# Periodic points, escape rates and escape measures

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ABSTRACT. For piecewise real analytic expanding Markov maps, with Markov hole, it is shown that the escape rate and corresponding escape measure can be rapidly approximated using periodic points.

## 1. Introduction

For a dynamical system $T : X \to X$, a non-empty subset $H \subset X$ induces an *escape time function*

$$e(x) = e_H(x) = \min \{n \geq 0 : T^n(x) \in H\} \ ,$$

the nomenclature motivated by interpreting $H$ as a *hole* in phase space $X$, through which points may escape under iteration. The sequence of super-level sets $E_n = \{x \in X : e(x) > n\}$ decreases with $n$, and for a probability measure $m$ on $X$ it is often the case that $m(E_n) \to 0$ as $n \to \infty$.

If $T$ is a suitable hyperbolic map and $m$ is for example Lebesgue measure, then the $m(E_n)$ approach zero at an exponential rate. In this case the exponential decay rate

$$\delta = \delta(T, H, m) = \lim_{n\to\infty} m(E_n)^{1/n}$$

is a quantity of interest; indeed

$$\varepsilon = \varepsilon(T, H, m) = -\log \delta(T, H, m)$$

is commonly referred to as the *escape rate*, and has been widely studied (see e.g. [**3, 4, 9, 10, 13, 14, 22**]). In certain special cases $\delta(T, H, m)$ can be found exactly[1], though in general this is not feasible, so there is interest in developing methods for its efficient approximation.

The purpose of this note is to describe, in the context of analytic expanding maps $T$, a method for rapidly approximating $\delta = \delta(T, H, m)$. It relies on locating all periodic

[1]e.g. if $T(x) = 3x \pmod 1$ on the interval $X = [0, 1]$, and $H$ is the 'middle third' $(1/3, 2/3)$, then $m(E_n) = (2/3)^n$ for each $n$, so that $\delta(T, H, m) = 2/3$.

points of $T$, up to a certain period $N$, say. This yields (see §3 for further details) an approximation $\delta_N \approx \delta$, where the error satisfies

$$|\delta - \delta_N| \le C\theta^{N^2}$$

for some $\theta \in (0,1)$, $C \ge 0$; in particular, the $\delta_N$ approximate $\delta$ *super-exponentially* fast.

For example if the map $T : [0,1] \to [0,1]$ is defined, as in [**4**], by

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \le x \le \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \le \frac{i+1}{10} \text{ for } 1 \le i \le 9\,, \end{cases}$$

and $H = [\frac{9}{10}, 1]$, we derive (see §5 for further details) the successive approximations

$$\delta_2 = 0.899376191482276109518851011534$$

$$\delta_3 = 0.901142928953763644891210358737$$

$$\delta_4 = 0.901139819292137417448614669069$$

$$\delta_5 = 0.901139820047631592907392158902$$

$$\delta_6 = 0.901139820047605710579196990120$$

$$\delta_7 = 0.901139820047605710706369756237$$

In fact these techniques also yield a means of rapidly approximating the corresponding *escape measure* $\mu$, the $T$-invariant measure supported on the *survivor set* $E_\infty = e_H^{-1}(\infty)$ and maximizing the quantity $h(m) - \int_{E_\infty} \log|T'|\, dm$ over all $T$-invariant probability measures $m$, where $h(\cdot)$ denotes metric entropy (see e.g. [**10, 14, 21**]). For example $\mu$ is completely determined by its sequence of $n$-th moments $\mu(n) = \int x^n\, d\mu(x)$, which in general are not known exactly, but the periodic points of $T$ can again be used (see §4 for the method, and §5 for an example) to derive a sequence $\mu_N(n)$, where $|\mu(n) - \mu_N(n)| = O(\theta^{N^2})$ as $N \to \infty$.

Using periodic points to calculate escape rates and related quantities is not a new idea. Indeed, there is a considerable body of work for rather general systems in the physics literature starting with [**1, 2**] (see also [**12, 19, 24**] for later developments and applications). Restricting attention to analytic expanding maps, however, we are able to rigorously justify the approach and to provide precise estimates for the speed of convergence of the approximations.

This article is organised as follows. After some preliminaries on transfer operators and their determinants in §2, the method for approximating the escape rate is described in §3, and for the escape measure in §4. In the final §5, the speed of convergence of these methods is illustrated using the map $T$ and hole $H$ defined above.

## 2. Transfer operators and determinants

Suppose the unit interval[2] $I = [0,1]$ is partitioned as $I = I_1 \cup \cdots \cup I_d$, $d \geq 2$, where the $I_i$ are closed intervals with pairwise disjoint interiors. We shall assume that $T : I \to I$ is such that $T|_{I_i}$ is real analytic, for each $i$, and *expanding* in the sense that $\min\{\,|T'(x)| : x \in I_i, 1 \leq i \leq d\,\} > 1$. We say that $T$ is *Markov* if for each $1 \leq i \leq d$ the closure of $T(I_i)$ is a union of elements of the partition $\alpha = \{I_1, \ldots, I_d\}$, in which case $\alpha$ is referred to as the *Markov partition.* For each $n \geq 1$, define the usual refined partition $\alpha^{(n)} = \vee_{i=0}^{n-1} T^{-i}\alpha$. By a *Markov hole* we mean a union of members of $\alpha^{(n)}$, for some $n \geq 1$. The fact that $T$ is expanding ensures that any sub-interval $H \subset I$ can be approximated arbitrarily well by a Markov hole[3]

Although the techniques described below apply, with slight modification, to general Markov holes $H$ for Markov maps $T$, for simplicity of exposition we shall henceforth assume that for each $1 \leq i \leq d$ the closure of $T(I_i)$ equals $I$ (the so-called *Bernoulli* case), and that the hole $H \subset I$ is a member of $\alpha$.

We denote by $T_i : I \to I_i$ $(1 \leq i \leq d)$ the contractions which are inverse branches to $T$. By the implicit function theorem the maps $T_i$ are real analytic, since each $T|_{I_i}$ is real analytic. In particular, we can choose a bounded open neighbourhood $U \subset \mathbb{C}$ containing $I$ such that

$$\overline{\cup_{i=1}^{d} T_i U} \subset U\,, \tag{2.1}$$

where here $T_i$ denotes the relevant holomorphic extension to $U$.

Let $A^2(U)$ denote the Hilbert space of analytic functions $f : U \to \mathbb{C}$ which are square-integrable with respect to 2-dimensional Lebesgue measure on $U$ equipped with the usual inner product.

We may now define a *transfer operator* $\mathcal{L}$ acting on $A^2(U)$ by

$$\mathcal{L}f(z) = \sum_{i=1}^{d} \epsilon_i T_i'(z) f(T_i z) \text{ where } f \in A^2(U)\,. \tag{2.2}$$

Here $\epsilon_i \in \{-1, 1\}$ denotes the sign of the derivative of $T_i$ on $I$.

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[2]For simplicity of exposition we restrict attention to one-dimensional dynamical systems, though in fact similar results apply to real analytic expanding Markov maps in higher dimensions. In dimension $D$ the rate of convergence (of $\delta_N$ to $\delta$, and of $\mu_N(n)$ to $\mu(n)$) can be shown to be $O(\theta^{N^{1+D^{-1}}})$ as $N \to \infty$, for some $0 < \theta < 1$; in particular it is super-exponential.

[3]This suggests the possibility of approximating the escape rate for non-Markov holes $H$, by using the methods of this paper for a sequence of Markov holes approximating $H$. More precisely, the escape rate can easily be seen to depend continuously on (the end points of) the hole, by a perturbation theorem of Keller and Liverani for the bounded variation semi-norm and $L^1$ (see [**20**]). Thus, for $\delta > 0$, provided $n$ is sufficiently large, we can choose intervals $H_1 \subset H \subset H_2$ where $H_1$, $H_2$ are unions of elements of $\alpha^{(n)}$ and such that $\varepsilon(T, H_1, m) \leq \varepsilon(T, H, m) \leq \varepsilon(T, H_2, m)$ satisfy $0 \leq \varepsilon(T, H, m) - \varepsilon(T, H_1, m), \varepsilon(T, H_2, m) - \varepsilon(T, H, m) \leq \delta$. However, whereas the values $\varepsilon(T, H_1, m)$, $\varepsilon(T, H_2, m)$ can be approximated quickly there is less explicit control of the dependence of $n$ on $\delta$.

Using (2.1) it is not difficult to see that $\mathcal{L}$ maps $A^2(U)$ continuously into itself. In fact, on this space the transfer operator has strong spectral properties, which will be crucial for the results to follow. The spectral properties are conveniently described in terms of the theory of exponential classes developed in [**5**], which we briefly recall. Given positive real numbers $a$ and $\gamma$, a bounded operator $L$ on a Hilbert space is said to belong to the exponential class $E(a,\gamma)$ if

$$\sup_{n\in\mathbb{N}} s_n(L)\exp(an^\gamma) < \infty\,,$$

where $s_n(L) = \inf\{\, \|L-K\| \,:\, \operatorname{rank}(K) < n \,\}$ denotes the $n$-th *approximation number* of $L$. We now have the following result.

Proposition 2.1. *The transfer operator $\mathcal{L} : A^2(U) \to A^2(U)$ given in (2.2) belongs to the exponential class $E(a,1)$ for some $a > 0$. In particular, $\mathcal{L}$ is trace class. Moreover, its eigenvalues decay at an exponential rate.*

Proof. The first assertion follows from [**7**, Theorem 5.9]. The second now follows since the approximation numbers of $\mathcal{L}$ are summable. The statement about the eigenvalue decay follows from [**7**, Lemma 5.11]. □

Given a hole $H \in \alpha$, without loss of generality assume that $H = I_d$. In order to analyse the corresponding escape rate we consider the following modified operator:

Definition 2.2. Define $\mathcal{L}_H$ by

$$\mathcal{L}_H f(z) = \sum_{i=1}^{d-1} \epsilon_i T_i'(z) f(T_i z) \text{ where } f \in A^2(U) \text{ and } z \in U\,. \tag{2.3}$$

Equivalently, we can think of $\mathcal{L}_H$ as the original transfer operator $\mathcal{L}$ with the term corresponding to $H$ removed. As a result, the modified transfer operator enjoys the same strong spectral properties as the original transfer operator.

Proposition 2.3. *The modified transfer operator $\mathcal{L}_H : A^2(U) \to A^2(U)$ given in (2.3) belongs to the exponential class $E(a,1)$ for some $a > 0$. In particular, $\mathcal{L}_H$ is trace class. Moreover, its eigenvalues decay at an exponential rate.*

Proof. See the proof of Proposition 2.1 □

Since $\mathcal{L}_H$ is trace class, it has a well-defined trace. Moreover, there is an explicit expression for the trace of any power of $\mathcal{L}_H$ in terms of fixed points of the iterates of the map:

Proposition 2.4. *For any $n \in \mathbb{N}$ we have*

$$\operatorname{tr}(\mathcal{L}_H^n) = \sum_{x \in \operatorname{Fix}_H(T^n)} \frac{\operatorname{sgn}((T^n)'(x))}{(T^n)'(x) - 1}\,,$$

*where $\operatorname{Fix}_H(T^n) = \left\{\, x \in [0,1] \,:\, T^n x = x, T^k x \notin H \text{ for } 0 \le k < n \,\right\}$ and $\operatorname{sgn}(\xi) \in \{-1,1\}$ denotes the sign of $\xi \in \mathbb{R}$.*

PROOF. This follows from [**8**, Theorem 4.2]. □

The traces can now be used to calculate the determinant of the operator $\mathcal{L}_H$.

PROPOSITION 2.5. *The function $z \mapsto \det(1 - z\mathcal{L}_H)$ given for $z$ of sufficiently small modulus by*

$$\det(I - z\mathcal{L}_H) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_H^n)\right) \tag{2.4}$$

*extends to an entire function, the zeros of which are exactly the reciprocals of the eigenvalues of $\mathcal{L}_H$ (counting algebraic multiplicities).*

*The Taylor coefficients $c_n$ of*

$$\det(I - z\mathcal{L}_H) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.5}$$

*satisfy both the recurrence relation*

$$c_n = -\frac{1}{n} \sum_{l=0}^{n-1} c_l \mathrm{tr}(\mathcal{L}_H^{n-l}) \text{ for } n \geq 1 \tag{2.6}$$

*with $c_0 = 1$, and Plemelj's formula*

$$c_n = \frac{(-1)^n}{n!} \det \begin{pmatrix} \mathrm{tr}(\mathcal{L}_H) & 1 & & & 0 \\ \mathrm{tr}(\mathcal{L}_H^2) & \mathrm{tr}(\mathcal{L}_H) & 2 & & \\ \vdots & \vdots & & \ddots & \\ \mathrm{tr}(\mathcal{L}_H^{n-1}) & \mathrm{tr}(\mathcal{L}_H^{n-2}) & \cdots & \mathrm{tr}(\mathcal{L}_H) & n-1 \\ \mathrm{tr}(\mathcal{L}_H^{n}) & \mathrm{tr}(\mathcal{L}_H^{n-1}) & \cdots & \mathrm{tr}(\mathcal{L}_H^2) & \mathrm{tr}(\mathcal{L}_H) \end{pmatrix}. \tag{2.7}$$

*Moreover, we have*

$$|c_n| = O(\theta^{n^2}) \quad \text{as } n \to \infty\,, \tag{2.8}$$

*for some $0 < \theta < 1$.*

PROOF. For the recurrence formula and Plemelj's formula see [**23**, Theorem 4.4.10]. The decay estimate for the Taylor coefficients is proved in [**7**, Theorem 6.1]. The remaining assertions follow from Lidskii's Trace Theorem (see, for example, [**15**, Theorem 8.4, Chapter III]). □

REMARK 2.6. Explicit estimates for $\theta$, in terms of geometric properties of $T_i(U)$, can be found in [**7**, Theorem 6.1].

PROPOSITION 2.7. *The following hold:*

(a) *The operator $\mathcal{L}_H$ has a simple eigenvalue $\delta \in (0, 1]$, strictly larger in modulus than all other eigenvalues, with corresponding eigenfunction $\varrho \in A^2(U)$, which is positive on $I$.*

(b) *There exists a probability measure* $\nu$ *supported on the survivor set* $E_\infty$ *satisfying*

$$\int_{E_\infty} \mathcal{L}_H f \, d\nu = \delta \int_{E_\infty} f \, d\nu \text{ for all } f \in A^2(U) \, .$$

(c) *The probability measure* $\mu = \varrho\nu$ *supported on the survivor set* $E_\infty$ *is* $T$*-invariant and coincides with the escape measure.*

(d) *The escape rate with respect to Lebesgue measure* $m$ *satisfies*

$$\varepsilon(T, H, m) = -\log \delta \, .$$

Proof. The assertions in (a), (b) and (c) follow from results in [**21**]. To be precise, the existence of the eigenmeasure $\nu$ in (b) follows immediately from Theorem A in [**21**]. For (a) observe that (b) together with the compactness of $\mathcal{L}_H$ imply the existence of an eigenvector $\varrho \in A^2(U)$ corresponding to $\delta$, which, by the positivity arguments used for the proof of Theorem A in [**21**], must have the stated properties. The same theorem also yields (c). Finally, (d) follows from the fact that

$$m(E_n) = \int_{I \setminus H} \mathcal{L}_H^n 1 \, dm$$

together with the spectral properties of $\mathcal{L}_H$ given in (a). □

## 3. Determining the escape rate

The results of §2 mean we can find the value $0 < \delta(T, H, m) \leq 1$ by considering the determinant:

Proposition 3.1. *The smallest zero (in modulus) of* $z \mapsto \det(I - z\mathcal{L}_H)$ *is simple, real, and equal to* $\delta(T, H, m)^{-1}$.

Proof. By Proposition 2.7 the value $\delta(T, H, m)$ is a simple eigenvalue of the transfer operator $\mathcal{L}_H$ and also the largest in modulus. Combining this with Proposition 2.5 the assertions follow. □

Setting $\delta = \delta(T, H, m)$, the expansion (2.5) now gives

$$0 = 1 + \sum_{n=1}^{\infty} c_n \delta^{-n} = 1 + \sum_{n=1}^{N} c_n \delta^{-n} + O(\theta^{N^2}).$$

leading naturally to the following definition:

Definition 3.2. For each $N \geq 1$ define $\delta_N$ to be the largest value (in modulus) such that

$$0 = 1 + \sum_{n=1}^{N} c_n \delta_N^{-n}.$$

This brings us to the first main result:

THEOREM 3.3. *The values $\delta_N$ converge to $\delta$ at a super-exponential rate; more precisely,*

$$\delta_N = \delta + O(\theta^{N^2}) \quad \textit{as } N \to \infty\,.$$

PROOF. By construction the sequence $\delta_N$ converges to $\delta$. In order to estimate the speed of convergence, fix $N \geq 1$ and write

$$\Delta(z) = \det(1 - z\mathcal{L}_H) = 1 + \sum_{n=1}^{\infty} c_n z^n\,,$$

$$\Delta_N(z) = 1 + \sum_{n=1}^{N} c_n z^n\,.$$

By the mean value theorem, there is $t_N$ on the line segment joining $\delta^{-1}$ and $\delta_N^{-1}$ such that

$$\left(\delta^{-1} - \delta_N^{-1}\right)\Delta_N'(t_N) = \Delta_N(\delta^{-1}) - \Delta_N(\delta_N^{-1}) = \Delta_N(\delta^{-1}) = \Delta_N(\delta^{-1}) - \Delta(\delta^{-1})\,.$$

But since $\Delta_N'(t_N) \to \Delta'(\delta^{-1}) \neq 0$ by Proposition 3.1 it follows that $|\Delta_N'(t_N)|$ is bounded away from zero. Thus

$$\left|\delta^{-1} - \delta_N^{-1}\right| \leq \frac{1}{|\Delta_N'(t_N)|} \sum_{n=N+1}^{\infty} |c_n|\,\delta^{-n} = O(\theta^{N^2}) \quad \text{as } N \to \infty$$

for some $0 < \theta < 1$. □

REMARK 3.4. The implied constant in Theorem 3.3 can if necessary be explicitly estimated, using bounds on the Taylor coefficients $c_n$.

## 4. Determining the escape measure

In order to approximate the escape measure, we first introduce the following weighted transfer operator.

DEFINITION 4.1. Let $\phi$ be a bounded holomorphic function on $U$. For $t$ in a bounded neighbourhood $V$ of $0 \in \mathbb{C}$, define the *weighted transfer operator* $\mathcal{L}_{H,t}$ by

$$\mathcal{L}_{H,t} f(z) = \sum_{i=1}^{d-1} \epsilon_i T_i'(z) e^{t\phi(z)} f(T_i z) \text{ where } f \in A^2(U) \text{ and } z \in U\,.$$

We now have analogues of the results from §2:

PROPOSITION 4.2. *The operators $\mathcal{L}_{H,t} : A^2(U) \to A^2(U)$ satisfy:*

(a) *For each $t \in V$ the operator $\mathcal{L}_{H,t}$ belongs to the exponential class $E(a,1)$ for some $a > 0$.*
(b) *The mapping $t \mapsto \mathcal{L}_{H,t}$ is holomorphic in the trace-class operator topology; in particular, the function $(z,t) \mapsto \det(1 - z\mathcal{L}_{H,t})$ is holomorphic on $\mathbb{C} \times V$.*

(c) *For any* $t \in V$ *and any* $n \in \mathbb{N}$ *we have*

$$\operatorname{tr}(\mathcal{L}_{H,t}^n) = \sum_{x \in \operatorname{Fix}_H(T^n)} \frac{\operatorname{sgn}\left((T^n)'(x)\right) e^{t\phi^{(n)}(x)}}{(T^n)'(x) - 1}.$$

*where* $\operatorname{Fix}_H(T^n) = \left\{ x \in [0,1] \,:\, T^n x = x, T^k x \notin H \text{ for } 0 \le k < n \right\}$ *and* $\phi^{(n)} = \sum_{k=0}^{n-1} \phi \circ T^k$.

(d) *The Taylor coefficients* $c_{n,\phi}(t)$ *of the determinant*

$$\det(I - z\mathcal{L}_{H,t}) = 1 + \sum_{n=1}^{\infty} c_{n,\phi}(t) z^n$$

*satisfy* $\sup_{t \in V} |c_{n,\phi}(t)| = O(\theta^{n^2})$ *as* $n \to \infty$ *for some* $0 < \theta < 1$.

Proof. Assertion (a) follows from [**7**, Theorem 5.9], while assertion (b) follows from (a) and [**15**, Section 1.9, Chapter IV]. The formula for the traces in (c) is a consequence of [**8**, Theorem 4.2], and (d) follows from [**7**, Theorem 6.1]. □

Remark 4.3. Setting $t = 0$ we see that $\mathcal{L}_{H,0} = \mathcal{L}$, hence $c_{n,\phi}(0) = c_n$ for all $n \ge 1$.

It turns out that the escape measure can be expressed as a quotient of the partial derivatives of $(z,t) \mapsto \det(I - z\mathcal{L}_{H,t})$. The proof of this relies on a formula for the derivative of a determinant which we briefly recall. Let $D \subset \mathbb{C}$ be an open neighbourhood of 0 and suppose that $D \ni s \mapsto L(s)$ is an operator-valued function which is holomorphic in the trace-class topology. If $\det(I - L(0)) \neq 0$, then

$$\frac{d}{ds} \det(I - L(s))|_{s=0} = -\det(I - L(0)) \operatorname{tr}(\dot{L}(0)(I - L(0))^{-1}), \tag{4.1}$$

where $\dot{L}(0) = \frac{d}{ds} L(s)|_{s=0}$. For a proof see [**23**, 4.3.1.9 Proposition] or [**15**, Section 1.9, Chapter IV].

The calculation of the escape measure relies on the following result.

Proposition 4.4. *We have*

$$\int_{E_\infty} \phi \, d\mu = \delta \frac{\frac{\partial}{\partial t} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}}{\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}}.$$

Proof. The proof is a simple application of formula (4.1), the only subtlety arising from the fact that both $\frac{\partial}{\partial t} \det(I - z\mathcal{L}_{H,t})$ and $\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,t})$ vanish for $t = 0$ and $z = 1/\delta$. This problem, however, can be circumvented by choosing $D$ to be a small punctured neighbourhood of $1/\delta$ such that $\det(1 - \zeta\mathcal{L}_{H,0}) \neq 0$ for $\zeta \in D$. We then apply formula (4.1) for $\zeta \in D$ and then take the limit $\zeta \to 1/\delta$.

We thus start by using (4.1) twice to obtain for any $\zeta \in D$

$$\frac{\partial}{\partial t} \det(I - \zeta\mathcal{L}_{H,t})|_{t=0} = -\det(I - \zeta\mathcal{L}_{H,0}) \operatorname{tr}(\zeta\dot{\mathcal{L}}_{H,0}(I - \zeta\mathcal{L}_{H,0})^{-1})$$

where $\dot{\mathcal{L}}_{H,0} = \frac{d}{dt}\mathcal{L}_{H,t}|_{t=0}$, and

$$\frac{\partial}{\partial z}\det(I - z\mathcal{L}_{H,0})|_{z=\zeta} = -\det(I-\zeta\mathcal{L}_{H,0})\mathrm{tr}(\mathcal{L}_{H,0}(I-\zeta\mathcal{L}_{H,0})^{-1})\,.$$

We now observe that $\frac{d}{dt}\mathcal{L}_{H,t}|_{t=0} = M_\phi\mathcal{L}_{H,0}$ where $M_\phi : A^2(U)\to A^2(U)$ is the operator of multiplication by $\phi$, that is, $M_\phi f = \phi f$ for $f\in A^2(U)$.

Before letting $\zeta\to 1/\delta$ we note that for $\zeta\in D$ we can write

$$\mathcal{L}_{H,0}(1-\zeta\mathcal{L}_{H,0})^{-1} = \frac{\delta}{1-\zeta\delta}\Pi + Q(\zeta)\,,$$

where $\Pi f = \int_{E_\infty} f\,d\nu\cdot\varrho$ denotes the spectral projection associated to the eigenvalue $\delta$ and $Q$ is a trace-class operator valued holomorphic function on $D$. This follows from standard spectral theory (see, for example, [**23**, 4.1.6 Theorem]) together with the fact that $\delta$ is a simple eigenvalue of $\mathcal{L}_{H,0}$ by Proposition 2.7.

Now

$$\begin{aligned}\frac{\frac{\partial}{\partial t}\det(I-z\mathcal{L}_{H,t})|_{t=0,z=1/\delta}}{\frac{\partial}{\partial z}\det(I-z\mathcal{L}_{H,t})|_{t=0,z=1/\delta}} &= \lim_{\zeta\to1/\delta}\frac{\frac{\partial}{\partial t}\det(I-\zeta\mathcal{L}_{H,0})|_{t=0}}{\frac{\partial}{\partial z}\det(I-z\mathcal{L}_{H,0})|_{z=\zeta}}\\ &= \lim_{\zeta\to1/\delta}\zeta\frac{\delta\mathrm{tr}(M_\phi\Pi)+(1-\zeta\delta)\mathrm{tr}(M_\phi Q(\zeta))}{\delta\mathrm{tr}(\Pi)+(1-\zeta\delta)\mathrm{tr}(Q(\zeta))} = \frac{1}{\delta}\frac{\mathrm{tr}(M_\phi\Pi)}{\mathrm{tr}(\Pi)}\end{aligned}$$

and the result follows by noting that $\mathrm{tr}(\Pi) = \int_{E_\infty}\varrho\,d\nu = 1$ and

$$\mathrm{tr}(M_\phi\Pi) = \int_{E_\infty}\phi\varrho\,d\nu = \int_{E_\infty}\phi\,d\mu\,.$$

□

Using Proposition 4.4 we can write

$$\int_{E_\infty}\phi\,d\mu = \delta\frac{\sum_{n=0}^\infty c'_{n,\phi}(0)\delta^{-n}}{\sum_{n=0}^\infty n\,c_{n,\phi}(0)\delta^{-(n-1)}} = \frac{\sum_{n=0}^N c'_{n,\phi}(0)\delta^{1-n}}{\sum_{n=0}^N n\,c_{n,\phi}(0)\delta^{1-n}} + O(\theta^{N^2})\,, \tag{4.2}$$

for some $0<\theta<1$. Here we have used the fact that $c'_{n,\phi}(0) = O(\theta^{n^2})$ as $n\to\infty$ for some $0<\theta<1$, which follows from Proposition 4.2 (d) and Cauchy's formula.

This leads naturally to the following definition:

Definition 4.5. For each $N\geq 1$, define $I_N(\phi)$ by

$$I_N(\phi) = \frac{\sum_{n=0}^N c'_{n,\phi}(0)\delta_N^{1-n}}{\sum_{n=0}^N n\,c_{n,\phi}(0)\delta_N^{1-n}} = \frac{\sum_{n=1}^N c'_{n,\phi}(0)\delta_N^{1-n}}{\sum_{n=1}^N n\,c_n\delta_N^{1-n}}\,.$$

This brings us to the second main result:

THEOREM 4.6. *The values $I_N(\phi)$ converge to $\int_{E_\infty} \phi \, d\mu$ at a super-exponential rate; more precisely,*

$$I_N(\phi) = \int_{E_\infty} \phi \, d\mu + O(\theta^{N^2}) \quad \text{as } N \to \infty$$

*for some $0 < \theta < 1$.*

PROOF. This follows from (4.2) and Theorem 3.3. □

REMARK 4.7. Similar approximating formulae, in the context of invariant measures equivalent to Lebesgue measure, have been derived in [**16, 17, 18**] using a slightly different approach.

Importantly, it is possible to efficiently calculate each $c'_{n,\phi}(0)$ using periodic points:

PROPOSITION 4.8. *Setting*

$$b_{\phi,n} = \frac{1}{n} \sum_{x \in \mathrm{Fix}_H(T^n)} \frac{\mathrm{sgn}((T^n)'(x))\phi^{(n)}(x)}{(T^n)'(x) - 1}\,, \tag{4.3}$$

*we have*

$$c'_{\phi,n}(0) = -\sum_{i=1}^{n} b_{\phi,i} c_{n-i} \quad \text{for all } n \geq 1\,. \tag{4.4}$$

PROOF. Let $z$ belong to a sufficiently small disc centred at the origin. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} \det(1 - z\mathcal{L}_{H,t})|_{t=0} &= -\det(1 - z\mathcal{L}_{H,0}) \sum_{m=1}^{\infty} \frac{z^m}{m} \frac{\partial}{\partial t} \mathrm{tr}(\mathcal{L}^n_{H,t})|_{t=0} \\ &= -(1 + \sum_{n=1}^{\infty} c_n z^n) \sum_{m=1}^{\infty} b_{\phi,m} z^m\,. \end{aligned} \tag{4.5}$$

On the other hand,

$$\frac{\partial}{\partial t} \det(1 - z\mathcal{L}_{H,t})|_{t=0} = 1 + \sum_{n=1}^{\infty} c'_n(0) z^n\,, \tag{4.6}$$

and the result now follows by comparing coefficients in (4.5) and (4.6). □

## 5. An example

As in [**4**], we consider the map

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \leq \frac{i+1}{10} \text{ for } 1 \leq i \leq 9 \end{cases}$$

and $H = [\frac{9}{10}, 1]$.

Note that the inverse branches $\{T_i\}_{0\le i\le 9}$ are given by

$$T_0(x) = \frac{x}{9+x}$$

and

$$T_i(x) = (x+i)/10 \quad \text{for } 1 \le i \le 9\,.$$

Writing

$$a_n = \frac{1}{n}\mathrm{tr}(\mathcal{L}_H^n) = \frac{1}{n}\sum_{x\in \mathrm{Fix}_H(T^n)} \frac{1}{(T^n)'(x)-1}\,,$$

these $a_n$ can be computed by locating the members of $\mathrm{Fix}_H(T^n)$, all of which are quadratic numbers.

For example there are 9 members of $\mathrm{Fix}_H(T)$, denoted $x_0, x_1, \ldots, x_8$, say. For each $1 \le i \le 8$ we see that

$$\frac{1}{T'(x_i)-1} = \frac{1}{10-1} = \frac{1}{9}\,,$$

whereas

$$\frac{1}{T'(x_0)-1} = \frac{1}{9-1} = \frac{1}{8}\,.$$

Therefore

$$a_1 = \sum_{i=0}^{8} \frac{1}{T'(x_i)-1} = \frac{1}{8} + \frac{8}{9} = \frac{73}{72}\,.$$

The computation of $a_2$ is only slightly more involved. For the fixed point 0 we have

$$\frac{1}{(T^2)'(0)-1} = \frac{1}{81-1} = \frac{1}{80}\,,$$

whereas for those 64 period-2 points $x_{ij} = (T_i \circ T_j)(x_{ij})$ with $1 \le i, j \le 8$, we have

$$\frac{1}{(T^2)'(x_{ij})-1} = \frac{1}{100-1} = \frac{1}{99}\,.$$

It remains to consider the 8 period-2 points of the form $x_{0i} = (T_0 \circ T_i)(x_{0i})$, and the 8 period-2 points of the form $x_{i0} = (T_i \circ T_0)(x_{i0})$, for $1 \le i \le 8$. In fact since $(T^2)'(x_{0i}) = (T^2)'(x_{i0})$, it suffices to consider the points $x_{0i}$, and a calculation gives

$$(T_0 \circ T_i)(x) = \frac{x+i}{x+90+i} \quad , \quad (T_0 \circ T_i)'(x) = \frac{90}{(x+90+i)^2}\,,$$

$$x_{0i} = 5\left(\sqrt{\left(9+\frac{i-1}{10}\right)^2 + \frac{i}{25}} - 9 - \frac{i-1}{10}\right)\,,$$

from which we compute

$$a_2 = \frac{1}{2}\left(\frac{1}{80} + \frac{64}{99} + 2\sum_{i=1}^{8} \frac{1}{(T^2)'(x_{0i})-1}\right) = 0.4109953458362511215886541628 58\ldots$$

Subsequent values $a_n$ can be computed similarly, for example:

$$a_3 = 0.2442479868723925943008958371 21\ldots$$

$$a_4 = 0.1648814849245365150739904169 86\ldots$$

$$a_5 = 0.1188496302501099446867937731 81\ldots$$

$$a_6 = 0.0892488434228904495807238896 12\ldots$$

$$a_7 = 0.0689361952898514484983035948 69\ldots$$

**5.1. The escape rate.** We are now in a position to compute the power series coefficients $c_i$ of the determinant $\det(I - z\mathcal{L}_H) = 1 + \sum_{i=1}^{\infty} c_i z^i$. Specifically, the formulae of Proposition 2.5 give

$$c_1 = -a_1$$

$$c_2 = -a_2 + \frac{a_1^2}{2}$$

$$c_3 = -a_3 + a_1a_2 - \frac{a_1^3}{6}$$

$$c_4 = -a_4 + \frac{a_2^2}{2} + a_1a_3 - \frac{a_1^2a_2}{2} + \frac{a_1^4}{24}$$

$$c_5 = -a_5 + a_1a_4 + a_2a_3 - \frac{a_1^2a_3}{2} - \frac{a_1a_2^2}{2} + \frac{a_1^3a_2}{6} - \frac{a_1^5}{120}$$

$$c_6 = -a_6 + \frac{a_3^2}{2} + a_1a_5 + a_2a_4 - \frac{a_1^2a_4}{2} - a_1a_2a_3 - \frac{a_2^3}{6} + \frac{a_1^3a_3}{6} + \frac{a_1^2a_2^2}{4} - \frac{a_1^4a_2}{24} + \frac{a_1^6}{720}$$

$$\begin{aligned} c_7 = -a_7 + a_1a_6 + a_2a_5 + a_3a_4 - \frac{a_1^2a_5}{2} - a_1a_2a_4 - \frac{a_1a_3^2}{2} - \frac{a_2^2a_3}{2} \\ + \frac{a_1^3a_4}{6} + \frac{a_1^2a_2a_3}{2} + \frac{a_1a_2^3}{6} - \frac{a_1^4a_3}{24} - \frac{a_1^3a_2^2}{12} + \frac{a_1^5a_2}{120} - \frac{a_1^7}{5040} \ . \end{aligned}$$

Substituting the above numerical values[4] of $a_n$ into the formulae for the $c_i$ then gives

$$c_1 = -\frac{73}{72}$$

$$c_2 = 0.1029899936699217179175186766 48\ldots$$

$$c_3 = -0.0012523806030019538195780390 57\ldots$$

$$c_4 = 1.9947545015369326142097604763 93\ldots \times 10^{-6}$$

$$c_5 = -4.3671179106583113436710356029 00\ldots \times 10^{-10}$$

$$c_6 = 1.3482155123568633996931879854 65\cdots \times 10^{-14}$$

$$c_7 = -5.9695594068695611598849476137 41\ldots \times 10^{-20} \ .$$

[4]Of course we use higher precision for the $a_n$, ensuring that the values $c_i$ are correct to the precision given.

These values of $c_i$ allow us to form, for $1 \leq N \leq 7$, the degree-$N$ polynomial approximation

$$\Delta_N(z) = 1 + \sum_{i=1}^{N} c_i z^i$$

to the determinant. The smallest root $z_N$ of $\Delta_N$ can then be computed as follows:

$$z_1 = 72/73 = 0.9863013698630136986301 36\ldots$$

$$z_2 = 1.1118817792495531842010120150 76\ldots$$

$$z_3 = 1.1097018773270633631804091112 27\ldots$$

$$z_4 = 1.1097057066965691821433921321 29\ldots$$

$$z_5 = 1.1097057057662183317744555833 03\ldots$$

$$z_6 = 1.1097057057662502044834822195 28\ldots$$

$$z_7 = 1.1097057057662502043268757295 70\ldots$$

and inverting these gives the same sequence of approximations $\delta_N = z_N^{-1}$ to $\delta(T, H, m)$ as listed in §1.

**5.2. The escape measure.** The escape measure $\mu$ is completely determined by its set of *moments* $\int_{E_\infty} x^n \, d\mu(x)$, $n \geq 0$. Each $n$-th moment can be rapidly approximated by setting $\phi(x) = x^n$, then using the approach described in §4. Here we shall illustrate this in the case $n = 1$: the first moment $\mu(1) = \int_{E_\infty} x \, d\mu(x)$ is often called the *barycentre*, or *resultant*, of the measure $\mu$.

Since $\phi(x) = x$ is fixed, we write $b_n = b_{\phi,n}$ (see (4.3)), so that

$$b_n = \frac{1}{n} \sum_{x \in \mathrm{Fix}_H(T^n)} \frac{\sum_{i=0}^{n-1} T^i x}{(T^n)'(x) - 1} \, .$$

We find that

$$b_1 = 4/9 = 0.4444444444444444444444444444\ldots$$

$$b_2 = 0.3631469799408668177106763906 86\ldots$$

$$b_3 = 0.3239456970780829020315869429 46\ldots$$

$$b_4 = 0.2915979181133540976000854333 02\ldots$$

$$b_5 = 0.2627386364233422819525263563 99\ldots$$

$$b_6 = 0.2367610955232243687892492780 48\ldots$$

$$b_7 = 0.2133545391130421480998947838 40\ldots$$

Recall that the coefficients $d_i = c'_{\phi,i}(0)$ (where $\phi(x) = x$) are given by formula (4.4). It follows that, for example, the first four[5] $d_i$ are given by:

$$d_1 = -b_1$$

$$d_2 = a_1 b_1 - b_2$$

$$d_3 = -b_3 + b_1 a_2 + a_1 b_2 - \frac{a_1^2 b_1}{2}$$

$$d_4 = a_2 b_2 + b_1 a_3 + a_1 b_3 - b_4 - a_1 b_1 a_2 - \frac{a_1^2 b_2}{2} + \frac{a_1^3 b_1}{6}$$

Substituting the numerical values of $a_n, b_n$ into the formulae for the $d_i$ gives us:

$$d_1 = -4/9 = -0.4444444444444444444444\ldots$$

$$d_2 = 0.0874703040097504662399408932 64\ldots$$

$$d_3 = -0.001528339602447030927159458 67\ldots$$

$$d_4 = 3.133193453094917698092477916170\ldots \times 10^{-6}$$

$$d_5 = -8.403901824081610940025293484 20\ldots \times 10^{-10}$$

$$d_6 = 3.090985019372664486353921814698\ldots \times 10^{-14}$$

$$d_7 = -1.602538948979713314526914251 40\ldots \times 10^{-19}$$

The approximations

$$\mu_N(1) = \frac{\sum_{n=1}^N d_n z_N^{n-1}}{\sum_{n=1}^N n c_n z_N^{n-1}}$$

to the integral $\mu(1) = \int_{E_\infty} x \, d\mu(x)$ are then:

$$\mu_2(1) = 0.4423543836746645322149291451 56\ldots$$

$$\mu_3(1) = 0.4421359775981968931136677480 55\ldots$$

$$\mu_4(1) = 0.4421366762978087220651252319 22\ldots$$

$$\mu_5(1) = 0.4421366760538653690481812498 45\ldots$$

$$\mu_6(1) = 0.4421366760538758472561048724 52\ldots$$

$$\mu_7(1) = 0.4421366760538758471975262144 97\ldots$$

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[5]In the calculation that follows we use $d_i$ for $1 \le i \le 7$, though the algebraic formulae for $d_i$ in terms of $a_n, b_n$ are a little long to conveniently give here (e.g. the analogous expression for $d_7$ consists of a sum of 30 terms).

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