Explicit a priori bounds on transfer operator eigenvalues

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Abstract. We provide explicit bounds on the eigenvalues of transfer operators defined in terms of holomorphic data.

Linear operators of the form $\mathcal{L}f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$, so-called *transfer operators* (see e.g. [**Bal, Rue1, Rue2**]), arise in a number of problems in dynamical systems. If the T_i are inverse branches of an expanding map T, and the weight functions w_i are positive, the spectrum of $\mathcal{L}$ has well-known interpretations in terms of the exponential mixing rate of an invariant Gibbs measure (see [**Bal**]). Applications also arise when the w_i are real-valued (e.g. [**CCR, JMS, Pol**]) or complex-valued (e.g. [**Dol, PS**]).

In this article we suppose that T_i and w_i are analytic functions of d variables, for each i in some countable[1] index set $\mathcal{I}$. Under suitable hypotheses on T_i and w_i the transfer operator $\mathcal{L}$ defines a compact operator on Hardy space $H^2(B)$, and we can give completely explicit bounds on its eigenvalue sequence[2] $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$:

Theorem 1. *Suppose there is a complex Euclidean ball $B \subset \mathbb{C}^d$ such that each $w_i : B \to \mathbb{C}$ is holomorphic with $\sum_{i\in\mathcal{I}} \sup_{z\in B} |w_i(z)| < \infty$, and each $T_i : B \to B$ is holomorphic with $\cup_{i\in\mathcal{I}} T_i(B)$ contained in the ball concentric with B whose radius is $r < 1$ times that of B.*

Then $\mathcal{L} : H^2(B) \to H^2(B)$ is compact and

$$|\lambda_n(\mathcal{L})| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}}\ n^{(d-1)/(2d)}\ r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \textit{for all } n \geq 1\,, \tag{1}$$

where $W := \sup_{z\in B} \sum_{i\in\mathcal{I}} |w_i(z)|$.

If $d = 1$ then

$$|\lambda_n(\mathcal{L})| \leq \frac{W}{\sqrt{1-r^2}}\ r^{(n-1)/2} \quad \textit{for all } n \geq 1\,. \tag{2}$$

Remark 2.
(i) An estimate of the form $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$ for some (undefined) constants $C > 0$, $\theta \in (0,1)$ is asserted, either implicitly or explicitly, in the work of several authors (e.g. [**FR, Fri, GLZ**]); the novelty here is that careful derivation of this bound renders explicit the constants C, θ.
(ii) Using different techniques, the bound $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$ can also be established in the case where B is an arbitrary open subset of $\mathbb{C}^d$ (see [**BJ**]), though here our expressions for C, θ are more complicated.

Example 3. If $\mathcal{L}f(z) = \sum_{n=1}^\infty \left(\frac{1}{n+z}\right)^2 f\left(\frac{1}{n+z}\right)$ (the Perron-Frobenius operator for the Gauss map $x \mapsto 1/x \pmod 1$, cf. [**May**]), $B \subset \mathbb{C}$ may be chosen as the open disc of radius $3/2$

[1] Subsequent results are new even when $\mathcal{I}$ is finite, but it is convenient to also allow countably infinite $\mathcal{I}$.

[2] Precisely, $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$ denotes the sequence of all eigenvalues of $\mathcal{L}$ counting algebraic multiplicities and ordered by decreasing modulus, with the usual convention (see e.g. [**Pie**, 3.2.20]) that distinct eigenvalues with the same modulus can be written in any order.

centred at the point 1. In this case $W = \sup_{z\in B}\sum_{n=1}^\infty |n+z|^{-2} = \sum_{n=1}^\infty (n-1/2)^{-2} = \pi^2/2$ and $r = 2/3$, so (2) yields

$$|\lambda_n(\mathcal{L})| \le \frac{3\pi^2}{2\sqrt{5}}(2/3)^{(n-1)/2} \quad \text{for all } n \ge 1\,.$$

NOTATION 4. For an open ball $D \subset \mathbb{C}^d$, let $H^\infty(D)$ denote the Banach space consisting of all bounded holomorphic $\mathbb{C}$-valued functions on D, with norm $\|f\|_{H^\infty(D)} := \sup_{z\in D}|f(z)|$.

Hardy space $H^2(D)$ (see [**Kra**, Ch. 8.3]) is the $L^2(\partial D,\sigma)$-closure of the set of those $f \in H^\infty(D)$ which extend continuously to the boundary ∂D, where σ denotes $(2d-1)$-dimensional Lebesgue measure on ∂D, normalised so that $\sigma(\partial D) = 1$. In particular, $H^2(D)$ is a Hilbert subspace of $L^2(\partial D,\sigma)$ with each element $f \in H^2(D)$ having a natural holomorphic extension to D (see [**Kra**, Ch. 1.5]).

In the sequel, no generality is lost by taking B in the statement of Theorem 1 to be the unit ball B_1, and the smaller concentric ball to be B_r, the ball of radius r centred at 0.

If $L : X_1 \to X_2$ is a continuous operator between Banach spaces then for $k \ge 1$, its *k-th approximation number* $a_k(L)$ is defined as

$$a_k(L) = \inf\{\|L-K\| \mid K : X_1 \to X_2 \text{ linear and continuous with } \operatorname{rank}(K) < k\}\,.$$

The proof of Theorem 1 hinges on the following two lemmas.

LEMMA 5. *If $J : H^2(B_1) \hookrightarrow H^\infty(B_r)$ denotes the canonical embedding, then J and $\mathcal{L}$ are compact and for all $n \ge 1$*

$$|\lambda_n(\mathcal{L})| \le W\prod_{k=1}^n a_k(J)^{1/n}\,. \tag{3}$$

PROOF. If $f \in H^2(B_1)$ and $z \in B_r$ then $|f(z)| \le (2/(1-r))^{d/2}$ by [**Rud**, Thm. 7.2.5], so $\{f \mid \|f\|_{H^2(B_1)} \le 1\}$ is a normal family in $H^\infty(B_r)$, hence relatively compact in $H^\infty(B_r)$ by Montel's Theorem (see [**Nar**, Ch. 1, Prop. 6]), thus J is compact.

Next observe that if $f \in H^\infty(B_1)$ then $f \in H^2(B_1)$ by [**Rud**, Thm. 5.6.8] and the canonical embedding $\hat{J} : H^\infty(B_1) \hookrightarrow H^2(B_1)$ is continuous of norm 1, because $\sigma(\partial B_1) = 1$. We claim that $\hat{\mathcal{L}}f := \sum_{i\in\mathcal{I}} w_i\cdot f\circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : H^\infty(B_r) \to H^\infty(B_1)$. To see this, fix $f \in H^\infty(B_r)$ and note that $w_i\cdot f\circ T_i \in H^\infty(B_1)$ with $\|w_i\cdot f\circ T_i\|_{H^\infty(B_1)} \le \|w_i\|_{H^\infty(B_1)}\|f\|_{H^\infty(B_r)}$ for every $i\in\mathcal{I}$. But since $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} \le \sum_{i\in\mathcal{I}}\|w_i\|_{H^\infty(B_1)}\|f\|_{H^\infty(B_r)}$ and $\sum_{i\in\mathcal{I}}\|w_i\|_{H^\infty(B_1)} < \infty$ by hypothesis, we conclude that $\hat{\mathcal{L}}f \in H^\infty(B_1)$ and that $\hat{\mathcal{L}}$ is continuous. Now $|f(T_i(z))| \le \|f\|_{H^\infty(B_r)}$ for every $z \in B_1$, $i \in \mathcal{I}$, so $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} = \sup_{z\in B_1}|(\hat{\mathcal{L}}f)(z)| \le \sup_{z\in B_1}\sum_{i\in\mathcal{I}}|w_i(z)|\,|f(T_i(z))| \le W\|f\|_{H^\infty(B_r)}$, and hence $\|\hat{\mathcal{L}}\| \le W$. Now clearly $\mathcal{L} = \hat{J}\hat{\mathcal{L}}J$, so $\mathcal{L}$ is compact, and

$$a_k(\mathcal{L}) \le \|\hat{J}\hat{\mathcal{L}}\|a_k(J) \le W a_k(J) \quad \text{for all } k \ge 1\,, \tag{4}$$

since in general $a_k(L_1L_2) \le \|L_1\|\,a_k(L_2)$ whenever L_1 and L_2 are bounded operators between Banach spaces (see [**Pie**, 2.2]). Moreover, since $\mathcal{L}$ is a compact operator on Hilbert space, Weyl's inequality (see [**Pie**, 3.5.1], [**Wey**]) asserts that $\prod_{k=1}^n|\lambda_k(\mathcal{L})| \le \prod_{k=1}^n a_k(\mathcal{L})$ for all $n \ge 1$. Together with (4) this yields (3), because $|\lambda_n(\mathcal{L})| \le \prod_{k=1}^n|\lambda_k(\mathcal{L})|^{1/n}$. □

LEMMA 6. *If $h_d(k) := \binom{k+d}{d}$ then for all $n \ge 1$,*

$$a_n(J)^2 \le \sum_{l=k}^\infty h_{d-1}(l)r^{2l} \quad \textit{where } k \ge 0 \textit{ is such that } \; h_d(k-1) < n \le h_d(k)\,. \tag{5}$$

Proof. $H^2(B_1)$ has reproducing kernel $K(z,\zeta) = (1-(z,\zeta)_{\mathbb{C}^d})^{-d}$ (see [**Kra**, Thm. 1.5.5][3]), where $(\cdot,\cdot)_{\mathbb{C}^d}$ denotes the Euclidean inner product, and $K(z,\zeta) = \sum_{n=1}^\infty p_n(z)\overline{p_n(\zeta)}$ whenever $\{p_n\}_{n=1}^\infty$ is an orthonormal basis for $H^2(B_1)$, the series converging pointwise for every $(z,\zeta) \in B_1 \times B_1$ (see [**Hal**, p. 19]).

Define $J_n : H^2(B_1) \to H^\infty(B_r)$ by $J_n f = \sum_{k=1}^{n-1} (f,p_k)\, p_k$. If $z \in B_r$ then

$$|Jf(z) - J_nf(z)|^2 = |f(z) - J_nf(z)|^2 = \left|\sum_{k=n}^\infty (f,p_k)\,p_k(z)\right|^2$$
$$\le \sum_{k=n}^\infty |(f,p_k)|^2 \sum_{k=n}^\infty |p_k(z)|^2 \le \|f\|^2_{H^2(B_1)} \,(K(z,z) - \sum_{k=1}^{n-1}|p_k(z)|^2)\,,$$

so

$$a_n(J)^2 \le \sup_{z\in B_r} \left(K(z,z) - \sum_{k=1}^{n-1} |p_k(z)|^2 \right) . \tag{6}$$

If $n=1$ then $k=0$, in which case (5) follows from (6) since $\sum_{l=0}^\infty h_{d-1}(l) r^{2l} = (1-r^2)^{-d}$. Now define the orthonormal basis $\{p_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}_0^d\}$ by (cf. [**Rud**, Prop. 1.4.8, 1.4.9])

$$p_{\underline{n}}(z) = K_{\underline{n}} z^{\underline{n}} \quad (\underline{n} \in \mathbb{N}_0^d)\,,$$

where $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)!\,\underline{n}!}}$, $\underline{n} = (n_1,\ldots,n_d)$, $z^{\underline{n}} = z_1^{n_1}\cdots z_d^{n_d}$, $\underline{n}! = n_1!\cdots n_d!$, $|\underline{n}| = n_1+\cdots+n_d$.

If $n \ge 2$ then there are $\binom{k+d-1}{d}$ multinomials of degree less than or equal to $k-1$, so

$$a_n(J)^2 \le \sup_{z\in B_r} \left(K(z,z) - \sum_{|\underline{n}|\le k-1} \left|p_{\underline{n}}(z)\right|^2 \right) = \sup_{z\in B_r} \sum_{l=k}^\infty \sum_{|\underline{n}|=l} \left|p_{\underline{n}}(z)\right|^2 \le \sum_{l=k}^\infty \frac{(l+d-1)!}{(d-1)!\,l!} r^{2l}$$

for all $n > \binom{k+d-1}{d}$, because $\sum_{|\underline{n}|=l} \frac{1}{\underline{n}!} |z^{\underline{n}}|^2 \le \frac{1}{l!} r^{2l}$ for $z \in B_r$ by the multinomial theorem. □

Proof of Theorem 1. By Lemma 5 it suffices to bound the geometric means $(\prod_{k=1}^n a_k)^{1/n}$, where $a_k := a_k(J)$. From Lemma 6 it follows that

$$a_n^2 \le \tilde\alpha_n \frac{r^{2\tilde\beta_n}}{(1-r^2)^d} \quad \text{for all } n \ge 1\,, \tag{7}$$

where

$$\begin{matrix} \tilde\alpha_n := h_{d-1}(k) \\ \tilde\beta_n := k \end{matrix} \qquad \text{for } h_d(k-1) < n \le h_d(k)\,,$$

because

$$\sum_{l=k}^\infty h_{d-1}(l) r^{2l} = h_{d-1}(k) r^{2k} \sum_{l=0}^\infty \frac{h_{d-1}(l+k)}{h_{d-1}(k)} r^{2l} \le h_{d-1}(k) r^{2k} \sum_{l=0}^\infty h_{d-1}(l) r^{2l} = h_{d-1}(k) \frac{r^{2k}}{(1-r^2)^d}\,.$$

Combining (7) with Lemma 5 gives, for all $n \ge 1$,

$$|\lambda_n(\mathcal{L})| \le W \alpha_n \frac{r^{\beta_n}}{(1-r^2)^{d/2}}\,, \tag{8}$$

where

$$\alpha_n := \prod_{l=1}^n \tilde\alpha_l^{1/(2n)}\,, \quad \beta_n := \frac{1}{n}\sum_{l=1}^n \tilde\beta_l\,.$$

[3]Note that the extra factor $(d-1)!/(2\pi^d)$ appearing in [**Kra**, Thm. 1.5.5] is due to a different normalisation of the measure σ on ∂B_1.

To obtain (1) and (2) from (8) we require an upper bound on α_n and a lower bound on β_n. We start with the bounds for α_n. Observe that

$$\tilde{\alpha}_1 = h_{d-1}(0) = 1\,, \text{ and } \tilde{\alpha}_l \le d(l-1)^{1-1/d} \text{ for } l \ge 2\,. \tag{9}$$

To see this note that

$$\frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} = \frac{(d!)^{1-1/d}}{(d-1)!}\left(\frac{\prod_{l=1}^{d-1}(k+l)^d}{\prod_{l=0}^{d-1}(k+l)^{d-1}}\right)^{1/d} = \frac{(d!)^{1-1/d}}{(d-1)!}\prod_{l=1}^{d-1}\left(1+\frac{l}{k}\right)^{1/d}$$

is decreasing in k, so if $h_d(k-1) < n \le h_d(k)$ then $\frac{\tilde{\alpha}_l}{(l-1)^{1-1/d}} \le \frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} \le \frac{h_{d-1}(1)}{h_d(0)^{1-1/d}} = d\,.$

The estimate (9) now yields the upper bound

$$\alpha_n = \prod_{i=1}^{n} \tilde{\alpha}_i^{1/(2n)} \le \sqrt{d}((n-1)!)^{(d-1)/(2dn)} \le \sqrt{d}\left(2\left(\frac{n}{e}\right)^n\right)^{(d-1)/(2dn)} \le \sqrt{d}n^{(d-1)/(2d)}\,, \tag{10}$$

where, for $n > 1$, we have used the estimate $(n-1)! \le 2\left(\frac{n}{e}\right)^n$ (i.e. $\log(n-1)! \le \int_{x=2}^n \log x\,dx \le n\log n - n + \log 2$).

We now turn to the bounds for β_n. If $h_d(k-1) < l \le h_d(k)$, so that $\tilde{\beta}_l = k$, then $l \le h_d(k) \le (d!)^{-1}(k+d)^d$, which implies $\tilde{\beta}_l = k \ge (d!)^{1/d} l^{1/d} - d$. Therefore

$$\beta_n = \frac{1}{n}\sum_{l=1}^{n} \tilde{\beta}_l \ge -d + (d!)^{1/d}\frac{1}{n}\sum_{l=1}^{n} l^{1/d} > -d + (d!)^{1/d}\frac{d}{d+1}n^{1/d}\,, \tag{11}$$

where we have used $\sum_{l=1}^n l^{1/d} > \int_{x=0}^n x^{1/d}\,dx = \frac{d}{d+1}n^{1+1/d}$.

Assertion (1) now follows from (8), (10), and (11). Finally, if $d = 1$ then $\beta_n = \frac{1}{n}\sum_{l=1}^n \tilde{\beta}_l = \frac{1}{n}\sum_{l=1}^n (l-1) = (n-1)/2$, and (10) becomes $\alpha_n \le 1$, so substituting into (8) yields (2). □

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