\mathcal{L})\zeta^n$, the lefthand side of (24) can be estimated in terms of $\alpha_n(\mathcal{L})$, $n > N$, which are bounded by Theorem 6.1.

7. Appendix A: A proof of Grothendieck's Remarque 9

In his thesis [**Gro**], Grothendieck proved that the eigenvalues of a bounded operator on a quasi-complete nuclear space decrease *rapidly* [**Gro**, Chap II, §2, No. 4, Corollaire 3]. He also noted that this result could be improved for certain spaces: in [**Gro**, Chap II, §2, No.4, Remarque 9] he provides a sketch of a proof that shows that the eigenvalue sequence $\lambda(L)$ of any bounded operator L on $\mathcal{H}(\Omega)$, $\Omega \in \mathcal{O}_d$, satisfies[3] $\lambda(L) \in \mathcal{E}(1/d)$.

The results of this paper allow us to give a short alternative proof of Grothendieck's Remarque 9. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a collection of members of $\mathcal{O}_d$ such that $\Omega_n \subset\subset \Omega_{n+1}$ for $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$. For $n \in \mathbb{N}$, define the seminorm p_n on $\mathcal{H}(\Omega)$ by $p_n(f) := \sqrt{\int_{\Omega_n} |f(z)|^2 \, dV(z)}$ (note that p_n gives the norm on $A^2(\Omega_n)$). Then $\{p_n\}$ forms a directed system of seminorms which turns $\mathcal{H}(\Omega)$ into a Fréchet space and which, by Lemma 2.4, coincides with the usual topology of uniform convergence on compact subsets of Ω. Moreover, since each identification $A^2(\Omega_{n+1}) \to A^2(\Omega_n)$ is nuclear by Theorem 4.7, the space $\mathcal{H}(\Omega)$ is nuclear.

Recall that a subset S of a topological vector space E is *bounded* if for each neighbourhood U of 0, we have $S \subset \alpha U$ for some $\alpha > 0$. A linear operator $L : E \to E$ is *bounded* if it takes a neighbourhood of 0 into a bounded set. We are now able to prove the following.

THEOREM 7.1. *[Grothendieck] Suppose $\Omega \in \mathcal{O}_d$, and $L : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ is a bounded linear operator. Then:*

(i) *There exists a sequence $\{s_k\}$ of positive numbers belonging to $\mathcal{E}(1/d)$, an equicontinuous sequence $\{f'_k\}$ in the topological dual $\mathcal{H}(\Omega)'$ of $\mathcal{H}(\Omega)$, and a bounded sequence $\{f_k\}$ in $\mathcal{H}(\Omega)$, such that L can be written*
$$Lf = \sum_k s_k \langle f, f'_k \rangle f_k \quad \text{for all } f \in \mathcal{H}(\Omega)\,.$$
Here, $\langle f, f' \rangle$ denotes the evaluation of $f' \in \mathcal{H}(\Omega)'$ at f.

(ii) $\lambda(L) \in \mathcal{E}(1/d)$.

PROOF. The two assertions will follow from a factorisation of L, which we shall first derive. Since L is bounded, there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, there is a constant M_n satisfying $p_n(Lf) \le M_n p_{n_0}(f)$ for all $f \in \mathcal{H}(\Omega)$. Fixing $m > n_0$, let $J_1 : \mathcal{H}(\Omega) \to A^2(\Omega_m)$ and $J_2 : A^2(\Omega_m) \to A^2(\Omega_{n_0})$ denote canonical identifications. Clearly, J_1 and J_2 are continuous. Let $\overline{J_2J_1\mathcal{H}(\Omega)}$ be the closure of $J_2J_1\mathcal{H}(\Omega)$ in the Hilbert space $A^2(\Omega_{n_0})$ and let $P : A^2(\Omega_{n_0}) \to \overline{J_2J_1\mathcal{H}(\Omega)}$ be the corresponding orthogonal projection. Then the linear map $f \in J_2J_1\mathcal{H}(\Omega) \mapsto Lf \in \mathcal{H}(\Omega)$ is well-defined and bounded, and therefore extends to a bounded linear map $\widetilde{L} : \overline{J_2J_1\mathcal{H}(\Omega)} \to \mathcal{H}(\Omega)$. The operator L therefore admits the factorisation
$$L = \widetilde{L}PJ_2J_1\,. \tag{25}$$
To prove (i), note that $J_2 \in E(1/d)$ by Theorem 4.7, so we have the Schmidt representation $J_2 f = \sum_k s_k(J_2)\,(f, a_k)_m\, b_k$, where $\{a_k\}$ and $\{b_k\}$ are orthonormal systems in $A^2(\Omega_m)$ and $A^2(\Omega_{n_0})$ respectively and $(\cdot,\cdot)_m$ denotes the inner product in $A^2(\Omega_m)$. Since $\widetilde{L}P$ is continuous,
$$Lf = \widetilde{L}PJ_2J_1 f = \sum_k s_k(J_2)\,(J_1 f, a_k)_m\,\widetilde{L}Pb_k\,,$$
which can be written as
$$Lf = \sum_k s_k(J_2)\,\langle f, J'_1 a'_k \rangle\,\widetilde{L}Pb_k\,, \tag{26}$$

[3]Grothendieck in fact asserted that $\lambda(L) \in \mathcal{E}(1)$, though his arguments can be modified so as to yield $\lambda(L) \in \mathcal{E}(1/d)$.

where J_1' denotes the adjoint of J_1, and a_k' the image of a_k under the canonical isomorphism of $A^2(\Omega_m)$ and its dual. In order to see that the representation (26) has the desired properties, we note that $\{\widetilde{L}Pb_k\}$ is bounded, since it is the continuous image of a bounded set. Furthermore, $\{J_1'a_k'\}$ is equicontinuous in the dual of $\mathcal{H}(\Omega)$, since

$$\left|\langle f, J_1'a_k'\rangle\right| = |(J_1 f, a_k)_m| \leq p_m(J_1 f)p_m(a_k) \leq p_m(f)\,.$$

Therefore (i) is proved.

To prove (ii) we again use the factorisation (25). By Pietsch's principle of related operators (see [**Pie1**, Satz 2]), $\lambda(L) = \lambda(\widetilde{L}PJ_2J_1) = \lambda(J_1\widetilde{L}PJ_2)$. But $J_1\widetilde{L}P : A^2(\Omega_{n_0}) \to A^2(\Omega_m)$ is a bounded operator between Hilbert spaces, and $J_2 \in E(1/d)$ by Theorem 4.7, so $J_1\widetilde{L}PJ_2 \in E(1/d)$ by Lemma 2.2, hence $\lambda(J_1\widetilde{L}PJ_2) \in \mathcal{E}(1/d)$ by Lemma 5.11, and (ii) follows. □

Remark 7.2. In our approach, assertion (ii) of Theorem 7.1 follows by combining Theorem 4.7 with Weyl's multiplicative inequality, whereas Grothendieck suggests to derive (ii) from (i) by considering the growth of the determinant $\det(I - \zeta L)$ at infinity and using Jensen's theorem to determine bounds on the distribution of its zeros. A more detailed analysis of this circle of ideas will be presented in the following §8.

8. Appendix B: Eigenvalue estimates via the determinant

Given a transfer operator $\mathcal{L}$ associated to a holomorphic map-weight system on $\Omega \in \mathcal{O}_d$, we have shown (Theorem 5.9) how to find explicit constants $a, A > 0$ such that

$$s_n(\mathcal{L}) \leq A\exp(-an^{1/d}) \quad \text{for all } n \in \mathbb{N}\,, \tag{27}$$

and used this (Theorem 5.13) to find explicit $b, B > 0$ for which

$$|\lambda_n(\mathcal{L})| \leq B\exp(-bn^{1/d}) \quad \text{for all } n \in \mathbb{N}\,. \tag{28}$$

The purpose of this appendix is to outline an alternative, less direct, method of obtaining eigenvalue bounds analogous to (28), again starting from the singular value estimate (27). This approach is based on an analysis of the growth of the determinant $\det(I-\zeta\mathcal{L})$, and was originally suggested by Grothendieck in [**Gro**, Chap. II, §2, No. 4, Remarque 9]. Further details of this strategy were given by Fried [**Fri**], and we shall offer some commentary on Fried's analysis, in particular his Lemma 6, adapted slightly to our Hilbert space setting.

A bound of the type (27) is not proved in [**Fri**], though does appear to be tacitly assumed [**Fri**, p. 506, line 8], on the basis of a suggested correction of [**Gro**, II, Remarque 9, p. 62–4] (see [**Fri**, p. 506, line 3], and our comments in Sections 1 and 7). With the singular value estimate (27) in hand, it is possible to analyse the growth properties of the function $\zeta \mapsto \det(I-\zeta\mathcal{L})$, which is entire because $\mathcal{L}$ is trace class (see §6). This is the content of [**Fri**, Lemma 6], which we now review, incorporating some refinements available in the Hilbert space setting. We start by writing

$$\det(I - \zeta\mathcal{L}) = \sum_{n=0}^{\infty} \alpha_n(\mathcal{L})\zeta^n\,.$$

As in Theorem 6.1 we use the formula

$$\alpha_n(\mathcal{L}) = \sum_{i_1<\ldots<i_n} \prod_{j=1}^{n} \lambda_{i_j}(\mathcal{L})\,,$$

and the inequality

$$\sum_{i_1<\ldots<i_n} \left|\prod_{j=1}^{n} \lambda_{i_j}(\mathcal{L})\right| \leq \sum_{i_1<\ldots<i_n} \prod_{j=1}^{n} s_{i_j}(\mathcal{L})\,,$$

to deduce that

$$|\alpha_n(\mathcal{L})| \leq A^n\beta_n(a,d)\,, \tag{29}$$

where $\beta_n(a,d)$ are the Taylor coefficients of the function $f_{a,1/d}$ defined by

$$f_{a,1/d}(\zeta) = \prod_{n=1}^{\infty}(1+\zeta\exp(-an^{1/d})) = \sum_{n=0}^{\infty}\beta_n(a,d)\zeta^n\,.$$

Note that (29) is sharper than the corresponding estimate in [**Fri**, p. 506], which contains an extra factor $n^{n/2}$. Following Fried, the coefficients $\beta_n = \beta_n(a,d)$ can be estimated, using Cauchy's theorem, by $\beta_n \leq r^{-n}M(r)$, where $M(r)$ is the maximum modulus of $f_{a,1/d}(\zeta)$ on $|\zeta| = r$. Using either the asymptotics

$$\log f_{a,1/d}(r) \sim a^{-d}\frac{1}{d+1}(\log r)^{1+1/d} \quad \text{as } r\to\infty$$

in [**Ban**, Proof of Proposition 3.1 (i)], or Fried's calculation that $\log 1/\beta_n \geq n\log r - a^{-d}P(\log r)$, where $P(x) := \sum_{j=0}^{d+1}\frac{d!}{j!}x^j$, we see that for any $\delta_0 > 1$,

$$\log 1/\beta_n \geq n\log r - \delta_0 a^{-d}\frac{1}{d+1}(\log r)^{1+1/d},$$

for r sufficiently large. Choosing $\log r = an^{1/d}$ gives $\log 1/\beta_n \geq \delta_1 a\frac{d}{d+1}n^{1+1/d}$ for n sufficiently large, where $\delta_1 = 1-(\delta_0-1)/d$. Therefore there exists $K>0$, depending on δ_1, such that

$$|\alpha_n(\mathcal{L})| \leq KA^n\exp\left(-\delta_1 a\frac{d}{d+1}n^{1+1/d}\right) \quad \text{for all } n\in\mathbb{N}\,.$$

Thus if $g(r) := \sum_{n=1}^{\infty} r^n\exp\left(-\delta_1 a\frac{d}{d+1}n^{1+1/d}\right)$ then

$$|\det(I-\zeta\mathcal{L})| \leq 1+K\sum_{n=1}^{\infty}|\zeta|^nA^n\exp\left(-\delta_1 a\frac{d}{d+1}n^{1+1/d}\right) = 1+Kg(A|\zeta|)\,.$$

To estimate the growth of g, define[4] $\mu(r) := \max_{1\leq n\leq\infty} r^n\exp\left(-\delta_1 a\frac{d}{d+1}n^{1+1/d}\right)$. This maximal term can be calculated explicitly using calculus (see [**Ban**, Proof of Proposition 3.1 (ii)]), and we obtain

$$\log\mu(r) \sim (\delta_1 a)^{-d}\frac{1}{d+1}(\log r)^{1+d} \quad \text{as } r\to\infty\,.$$

But g is an entire function of finite order, so $\log\mu(r)\sim\log g(r)$ as $r\to\infty$ (see e.g. [**PS**, Problem 54]), hence $\log g(r) \sim (\delta_1 a)^{-d}\frac{1}{d+1}(\log r)^{1+d}$ as $r\to\infty$. Therefore, for $|\zeta|$ sufficiently large and $\delta_2 \geq \delta_1^{-d}$,

$$\log|\det(1-\zeta\mathcal{L})| \leq \delta_2 a^{-d}\frac{1}{d+1}(\log|\zeta|A)^{1+d}\,. \tag{30}$$

The bound (30) allows us to estimate the speed with which the zeros of $\det(1-\zeta\mathcal{L})$ tend to infinity. Specifically, if $n(r)$ denotes the number of zeros of $\det(1-\zeta\mathcal{L})$ in the disk of radius r centred at 0, and $N(r) := \int_0^r t^{-1}n(t)\,dt$, Jensen's theorem (see e.g. [**Boa**, p. 2]) gives

$$N(r) \leq \delta_2 a^{-d}\frac{1}{d+1}(\log rA)^{1+d} \tag{31}$$

for r sufficiently large. We now require the following lemma:

Lemma 8.1. *If $N(r)\leq K(\log r)^{1+d}$ for some positive real number d, then*

$$n(r) \leq K\frac{(1+d)^{1+d}}{d^d}(\log r)^d.$$

[4]Alternatively one could proceed as in [**Fri**], but the method there is a little less sharp.

Proof. If $p > 1$ then $(p-1)n(r)\log r = n(r)\int_r^{r^p} t^{-1}\,dt \le \int_r^{r^p} t^{-1}n(t)\,dt \le N(r^p)$, so

$$n(r) \le \frac{N(r^p)}{(p-1)\log r} \le \frac{Kp^{1+d}(\log r)^{1+d}}{(p-1)\log r}\,.$$

The assertion follows by choosing $p = 1 + 1/d$. $\square$

Combining (31) and Lemma 8.1 gives $n(r) \le \delta_3 a^{-d}\left(\frac{1+d}{d}\right)^d(\log rA)^d$ for r sufficiently large. But the zeros of $\det(I - \zeta\mathcal{L})$ are precisely the numbers $\lambda_1(\mathcal{L})^{-1}$, $\lambda_2(\mathcal{L})^{-1}, \ldots$, ordered by modulus, so for n sufficiently large, $n \le \delta_3 a^{-d}\left(\frac{1+d}{d}\right)^d(\log A|\lambda_n(\mathcal{L})|^{-1})^d$, and finally we deduce the required eigenvalue bound

$$|\lambda_n(\mathcal{L})| \le A\exp\left(-\delta_3^{-1/d} a\frac{d}{1+d}n^{1/d}\right) \quad \text{for } n \text{ sufficiently large.} \tag{32}$$

Since δ_3 can be chosen arbitrarily close to 1, (32) can be made arbitrarily close to the bound of Lemma 5.11. Note, however, that (32) only holds for $n \ge N$, for some unknown N, whereas the bound of Theorem 5.13 is valid for all $n \in \mathbb{N}$.

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