EIGENVALUES OF TRANSFER OPERATORS FOR DYNAMICAL SYSTEMS WITH HOLES

O. F. BANDTLOW & O. JENKINSON

Abstract. For real-analytic expanding open dynamical systems in arbitrary finite dimension, we establish rigorous explicit bounds on the eigenvalues of the corresponding transfer operators acting on spaces of holomorphic functions. In dimension 1 the eigenvalue decay rate is exponentially fast, while in dimension d it is $O(\theta^{n^{1/d}})$ as $n \to \infty$ for some $0 < \theta < 1$.

1. Introduction

For an expanding map $T : X \to X$, the Perron-Frobenius operator $\mathcal{P}$ defined by

$$\mathcal{P}f(x) = \sum_{Ty=x} \frac{f(y)}{|T'(y)|},$$

and more general transfer operators $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \sum_{Ty=x} e^{\varphi(y)} f(y)$$

with potential function $\varphi : X \to \mathbb{R}$, are important objects in the thermodynamic formalism approach to ergodic theory.

Given a subset $H \subset X$, which we regard as a *hole* in X, it is natural to consider modified operators $\mathcal{P}_H$ and $\mathcal{L}_H$, defined by $\mathcal{P}_H f = \mathcal{P}(f\chi_{X\setminus H})$ and $\mathcal{L}_H f = \mathcal{L}(f\chi_{X\setminus H})$, in view of their connections with escape rate (see, for example, [6, 9]) and various equilibrium measures supported by the survivor set $X_\infty = \cap_{n=0}^{\infty} T^{-n}(X \setminus H)$.

The purpose of this note is to describe, in the case where T is piecewise analytic and H is a suitable hole, explicit estimates on the spectral asymptotics of $\mathcal{P}_H$ and $\mathcal{L}_H$ when acting on various Banach spaces of holomorphic functions.[1]

[1] When acting on these spaces, $\mathcal{P}_H$ has a strictly positive spectral radius δ, with $\delta > 0$ an eigenvalue such that $-\log \delta$ is the corresponding escape rate (see e.g. [14] for one-dimensional maps); thus escape is at an exponential rate, rather than anything faster. Moreover, $\delta^{-n}\mathcal{P}_H^n 1 \to \varrho$, where ϱ is the density function for the Pianigiani-Yorke measure [15].

Specifically, we take $X \subset \mathbb{R}^d$ to be compact and connected, and $\mathcal{X} = \{X_i\}_{i\in\mathcal{I}}$ a finite partition (consisting of non-empty pairwise disjoint subsets of X, each one open in $\mathbb{R}^d$, whose union is dense in X). The map $T : X \to X$ is assumed Borel measurable, with $T(X_i)$ open in $\mathbb{R}^d$ for each $i \in \mathcal{I}$, and $T|_{X_i} : X_i \to T(X_i)$ a C^1 diffeomorphism which can be extended to a C^1 map on $\overline{X_i}$. We assume that T is *full branch*, i.e. $\overline{T(X_i)} = X$ for all $i \in \mathcal{I}$, and *expanding*, i.e. there exists $\beta > 1$ such that if $x, y \in X_i$ for some $i \in \mathcal{I}$ then $\|T(x) - T(y)\| \geq \beta \|x - y\|$. Each $T|_{X_i}$ has an *inverse branch* T_i, defined so that $T \circ T_i$ is the identity on the interior of X, and $T_i \circ T$ the identity on X_i, and satisfying $\sup_{x\in int(X)} \|T_i'(x)\|_{L(\mathbb{R}^d)} \leq \beta^{-1}$ for all $i \in \mathcal{I}$, where $\|\cdot\|_{L(\mathbb{R}^d)}$ denotes the induced operator norm on $L(\mathbb{R}^d) = L((\mathbb{R}^d, \|\cdot\|))$. We assume that $T : X \to X$ is *real analytic*, i.e. there is a bounded connected open set $D \subset \mathbb{C}^d$, with $X \subset D$, such that each T_i has a holomorphic extension to D.

For simplicity we shall take the hole H to be a union of some (but not all) elements of $\mathcal{X}$. In fact with some extra effort, and more cumbersome notation, the techniques described here extend to the case where H is a union of members of some refinement $\vee_{i=0}^{n-1} T^{-i}\mathcal{X}$ (a so-called Markov hole). Let $\mathcal{J} \subset \mathcal{I}$ be such that $\cup_{i\in\mathcal{J}} X_i = X \setminus H$. Transfer operators $\mathcal{L}_H$ for the open dynamical system $T|_{X\setminus H}$ then take the form

$$\mathcal{L}_H f = \sum_{i\in\mathcal{J}} w_i f \circ T_i \,, \tag{1}$$

where the *weight functions* w_i are related to the potential function φ by $w_i = \exp(\varphi \circ T_i)$ on X, and assumed to admit a holomorphic extension to D which in turn extends continuously to $\overline{D}$. In the particular case $\varphi = -\log |T'|$, when w_i are the holomorphic extensions to D of $|T_i'|$ on X, the corresponding transfer operator is precisely the modified Perron-Frobenius operator $\mathcal{P}_H$. We shall always assume that D has the property that the closure of $\cup_{i\in\mathcal{J}} T_i(D)$ lies inside D itself, referring to such domains D as being *admissible* for the map T; this technical requirement, which we always assume without further comment, will ensure that $\mathcal{L}_H$ preserves suitable Banach spaces of functions holomorphic on D.

The structure of the article is as follows. We begin in §2 by considering transfer operators $\mathcal{L}_H$ acting on the Banach space[2] $U(D)$ of those holomorphic functions on D which extend continuously to $\overline{D}$ equipped with the usual supremum norm $\|w\|_{U(D)} = \sup_{z\in D} |w(z)|$. We show

[2]The study of transfer operators on this space $U(D)$ was inaugurated by Ruelle [18].

(Theorems 1 and 2) that in dimension $d = 1$, the eigenvalues $\lambda_n(\mathcal{L}_H)$ (arranged in order of decreasing modulus) converge to zero exponentially fast, deriving an explicit bound for $|\lambda_n(\mathcal{L}_H)|$. In higher dimensions $d \geq 2$ there are similar explicit bounds (see Theorem 3), though here the convergence to zero is[3] $O(\theta^{n^{1/d}})$ as $n \to \infty$, for some $\theta \in (0,1)$. In §3 we show that in fact the eigenvalues for $\mathcal{L}_H : U(D) \to U(D)$ are identical to those for $\mathcal{L}_H$ acting on a variety of Banach spaces $A(D)$ of holomorphic functions. This suggests the possibility of improving the bounds of §2 by judicious choice of $A(D)$, a strategy we pursue in §4 where $A(D)$ is chosen to be Hilbert Hardy space $H^2(D)$, yielding Theorems 6 and 7.

2. Eigenvalue estimates via Weyl's inequality

We begin with an explicit estimate on the eigenvalues of the modified Perron-Frobenius operator in dimension $d = 1$:

Theorem 1. *For an expanding interval map, the eigenvalues of the modified Perron-Frobenius operator $\mathcal{P}_H : U(D) \to U(D)$ satisfy*

$$|\lambda_n(\mathcal{P}_H)| \leq \theta^{n-1}\sqrt{n} \sup_{z\in D} \sum_{i\in\mathcal{J}} |T_i'(z)| \quad \textit{for all } n \geq 1\,, \tag{2}$$

provided each T_i' extends holomorphically to a disc $D \subset \mathbb{C}$, where $\theta < 1$ is such that $\cup_{i\in\mathcal{J}} T_i(D)$ is contained in the concentric disc whose radius is θ^2 times that of D.

The bounds in Theorem 1 are readily computed for specific maps T:

Example 1. As in [2], we consider the map

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \leq \frac{i+1}{10}, \text{ for } 1 \leq i \leq 9 \end{cases}$$

Note that the inverse branches $\{T_i\}_{0\leq i\leq 9}$ are given by

$$T_0(x) = \frac{x}{9+x}$$

and

$$T_i(x) = (x+i)/10 \quad \text{for } 1 \leq i \leq 9\,.$$

Choosing Markov hole $H = [1/5, 1]$ corresponds to setting $\mathcal{J} = \{0, 1\}$.

[3]Ruelle [18], following Grothendieck [11], stated the asymptotics were $O(\theta^n)$ as $n \to \infty$, independent of the dimension d, though Fried [10] corrected this to $O(\theta^{n^{1/d}})$. One novelty of our results, relative to Fried and Ruelle, is that the constant θ, as well as the implicit constant in the big-O asymptotics, are rendered explicit.

We claim that the eigenvalues of the modified Perron-Frobenius operator $\mathcal{P}_H : U(D) \to U(D)$ are bounded by

$$|\lambda_n(\mathcal{P}_H)| \le \frac{77}{320}\sqrt{5n}\left(\frac{1}{\sqrt{5}}\right)^n \quad \text{for all } n \ge 1\,. \tag{3}$$

In particular, note that the case $n = 1$ yields a bound on the escape rate γ (see, for example, [6]) for this open dynamical system, namely $\gamma \ge -\log 77/320$.

Let D be the disc of radius 1 centred at 0. Noting that $T_0(-1) = -1/8$, $T_1(-1) = 0$, $T_0(1) = 1/10$, and $T_1(1) = 1/5$, we see that $\cup_{i\in\mathcal{J}} T_i(D)$ is contained in the disc of radius $1/5$ centred at 0. This means we may set $\theta = 1/\sqrt{5}$ in Theorem 1. Note that $|T_0'(z)|+|T_1'(z)| = \frac{9}{|9+z|^2} + \frac{1}{10}$, and the supremum of this expression on D is the value $77/320$, attained (on the boundary of D) at $z = -1$. The bound (3) then follows from (2).

In fact Theorem 1 is a special case of the following one-dimensional result:

Theorem 2. *For an expanding interval map, the eigenvalues of the transfer operator $\mathcal{L}_H : U(D) \to U(D)$ satisfy:*

$$|\lambda_n(\mathcal{L}_H)| \le \theta^{n-1}\sqrt{n}\,\sup_{z\in D}\sum_{i\in\mathcal{J}} |w_i(z)| \quad \textit{for all } n \ge 1\,, \tag{4}$$

provided each w_i and T_i extend holomorphically to the disc $D \subset \mathbb{C}$, where $\theta < 1$ is such that $\cup_{i\in\mathcal{J}} T_i(D)$ is contained in the concentric disc whose radius is θ^2 times that of D.

Proof. Let D' denote the concentric disc whose radius is $r = \theta^2$ times that of D. First we observe that $\hat{\mathcal{L}}_H f := \sum_{i\in\mathcal{J}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}}_H : U(D') \to U(D)$. To see this, fix $f \in U(D')$ and note that $w_i \cdot f \circ T_i \in U(D)$ with $\|w_i \cdot f \circ T_i\|_{U(D)} \le \|w_i\|_{U(D)} \|f\|_{U(D')}$ for every $i \in \mathcal{J}$. But $\|\hat{\mathcal{L}}_H f\|_{U(D)} \le \sum_{i\in\mathcal{J}} \|w_i\|_{U(D)} \|f\|_{U(D')}$, so $\hat{\mathcal{L}}_H f \in U(D)$ and $\hat{\mathcal{L}}_H$ is continuous. Now $\|\hat{\mathcal{L}}_H\| \le W =: \sup_{z\in D} \sum_{i\in\mathcal{J}} |w_i(z)|$, because for $f \in U(D')$ we have $|f(T_i(z))| \le \|f\|_{U(D')}$ for every $z \in D$, $i \in \mathcal{J}$; thus by the maximum modulus principle $\|\hat{\mathcal{L}}_H f\|_{U(D)} = \sup_{z\in D} |(\hat{\mathcal{L}}_H f)(z)| \le \sup_{z\in D} \sum_{i\in\mathcal{J}} |w_i(z)|\,|f(T_i(z))| \le W\,\|f\|_{U(D')}$.

Recall that if $L : B_1 \to B_2$ is a continuous operator between Banach spaces then for $k \ge 1$, its *k-th approximation number* $a_k(L)$ is defined by $a_k(L) = \inf\{\|L - K\| \mid K : B_1 \to B_2 \text{ linear with rank}(K) < k\}$, and in general $a_k(L_1L_2) \le \|L_1\|\,a_k(L_2)$ (see [16, 2.2]).

Now clearly $\mathcal{L}_H = \hat{\mathcal{L}}_H J$, where $J : U(D) \hookrightarrow U(D')$ denotes the canonical embedding, so

$$a_k(\mathcal{L}_H) \le \|\hat{\mathcal{L}}_H\| a_k(J) \le W a_k(J) \quad \text{for all } k \ge 1\,. \tag{5}$$

Moreover, it can be shown that $\mathcal{L}_H$ is compact; in fact, it is of exponential class (see [3]), and in particular nuclear of any order.

Before proceeding recall that Weyl's inequality (see, for example, [12]) asserts that $\prod_{k=1}^n |\lambda_k(\mathcal{L}_H)| \le n^{n/2} \prod_{k=1}^n a_k(\mathcal{L}_H)$ for every $n \in \mathbb{N}$.[4] Together with (5) this yields the inequality

$$|\lambda_n(\mathcal{L}_H)| \le W n^{1/2} \prod_{k=1}^n a_k(J)^{1/n} \quad \text{for all } n \ge 1\,, \tag{6}$$

because $|\lambda_n(\mathcal{L}_H)| \le \prod_{k=1}^n |\lambda_k(\mathcal{L}_H)|^{1/n}$.

Using a result originally due to Babenko (see [1] or [17, Theorem VIII.2.1]) we see

$$a_l(J) \le r^{l-1} \quad \text{for all } l \ge 1\,,$$

hence $\prod_{l=1}^n a_l(J)^{1/n} \le r^{\frac{1}{n}\sum_{l=1}^n l-1} = r^{(n-1)/2}$, so (6) becomes

$$|\lambda_n(\mathcal{L}_H)| \le W n^{1/2} r^{(n-1)/2}\,,$$

which is the desired bound (4). □

In higher dimension d the rate of eigenvalue decay is slower than exponential, and can be shown to be $O(\theta^{n^{1/d}})$ as $n \to \infty$, for some $\theta \in (0,1)$. The main new ingredient in the following result, proved in [5], is an estimate due to Farkov [8] on the approximation numbers of the embedding operator J in higher dimensions, namely $a_l(J) \le r^{t_l}$, where $t_l := k$ for $\binom{k-1+d}{d} < l \le \binom{k+d}{d}$.

Theorem 3. *In dimension $d \ge 1$, suppose the Euclidean ball $D \subset \mathbb{C}^d$ is such that $\cup_{i\in\mathcal{J}} T_i(D)$ is contained in the concentric ball whose radius is $r < 1$ times that of D. Setting $W := \sup_{z\in D} \sum_{i\in\mathcal{J}} |w_i(z)|$, the eigenvalues of $\mathcal{L}_H : U(D) \to U(D)$ can be bounded by*

$$|\lambda_n(\mathcal{L}_H)| < \frac{W}{r^d} n^{1/2}\, r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \textit{for all } n \ge 1\,. \tag{7}$$

[4] This is a Banach space version of Weyl's original inequality [19] in Hilbert space; the constant $n^{n/2}$ is optimal (see [12]).

3. The common spectrum

It turns out that a more oblique approach yields different, and sometimes better, bounds on the eigenvalues of $\mathcal{L}_H : U(D) \to U(D)$. This approach consists of varying the space upon which $\mathcal{L}_H$ acts. Clearly, in general $U(D)$ is not the only function space preserved by a transfer operator $\mathcal{L}_H$, and we would expect the spectrum of $\mathcal{L}_H$ to vary according to the space on which it acts. There is interest, however, in identifying a class of spaces $A(D)$ which are sufficiently closely related to $U(D)$ to ensure that the spectrum of $\mathcal{L}_H$ on these spaces is precisely the same as that of $\mathcal{L}_H : U(D) \to U(D)$. This motivates the following definition:

Definition 1. For a non-empty open connected set $D \subset \mathbb{C}^d$, a Banach space $A(D)$ of holomorphic functions $f : D \to \mathbb{C}$ is called *favourable* if it contains $U(D)$, with the natural embedding $U(D) \hookrightarrow A(D)$ having norm 1, and if $f \mapsto f(z)$ is continuous on $A(D)$ for each $z \in D$.

Transfer operators $\mathcal{L}_H$ can be shown (see [5]) to preserve all favourable spaces[5] $A(D)$, with the eigenvalues of $\mathcal{L}_H : A(D) \to A(D)$ related to a certain entire function:

Theorem 4. *The transfer operator $\mathcal{L}_H$ defined by (1) preserves every favourable space $A(D)$ of holomorphic functions on D. It has a well-defined spectral trace $\tau_{A(D)}(\mathcal{L}_H) = \sum_{n=1}^\infty \lambda_n(\mathcal{L}_H|_{A(D)})$ and spectral determinant $\det_{A(D)}$, related by*

$$\det_{A(D)}(I - z\mathcal{L}_H|_{A(D)}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \tau_{A(D)}(\mathcal{L}_H^n)\right), \tag{8}$$

for all $z \in \mathbb{C}$ in a suitable neighbourhood of 0, and such that, counting multiplicities, the zeros of the entire function $z \mapsto \det(I - z\mathcal{L}_H|_{A(D)})$ are precisely the reciprocals of the eigenvalues of $\mathcal{L}_H : A(D) \to A(D)$.

Motivated by the possibility that the trace and determinant do not in fact vary with the choice of favourable space $A(D)$, we follow Ruelle [18] in considering the following function:

Definition 2. For given weight functions w_i, $i \in \mathcal{I}$, the associated *dynamical determinant* is the entire function $\Delta : \mathbb{C} \to \mathbb{C}$, defined for all z of sufficiently small modulus by

$$\Delta(z) = \exp\left(-\sum_{n\in\mathbb{N}} \frac{z^n}{n} \sum_{\underline{i}\in\mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T_{\underline{i}}'(z_{\underline{i}}))}\right), \tag{9}$$

[5]As always, we are making the standing assumption that D is an admissible domain, i.e. that the closure of $\cup_{i\in\mathcal{I}} T_i(D)$ lies in D.

where for $\underline{i} = (i_1, \ldots, i_n) \in \mathcal{J}^n$ we set $T_{\underline{i}} := T_{i_n} \circ \cdots \circ T_{i_1}$, and $w_{\underline{i}} := \prod_{k=1}^n w_{i_k} \circ T_{P_{k-1}\underline{i}}$, where $P_k : \mathcal{J}^n \to \mathcal{J}^k$ denotes the projection $P_k\underline{i} = (i_1, \ldots, i_k)$ with the convention that $T_{P_0\underline{i}} = \mathrm{id}$, and $z_{\underline{i}}$ denotes the (unique, by [7]) fixed-point of $T_{\underline{i}}$ in D.

Theorem 5. *For every favourable space $A(D)$, the determinant of the transfer operator $\mathcal{L}_H : A(D) \to A(D)$ defined by (1) is precisely the dynamical determinant Δ, and its eigenvalue sequence is precisely the same as for $\mathcal{L} : U(D) \to U(D)$.*

Proof. The common trace formula

$$\tau_{A(D)}(\mathcal{L}_H^n) = \sum_{\underline{i} \in \mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))} \quad \text{for all } n \geq 1 \tag{10}$$

can be established (see [5]), valid for every favourable space $A(D)$ on which $\mathcal{L}_H$ acts, so that equality of determinants follows from comparison of (8) and (9). The equality of the eigenvalue sequences follows from the fact that the determinants are spectral. □

4. Hilbert Hardy space

In view of Theorem 5 we are now at liberty to make particular choices of favourable spaces, in the hope of obtaining interesting new bounds on the eigenvalues of the transfer operator $\mathcal{L} : U(D) \to U(D)$.

For $p \in [1, \infty)$, the *Hardy space* $H^p(D)$ (see [13, Ch. 8.3]) is a favourable space, and we will be particularly interested in the Hilbert Hardy space $H^2(D)$.[6] The following eigenvalue bounds, valid in dimension 1, are obtained by choosing favourable space $A(D) = H^2(D)$ for $D \subset \mathbb{C}$ a disc:

Theorem 6. *With the hypotheses and notation of Theorem 2,*

$$|\lambda_n(\mathcal{L}_H)| \leq \frac{W}{\sqrt{1 - \theta^4}}\, \theta^{n-1} \quad \textit{for all } n \geq 1\,. \tag{11}$$

Proof. As in the proof of Theorem 3, let D' denote the concentric disc whose radius is $r = \theta^2$ times that of D, and let $J : H^2(D) \hookrightarrow H^\infty(D')$ denote canonical embedding. It can be shown that, for all $n \geq 1$,

$$|\lambda_n(\mathcal{L}_H)| \leq W \prod_{k=1}^n a_k(J)^{1/n}\,, \tag{12}$$

[6]If D has C^2 boundary then $H^2(D)$ can be identified with the $L^2(\partial D, \sigma)$-closure of $U(D)$, where σ denotes $(2d-1)$-dimensional Lebesgue measure on the boundary ∂D, normalised so that $\sigma(\partial D) = 1$. The inner product in $H^2(D)$ is given (see [13, Ch. 1.5 and 8]) by $(f, g) = \int_{\partial D} f^*\, \overline{g^*}\, d\sigma$, where, for $h \in H^2(D)$, the symbol h^* denotes the corresponding nontangential limit function in $L^2(\partial D, \sigma)$.

an inequality which is superior to (6), by virtue of the original Hilbert space version of Weyl's inequality, namely $\prod_{k=1}^n |\lambda_k(L)| \le \prod_{k=1}^n a_k(L)$ (see [16, 3.5.1], [19]). An argument (see [4]) exploiting the interplay between the reproducing kernel of $H^2(D)$, and an orthonormal basis for $H^2(D)$ then allows the estimate

$$a_n(J) \le \frac{r^{n-1}}{\sqrt{1-r^2}}\,, \tag{13}$$

and substituting into (12) yields the result. □

Example 2. Comparing (12) with (4), we see that Theorem 6 leads to improved eigenvalue bounds whenever $n > 1/(1-\theta^4)$. In Example 1 we can choose $\theta = 1/\sqrt{5}$, therefore for all $n \ge 2 > 25/24$, the estimate

$$|\lambda_n(\mathcal{P}_H)| \le \frac{77}{320}\frac{5}{\sqrt{24}}\left(\frac{1}{\sqrt{5}}\right)^{n-1}$$

derived from (12) is sharper than the previous bound (3) on the eigenvalues of the modified Perron-Frobenius operator.

A more elaborate version of the proof of Theorem 6 (see [4] for details) gives the following higher dimensional analogue, which for sufficiently large values of n yields estimates which are superior to those of Theorem 3:

Theorem 7. *With the hypotheses and notation of Theorem 3,*

$$|\lambda_n(\mathcal{L}_H)| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}}\; n^{(d-1)/(2d)}\; r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}} \quad \textit{for all } n \ge 1\,. \tag{14}$$

References

[1] K. I. Babenko, Best approximations to a class of analytic functions (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.*, **22** (1958), 631–640
[2] W. Bahsoun and C. Bose, Quasi-invariant measures, escape rates and the effect of the hole, *Discrete Contin. Dynam. Sys.*, **27** (2010), 1107–1121.
[3] O. F. Bandtlow, Resolvent estimates for operators belonging to exponential classes, *Integr. Equat. Oper. Th.*, **61** (2008), 21–43.
[4] O. F. Bandtlow and O. Jenkinson, Explicit a priori bounds on transfer operator eigenvalues, *Comm. Math. Phys.*, **276** (2007), 901–905.
[5] O. F. Bandtlow and O. Jenkinson, On the Ruelle eigenvalue sequence, *Ergod. Theor. Dyn. Syst.*, **28** (2008), 1701–1711.
[6] M. Demers & L.-S. Young, Escape rates and conditionally invariant measures, *Nonlinearity*, **19** (2006), 377–397.
[7] C. J. Earle & R. S. Hamilton, A fixed point theorem for holomorphic mappings, in *Global Analysis* (S. Chern & S. Smale, Eds.), Proc. Symp. Pure Math., Vol. XVI, pp. 61–65, American Mathematical Society, Providence R.I., 1970.

[8] Yu. A. Farkov, The N-widths of Hardy-Sobolev spaces of several complex variables, *J. Approx. Theory*, **75** (1993), 183–197.
[9] A. Ferguson & M. Pollicott, Escape rates for Gibbs measures, *Ergod. Theor. Dyn. Syst.*, **32** (2012), 961–988.
[10] D. Fried, Zeta functions of Ruelle and Selberg I, *Ann. Sci. Ec. Norm. Sup.* **9** (1986), 491–517
[11] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**, 1955.
[12] A. Hinrichs, Optimal Weyl inequality in Banach spaces, *Proc. Amer. Math. Soc.*, **134** (2006), 731–735.
[13] S. G. Krantz, *Function theory of several complex variables*, 2nd edition, AMS Chelsea, 1992.
[14] C. Liverani & V. Maume-Deschamps, Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set, *Ann. Inst. H. Poincaré Probab. Statist.*, **39** (2003), 385–412.
[15] G. Pianigiani & J. A. Yorke, Expanding maps on sets which are almost invariant: decay and chaos, *Trans. Amer. Math. Soc.*, **252** (1979), 351–366.
[16] A. Pietsch, *Eigenvalues and s-numbers*, CUP, Cambridge, 1987.
[17] A. Pinkus, n-widths in approximation theory, Springer-Verlag, 1985.
[18] D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, **34** (1976), 231–242
[19] H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. Nat. Acad. Sci. USA*, **35** (1949), 408–411.

Oscar F. Bandtlow; School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, UK.
ob@maths.qmul.ac.uk

Oliver Jenkinson; School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, UK.
omj@maths.qmul.ac.uk