

Resonances for rational Anosov maps on the torus

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Abstract

A complete description of resonances for rational toral Anosov diffeomorphisms preserving certain Reinhardt domains is presented. As a consequence it is shown that every homotopy class of two-dimensional Anosov diffeomorphisms contains maps with the sequence of resonances decaying stretched-exponentially. This is achieved by introducing a certain group of rational toral diffeomorphisms and computing the resonances of the respective composition operator considered on suitable anisotropic spaces of hyperfunctions. The class of examples is sufficiently rich to also include non-linear Anosov maps with trivial resonances, or resonances decaying exponentially, as well as with or without area-preservation or reversing symmetries.

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1 Introduction

Anosov diffeomorphisms are the simplest truly hyperbolic dynamical systems, whose long term asymptotic behaviour characterized by correlation decay or rates of mixing are classical topics in smooth ergodic theory and statistical physics. The main tool for studying statistical properties for Anosov maps T acting on a compact manifold M is the *weighted composition operator*¹ defined by

$$C_{T,w}: f \mapsto w \cdot f \circ T,$$

where w is a smooth function on M and $C_{T,w}$ acts on a suitable space of distributions. There is a considerable amount of recent literature devoted to the construction of so-called anisotropic spaces for hyperbolic dynamical systems, on which the above operator is quasicompact, implying a spectral gap and exponential decay of correlations. The general idea behind the construction of these spaces is to create sufficient smoothness in the expanding direction and dual smoothness in the contracting direction, see [Rug, BKL, GL, BT, FRS, B], to name but a few.

If the Anosov map is real-analytic, it is sometimes possible to prove compactness and even nuclearity of the above operator, see [Rug, J, FR], implying that its non-zero spectrum is a sequence of isolated eigenvalues known as *Pollicott-Ruelle resonances*, which determine all intrinsic exponential mixing rates of the given system. However, quantitative results such as location, or the very existence, of non-trivial resonances are rare, a few instances in different hyperbolic settings being [FGL, DFG]. Even for Anosov diffeomorphisms on the torus \mathbb{T}^2 , arguably the simplest setting of uniformly hyperbolic dynamical systems, it was established only recently in [A] (after an idea of F. Naud) that non-trivial resonances exist generically. In [SBJ] the authors presented a one-parameter family of Anosov maps, proving the presence of infinitely many distinct resonances by calculating their location explicitly, and also conjectured locations of resonances for another family of Anosov maps, which was recently proven in [PoS]. In both cases, after establishing compactness of the transfer operator on a suitable anisotropic Hilbert space (originally introduced in [FR]), the eigenvalues are read off from an upper triangular matrix representation of the operator with respect to a weighted Fourier basis. These results, though valuable for rigorously establishing the location of resonances for these particular families, are rather ad hoc as they do not reveal the underlying spectral structure of the associated operator. In this context, a key contribution of this article is to explain the underlying structure of resonances for a class of rational Anosov diffeomorphisms. For this it will be helpful to work with simple anisotropic spaces, which, albeit less general, interact well with the analytic structure of the underlying map.

For analytic Anosov maps on \mathbb{T}^2 with constant invariant stable and unstable cone fields we define anisotropic Hilbert spaces of hyperfunctions as a closure of Laurent polynomials under a weighted L^2 norm with the weight function adapted to the invariant cones, and show that the respective weighted composition operator is well defined and trace-class. These spaces are isometrically isomorphic to a direct sum of Hardy-Hilbert spaces on log-conical Reinhardt domains, with the logarithmic base induced by the stable and unstable cones. Using this viewpoint and assuming additionally that these Anosov maps extend holomorphically to certain domains in $\hat{\mathbb{C}}^2$, we are able to explicitly compute all the eigenvalues of the respective composition operator. In addition we present a group of (non-linear) toral diffeomorphisms which satisfy these assumptions and provide examples in each homotopy class of toral Anosov diffeomorphisms for which the composition operator has infinitely many distinct non-trivial resonances $(\lambda_n)_{n \in \mathbb{N}}$ whose rate of decay is stretched-exponential, that is, the upper bound of $\exp(-an^{1/2})$ for some $a > 0$ is tight.

Results in [BJS] on resonances for analytic expanding circle maps of degree d , with $|d| > 2$, arising from finite Blaschke and anti-Blaschke products, made it possible to establish exponential lower bounds for the decay rate on a dense set of analytic expanding circle maps [BN]. In the same vein, the class of non-linear toral diffeomorphisms presented in this paper will be an essential

¹This operator is also referred to as the transfer operator or Koopman operator, depending on the weight.

ingredient in proving that, generically, the stretched-exponential decay rate of resonances is optimal within the class of analytic Anosov diffeomorphisms on \mathbb{T}^2 . Moreover, as all constructions are very explicit most of our results should extend to Anosov maps in higher dimensions. These are, however, beyond the current scope and will be pursued in subsequent works.

1.1 Statement of results

We will only consider toral Anosov diffeomorphisms with constant expanding and contracting cone fields. Restricting to constant invariant cone fields enables us to work with simple anisotropic Hilbert spaces H_ν , which can be seen as the completion of Laurent polynomials under the ν -weighted L_ν^2 inner product for a *cone-wise exponential weight* function $\nu: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$, that is $\nu(n) = e^{f(n)}$ where $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is piecewise linear, with the pieces being cones in \mathbb{R}^2 . We will make an additional assumption on the invariant cone fields, termed *strongly expanding constant invariant cone field* condition or short-hand (*sec*), see Definition 2.12 for the precise definition. If the (constant) unstable invariant cone field can be chosen to correspond to the positive/negative quadrants $\pm\mathbb{R}_{>0}^2$, we refer to these as positive, and call the respective condition (*p-sec*).

Theorem 1.1. *Let T be an analytic Anosov diffeomorphism of \mathbb{T}^2 satisfying the (*sec*) assumption and $w: \mathbb{T}^2 \rightarrow \mathbb{C}$ an analytic function. Then there exists a cone-wise exponential $\nu: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ such that*

$$f \mapsto w \cdot f \circ T$$

is a well-defined trace-class operator on H_ν .

For a special subclass of analytic Anosov diffeomorphisms where cone fields can be chosen to correspond to quadrants of \mathbb{R}^2 and the diffeomorphisms extend holomorphically to certain domains of $\hat{\mathbb{C}}^2$, we are able to compute all eigenvalues of the above operator explicitly in terms of multipliers of fixed points on these domains. To state the results we introduce the notation $\Sigma = \{\sigma = (\sigma_1, \sigma_2): \sigma_1, \sigma_2 \in \{\pm 1\}\}$, and $D^\sigma = \{z \in \hat{\mathbb{C}}^2: |z_1|^{\sigma_1} > 1, |z_2|^{\sigma_2} > 1\}$ with $\sigma \in \Sigma$ for the four bidisks in $\hat{\mathbb{C}}^2$. It will further be convenient to write $\mathcal{N}^1 = \mathbb{N}_0^2 \setminus \{(0, 0)\}$, $\mathcal{N}^{-1} = \mathbb{N}^2$, and decompose Σ as $\Sigma = \Sigma^1 \cup \Sigma^{-1}$ with $\Sigma^\ell = \{\sigma \in \Sigma: \sigma_1 \cdot \sigma_2 = \ell\}$, $\ell \in \{\pm 1\}$. The notation T^ℓ for $\ell = -1$ stands for the inverse of T . We write $C_T = C_{T,1}$ for the unweighted composition operator.

Theorem 1.2. *Let T be an analytic Anosov diffeomorphism on \mathbb{T}^2 satisfying the (*p-sec*) assumption. Then there exists a cone-wise exponential $\nu: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ such that C_T is a well-defined trace-class operator on H_ν , and its spectral determinant is an entire function of the form*

$$\det(\text{Id} - zC_T) = (1 - z)\chi_T(z),$$

where χ_T is an entire function with zeros outside of $\text{cl}(\mathbb{D})$. Moreover, if T^ℓ holomorphically extends² to D^σ for all $\sigma \in \Sigma^\ell$ and $\ell \in \{\pm 1\}$ then χ_T is the product of two entire functions χ_T^{+1} and χ_T^{-1} , whose zeros are given explicitly. Specifically, for every $\ell \in \{\pm 1\}$ exactly one of the following cases holds:

(i) *if $T^\ell(D^\sigma) \subseteq D^\sigma$ for all $\sigma \in \Sigma^\ell$, then*

$$\chi_T^\ell(z) = \prod_{n \in \mathcal{N}^\ell} \prod_{\sigma \in \Sigma^\ell} (1 - z^\ell \lambda_\sigma^n)$$

where $\lambda_\sigma = (\lambda_{\sigma,1}, \lambda_{\sigma,2})$ are the multipliers³ of the unique attracting fixed point $z_\sigma^ \in D^\sigma$ of T^ℓ , and $s = 0$ if T is orientation-preserving and $s = 1$ if it is orientation-reversing. Additionally, it holds that $z_\sigma^* = 1/\overline{z_{-\sigma}^*}$ and $\lambda_\sigma = \overline{\lambda_{-\sigma}}$ for $\sigma \in \Sigma^\ell$.*

²Here we view \mathbb{T}^2 as a subset of $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and consider extensions of T and T^{-1} to subsets of $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ as holomorphic in the same sense as for mappings of the Riemann sphere.

³The multipliers of T at a point z are the eigenvalues $\lambda = (\lambda_1, \lambda_2)$ of $D_z T = \left(\frac{\partial T_k}{\partial z_l}\right)_{k,l}$ at z . We write $\lambda^n = \lambda_1^{n_1} \cdot \lambda_2^{n_2}$ for $n \in \mathbb{Z}^2$.

(ii) if $T^\ell(D^\sigma) \subseteq D^{-\sigma}$ for all $\sigma \in \Sigma^\ell$, then

$$\chi_T^\ell(z) = \prod_{n \in \mathcal{N}^\ell} \prod_{\sigma \in \Sigma^\ell} (1 - z^2 \lambda_\sigma^n)^{1/2},$$

where $\lambda_\sigma = \lambda_{-\sigma}$ are the multipliers of the unique attracting fixed point $z_\sigma^* \in D^\sigma$ of $T^{2\ell}$. Additionally, $\lambda_{\sigma,1}$ and $\lambda_{\sigma,2}$ are either real, or complex conjugates of each other.

Clearly, certain linear toral diffeomorphisms satisfy the assumptions of this theorem. For example, the well-known cat map $(z_1, z_2) \mapsto (z_1^2 z_2, z_1 z_2)$ satisfies assumption (i) and hence we can compute all fixed points and their multipliers, which however are all trivial (that is, zero) in this case. The group $\text{Aut}(\mathbb{T}^2)$ of linear diffeomorphisms (automorphisms) of the torus is isomorphic to $\text{GL}_2(\mathbb{Z})$. In order to construct non-linear maps, we shall consider a group of diffeomorphisms generated by a finite set Γ of generators of $\text{Aut}(\mathbb{T}^2)$ and a (uncountably infinite) set \mathcal{G} of certain rational diffeomorphisms preserving \mathbb{T}^2 . One particular choice for this set is

$$\mathcal{G} = \{(z_1, z_2) \mapsto (b_a(z_1), z_2) : a \in \mathbb{D}\}$$

with $b_a(z) = (z - a)/(1 - \bar{a}z)$. We call \mathcal{F} the group of diffeomorphisms generated by $\Gamma \cup \mathcal{G}$. A certain subset of \mathcal{F} comprises of hyperbolic diffeomorphisms satisfying the assumptions of Theorem 1.2. The explicit construction of elements of this set (see Section 5.1) allows us to compute the sequence of eigenvalues of the corresponding composition operators and present examples with qualitatively different decay rates of this sequence (stretched-exponential, exponential, and trivially super-exponential/all-zero). Using the structure of conjugacy classes of $\text{GL}_2(\mathbb{Z})$ we obtain the following theorem.

Theorem 1.3. *Every homotopy class of analytic Anosov diffeomorphisms on \mathbb{T}^2 contains (non-linear) Anosov diffeomorphisms $T \in \mathcal{F}$, such that for suitable H_ν the corresponding operator C_T is well defined and trace-class, with the entire function $z \mapsto \det(\text{Id} - zC_T) = (1 - z)\chi_T(z)$ as its spectral determinant. In particular, denoting by $(\lambda_n)_{n \in \mathbb{N}}$ the sequence of eigenvalues of C_T ordered by modulus in decreasing order, and counted with multiplicities, we obtain the following.*

(i) *For every homotopy class \mathcal{H} and $\eta > 0$, there exists $T \in \mathcal{H} \cap \mathcal{F}$ such that the eigenvalue sequence of C_T satisfies*

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n^{1/2}} = \eta.$$

(ii) *For every homotopy class \mathcal{H} of orientation-preserving Anosov diffeomorphisms and $\eta > 0$, there exists $T \in \mathcal{H} \cap \mathcal{F}$ such that the eigenvalue sequence of C_T satisfies*

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n} = \eta.$$

(iii) *Every homotopy class \mathcal{H} of Anosov diffeomorphisms not containing a linear conjugate of one of $\{(z_1, z_2) \mapsto (z_1^k z_2, z_1), k \in \mathbb{N}\}$ has an element $T \in \mathcal{H} \cap \mathcal{F}$ not smoothly conjugated to a linear Anosov diffeomorphism, such that*

$$\chi_T(z) = 1$$

for all $z \in \mathbb{C}$.

Remark 1.4. We note that the maps $T \in \mathcal{F}$ above can be written in closed form. In contrast to the more common setting of analytic perturbations of linear maps (see, for example, [A, FR]), these maps are not required to be C^1 close to the respective linear automorphisms, that is, the C^1

distance can be arbitrarily large. In fact, as the proof of Theorem 1.3 will show, the spectral gap for the composition operator associated to the maps in (i) and (ii) can take an arbitrary value in $(0, 1)$. Moreover, under the additional assumption that T is orientation-preserving, in case (i) the second-largest eigenvalue $\lambda_2 \in \mathbb{D}$ and the decay rate $\eta > 0$ can be chosen arbitrarily, independently of each other. The behaviour in (i) is believed to be generic for two-dimensional Anosov maps, whereas the cases (ii) and (iii) are exceptional. We also note that under the assumptions of Theorem 1.2 (as well as of Theorem 1.3), the spectral determinant can be shown to coincide with the usual dynamical determinant, as the usual trace formula (see, for example, [FR, Proposition 5]) can be established in this setting.

Remark 1.5. As shown in [FG], any homotopy class \mathcal{H} of analytic Anosov diffeomorphisms on \mathbb{T}^2 is path-connected, so that in fact, in the cases (i)-(iii) in Theorem 1.3, any $T' \in \mathcal{H}$ is homotopic via a continuous path of toral diffeomorphisms to a $T \in \mathcal{H}$ with the respective spectral property. Moreover, by the famous Franks-Newhouse classification theorem, every Anosov diffeomorphism of codimension 1 on a closed Riemannian manifold⁴ is topologically conjugate to a hyperbolic toral automorphism, see for example [Hi]. Thus, since any two smooth surfaces that are homeomorphic are also diffeomorphic, in the above theorem \mathbb{T}^2 can be replaced with any two-dimensional closed Riemannian manifold.

This paper is organised as follows. Starting with a motivational example of the composition operator for the cat map in Section 2.1, we discuss its boundedness and compactness on an anisotropic space of hyperfunctions relevant to the current work. In Section 2.2 we consider a class of analytic toral Anosov diffeomorphisms with strongly expanding constant invariant cone (*sec*) fields and show how properties on the tangent bundle are translated into properties of the map in a small complex neighbourhood of the torus. We start Section 3 by summarising properties of Hardy-Hilbert spaces on log-conical Reinhardt domains and realizing the anisotropic Hilbert spaces of hyperfunctions from Section 2.1 as direct sums of these Hardy-Hilbert spaces. The main theorem of this section is Theorem 3.24, which together with a standard factorisation argument yields a trace-class weighted composition operator for analytic Anosov maps with the (*sec*) property, thus proving Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2, that is, the computation of resonances of the composition operator associated to rational Anosov diffeomorphisms satisfying the (*p-sec*) condition and preserving certain polydisks. In Section 5 we prove Theorem 1.3. For this, we first introduce in Section 5.1 the group of diffeomorphisms \mathcal{F} , discuss their properties and present examples of Anosov maps with and without additional properties such as area-preservation and symmetry reversal. In Section 5.2, using conjugacy classes of toral Anosov automorphisms, we construct non-linear Anosov maps in \mathcal{F} satisfying the assertions of Theorem 1.3.

2 Cone conditions for Anosov diffeomorphisms on \mathbb{T}^2

2.1 A pedestrian approach

In this section we want to explore functional-analytic properties of composition operators associated to Anosov automorphisms on the torus on Hilbert spaces that can be defined as the completion of the space of Laurent polynomials with a norm depending on a certain weight function. We take the well-known cat map as an example of a toral automorphism, and discuss boundedness and compactness of the associated composition operator depending on the weight function. The main result of this section is to show that for a suitable weight function the composition operator is Hilbert-Schmidt. Since most of our work in this paper focuses on the two-dimensional case, for convenience we introduce the following shorthands.

⁴An Anosov diffeomorphism is said to be of codimension 1 if the codimension of either its stable or unstable foliations has dimension 1.

Notation 2.1. For $\alpha, \beta \in \mathbb{R}^2$ (or \mathbb{Z}^2) we write $\alpha > \beta$ if $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, and moreover for $c \in \mathbb{R}$ the notation $\alpha > c$ will be used as a shorthand for $\alpha_1 > c$ and $\alpha_2 > c$ (and analogously for other comparison operators). For $z \in \mathbb{C}^2$ (or $\mathbb{R}^2, \mathbb{Z}^2$) and $n \in \mathbb{Z}^2$ we use the multiindex notation $z^n = z_1^{n_1} z_2^{n_2}$, and we write $|z| = (|z_1|, |z_2|)$.

2.1.1 Weighted Hilbert spaces

Let $\mathbb{T}^2 = \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$, and let \mathcal{P} denote the space of Laurent polynomials on \mathbb{T}^2 ,

$$\mathcal{P} = \{f : \mathbb{T}^2 \rightarrow \mathbb{C} : f(z) = \sum_{n \in \mathbb{Z}^2, |n| \leq N} f_n z^n, \text{ with } f_n \in \mathbb{C}, N \in \mathbb{N}\}.$$

For any $\nu : \mathbb{Z}^2 \rightarrow \mathbb{R}_{>0}$, we define an inner product on \mathcal{P} by

$$\langle f, g \rangle_\nu = \sum_{n \in \mathbb{Z}^2} f_n \bar{g}_n \nu(n)^2,$$

where $(f_n)_{n \in \mathbb{Z}^2}$ and $(g_n)_{n \in \mathbb{Z}^2}$ are the Fourier coefficients of f and g , and we denote by $\|\cdot\|_\nu$ the corresponding norm. Note that $\nu(n) = \|p_n\|_\nu$ for $n \in \mathbb{Z}^2$, where p_n is the monomial $z \mapsto z^n$.

Definition 2.2. We write \mathcal{H}_ν for the completion of \mathcal{P} with respect to the norm $\|\cdot\|_\nu$.

It turns out that \mathcal{H}_ν is a separable Hilbert space, which contains the Laurent polynomials as a dense subset, with an orthonormal basis given by the normalised monomials

$$e_n(z) = \frac{z^n}{\nu(n)}, \quad n \in \mathbb{Z}^2.$$

2.1.2 Composition operator for the cat map

The hyperbolic matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

induces the well-known toral automorphism $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by $(z_1, z_2) \mapsto (z_1^2 z_2, z_1, z_2)$, known as the cat map. We shall study the properties of the composition operator C_T associated to T when considered on different Hilbert spaces H_ν . In particular, we will see that for certain choices of ν the operator C_T is Hilbert-Schmidt. A bounded operator L on a Hilbert space H is Hilbert-Schmidt if $\sum_n \|L e_n\|^2 < \infty$ where $\{e_n\}$ is an orthonormal basis of H . A sufficient condition for this is that there are $\delta, c > 0$ such that for all n it holds that

$$\|L e_n\| \leq \delta \exp(-c|n|). \quad (1)$$

We can compute $\|C_T e_n\|_\nu$ explicitly, as

$$\|C_T e_n\|_\nu^2 = \sum_{m \in \mathbb{Z}^2} \left(\frac{\nu(m)}{\nu(n)} \right)^2 \delta_{2n_1+n_2, m_1} \delta_{n_1+n_2, m_2} = \left(\frac{\nu(An)}{\nu(n)} \right)^2. \quad (2)$$

We first consider weight functions ν adapted to the dynamics of T introduced in [FR]. For this, we denote the unstable/stable eigenvalues of A by $\lambda_{u/s} = \varphi^{\pm 2}$, where $\varphi = (1 + \sqrt{5})/2$ is the golden mean, and write $V = (v_u, v_s)$ for the matrix with the corresponding (normalised) unstable/stable eigenvectors as its column vectors. We write $\langle \cdot, \cdot \rangle$ for the standard inner product on \mathbb{R}^2 .

Lemma 2.3. *For any $a = (a_1, a_2) > 0$ and $\nu(n) = \nu_a(n) = \exp(-a_1|\langle n, v_u \rangle| + a_2|\langle n, v_s \rangle|)$, the composition operator C_T is a well-defined Hilbert-Schmidt operator on $(H_\nu, \|\cdot\|_\nu)$.*

Proof. By using $\langle An, v_{u/s} \rangle = \langle n, A^T v_{u/s} \rangle = \lambda_{u/s} \langle n, v_{u/s} \rangle$ and (2), we obtain

$$\begin{aligned} \|C_T e_n\|_\nu &= \exp(-a_1 \lambda_u |\langle n, v_u \rangle| + a_2 \lambda_s |\langle n, v_s \rangle|) \cdot \exp(a_1 |\langle n, v_u \rangle| - a_2 |\langle n, v_s \rangle|) \\ &= \exp(-a_1 (\lambda_u - 1) |\langle n, v_u \rangle| - a_2 (1 - \lambda_s) |\langle n, v_s \rangle|). \end{aligned}$$

As $(\lambda_u - 1) > 0$, $(1 - \lambda_s) > 0$, and all norms on \mathbb{R}^2 are equivalent, we obtain inequality (1). \square

It turns out that replacing V by the identity matrix yields an operator that is not even compact. We omit a proof, as this follows by a direct calculation.

Lemma 2.4. *Let $a = (a_1, a_2) \in \mathbb{R}_{>0}^2$ and $\nu(n) = \nu_a(n) = \exp(-a_1 |n_1| + a_2 |n_2|)$. The operator C_T is bounded on $(H_\nu, \|\cdot\|_\nu)$ if and only if $a_1 = a_2$. It is never compact on $(H_\nu, \|\cdot\|_\nu)$.*

On the other hand, as will become apparent in the next section, working with a diagonal matrix V yields more convenient function spaces (specifically, Hilbert spaces of holomorphic functions on polydisks). We can restore the nice properties of the composition operator in this setting, by allowing a to be a function of $n \in \mathbb{Z}^2$.

Definition 2.5 (Quadrant-wise exponential weight). Let $\alpha, \gamma \in \mathbb{R}^2$ and define $a: \mathbb{Z}^2 \rightarrow \mathbb{R}^2$ as

$$a_{\alpha, \gamma}(n) = \begin{cases} \alpha, & \text{if } n_1 \cdot n_2 \geq 0 \\ \gamma, & \text{if } n_1 \cdot n_2 < 0, \end{cases}$$

and $\nu(n) = \nu_{\alpha, \gamma}(n) = \exp(-\langle a_{\alpha, \gamma}(n), |n| \rangle)$. We call such weight function *quadrant-wise exponential*.

Lemma 2.6. *Let $\nu = \nu_{\alpha, \gamma}$ be a quadrant-wise exponential weight. If $\alpha \in \mathbb{R}_{>0}^2$ and $\gamma \in \mathbb{R}_{<0}^2$, then the operator C_T is a well-defined Hilbert-Schmidt operator on H_ν .*

Proof. For $n \in \mathbb{Z}^2$ we denote

$$\varphi(n) = \langle a_{\alpha, \gamma}(n), |n| \rangle - \langle a_{\alpha, \gamma}(An), |An| \rangle,$$

so that

$$\|C_T e_n\|_\nu = \frac{\nu(An)}{\nu(n)} = \exp(\varphi(n)).$$

To prove the lemma, it suffices to show that there exists $c > 0$ such that

$$\varphi(n) < -c(|n_1| + |n_2|) \quad (\forall n \in \mathbb{Z}^2). \quad (3)$$

Set $m = An$. We note that since $\|n\|_2 \leq \|A^{-1}\| \|m\|_2$ and by the equivalence of norms in \mathbb{R}^2 , there exists $\tilde{c} > 0$ such that $|m_1| + |m_2| = \|m\|_1 \geq \tilde{c} \|n\|_1 = \tilde{c}(|n_1| + |n_2|)$, for all $n \in \mathbb{Z}^2$ and $m = An$. There are three cases we need to take care of.

(i) $n_1 \cdot n_2 \geq 0$. Note that in this case $|m| = |An| = A|n|$ and $m_1 \cdot m_2 \geq 0$. It follows that

$$\varphi(n) = \langle \alpha, |n| - A|n| \rangle = \langle \alpha - A^T \alpha, |n| \rangle = -(\alpha_1 + \alpha_2)|n_1| - \alpha_1 |n_2| < -c_1(|n_1| + |n_2|),$$

for any $0 < c_1 < \alpha_1$.

(ii) $m_1 \cdot m_2 < 0$, which implies $n_1 \cdot n_2 < 0$ as $n = A^{-1}m$ and $n_1 \cdot n_2 = (m_1 - m_2)(2m_2 - m_1) = 3m_1 m_2 - 2m_2^2 - m_1^2 < 0$. Noting that $|n_1| = |m_1 - m_2| = |m_1| + |m_2|$ and $|n_2| = |2m_2 - m_1| = 2|m_2| + |m_1|$ we obtain

$$\varphi(n) = \langle \gamma, |n| - |m| \rangle = \gamma_1 |m_2| + \gamma_2 (|m_1| + |m_2|) < -c_2(|n_1| + |n_2|),$$

for any $0 < c_2 < -\gamma_2 \tilde{c}$.

(iii) $n_1 \cdot n_2 < 0$ and $m_1 \cdot m_2 \geq 0$. In this case

$$\varphi(n) = \langle \gamma, |n| \rangle - \langle \alpha, |m| \rangle \leq \max(\gamma_1, \gamma_2)(|n_1| + |n_2|) - \min(\alpha_1, \alpha_2)(|m_1| + |m_2|) < -c_3(|n_1| + |n_2|),$$

for any $0 < c_3 < 2 \min(-\gamma_1, -\gamma_2, \alpha_1 \tilde{c}, \alpha_2 \tilde{c})$.

Combining the above, we have that (3) holds for all $n \in \mathbb{Z}^2$ with $c = \min(c_1, c_2, c_3)$. \square

Remark 2.7. It turns out that the use of quadrant-wise exponential weight functions is not restricted to linear toral Anosov diffeomorphisms, but can also be applied to certain non-linear maps. It is possible to show that for any map in the family of non-linear maps studied in [SBJ], one can find suitable α and γ , such that the associated composition operator considered on the Hilbert space H_ν with $\nu = \nu_{\alpha, \gamma}$ is Hilbert-Schmidt, and even trace-class. The proof is a straightforward but lengthy calculation involving properties of the underlying map summarized in [SBJ, Lemma 2.3]. See also [PoS] for this and related results.

2.2 Anosov diffeomorphisms with strong mapping conditions

In this section, we establish more general conditions on the toral diffeomorphisms which will be sufficient to prove that the associated composition operator is trace-class on a suitable weighted Hilbert space. In particular, we characterise maps that satisfy the conditions of the main result of the next section (Theorem 3.24). We start by recalling some well-known facts about cones and Reinhardt domains.

2.2.1 Convex cones in \mathbb{R}^2

A *cone* $\Lambda \subset \mathbb{R}^2$ is a set such that if $v \in \Lambda$, then $\lambda v \in \Lambda$ for all $\lambda > 0$. A cone shifted by a vector, that is a set of the form $x + \Lambda$, where $x \in \mathbb{R}^2$ and $\Lambda \subset \mathbb{R}^2$ is a cone, is called an *affine cone*. For $p_u, p_s \in \mathbb{R}^2$ denote by $P = (p_u, p_s)$ the matrix having p_u and p_s as its column vectors. For an invertible matrix P denote by Λ_P the convex open polyhedral cone in \mathbb{R}^2 (positively) spanned by p_u and p_s , that is,

$$\Lambda_P = \{Px = x_1 p_u + x_2 p_s : x > 0\} = P(\mathbb{R}_{>0}^2).$$

Writing $W = (w_u, w_s) = (P^T)^{-1}$, this can equivalently be expressed as

$$\Lambda_P = \{x \in \mathbb{R}^2 : \langle w_u, x \rangle > 0, \langle w_s, x \rangle > 0\}.$$

Its *polar cone* is given by

$$(\Lambda_P)^\circ = \{x \in \mathbb{R}^2 : \langle x, p \rangle \leq 0 \text{ for all } p \in \Lambda_P\} = \{x \in \mathbb{R}^2 : P^T x \leq 0\} = W(\mathbb{R}_{\leq 0}^2).$$

While Λ_P is the image of the positive quadrant of \mathbb{R}^2 under P , it will later be useful to consider the images of all quadrants. For this, we write $I^\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ for $\sigma \in \Sigma$, where⁵

$$\Sigma := \{\sigma = (\sigma_1, \sigma_2) : \sigma_1, \sigma_2 \in \{\pm 1\}\},$$

and denote by Λ_P^σ the image of the quadrant $R^\sigma = I^\sigma(\mathbb{R}_{>0}^2)$ under P , that is $\Lambda_P^\sigma = P(R^\sigma)$ for $\sigma \in \Sigma$. A short calculation reveals that the cone Λ_P^σ can be written as

$$\Lambda_P^\sigma = \{x \in \mathbb{R}^2 : \sigma_1 \langle w_u, x \rangle > 0, \sigma_2 \langle w_s, x \rangle > 0\}.$$

All the above cones have $(0, 0)$ as their apex. As we shall see shortly we will need to work with cones translated by a vector. For this, let us denote by R_δ^σ the image under I^σ of the first quadrant translated by $\delta \in \mathbb{R}^2$, that is $R_\delta^\sigma = I^\sigma(\mathbb{R}_{>0}^2 + \delta) = R^\sigma + v_\delta^\sigma$ with apex $v_\delta^\sigma = I^\sigma \delta$.

⁵For brevity, when using $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ as an index, we will often just write out its signs, e.g. writing R^{++} for $R^{(+1, +1)}$, R^{+-} for $R^{(+1, -1)}$, etc.

Definition 2.8. For $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$ and $P \in \text{GL}_2(\mathbb{R})$ we denote $\Lambda_{P,\delta}^\sigma$ the convex affine cone

$$\Lambda_{P,\delta}^\sigma = P(R_\delta^\sigma) = \Lambda_P^\sigma + v_{P,\delta}^\sigma, \quad (4)$$

with apex $v_{P,\delta}^\sigma = Pv_\delta^\sigma$.

We note that $\sigma_1\langle w_u, v_{P,\delta}^\sigma \rangle = \sigma_1\langle P^T w_u, I^\sigma \delta \rangle = \sigma_1\langle (1,0)^T, I^\sigma \delta \rangle = \delta_1$ and $\sigma_2\langle w_s, v_{P,\delta}^\sigma \rangle = \delta_2$. Using these equalities, $\Lambda_{P,\delta}^\sigma$ can be rewritten as

$$\begin{aligned} \Lambda_{P,\delta}^\sigma &= \{x \in \mathbb{R}^2: \sigma_1\langle w_u, x - v_{P,\delta}^\sigma \rangle > 0, \sigma_2\langle w_s, x - v_{P,\delta}^\sigma \rangle > 0\} \\ &= \{x \in \mathbb{R}^2: \sigma_1\langle w_u, x \rangle > \delta_1, \sigma_2\langle w_s, x \rangle > \delta_2\}. \end{aligned} \quad (5)$$

2.2.2 Log-conical Reinhardt domains of $\hat{\mathbb{C}}^2$

The translated convex cones in (4) will be useful for defining certain two-dimensional complex domains. For this we first require some definitions.

Notation 2.9. We denote the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and write $\hat{\mathbb{C}}^2 = \hat{\mathbb{C}} \times \hat{\mathbb{C}}$. For $z \in \hat{\mathbb{C}}^2$, $v \in \mathbb{R}^2$ and $a \in \mathbb{Z}^2$ we write $|z| = (|z_1|, |z_2|)$, $z^a = z_1^{a_1} z_2^{a_2}$, $e^v = (e^{v_1}, e^{v_2})$, and for $z \in (\mathbb{C} \setminus \{0\})^2$ we write $\log |z| = (\log |z_1|, \log |z_2|)$. For any domain $D \subseteq \hat{\mathbb{C}}^2$, we write $D^+ = D \cap (\mathbb{C} \setminus \{0\})^2$.

Definition 2.10. A domain $D \subset \hat{\mathbb{C}}^2$ is called *polycircular* or a *Reinhardt domain* if it is invariant under polyrotations, that is, if $z \in D$ implies $\omega z = (\omega_1 z_1, \omega_2 z_2) \in D$ for all $\omega \in \mathbb{T}^2$. The set $|D| := \{|z|: z \in D\} \subset (\mathbb{R}_{\geq 0} \cup \{\infty\})^2$ is called *absolute domain* of D , and $\Lambda = \log |D^+| := \{\log |z|: z \in D^+\} \in \mathbb{R}^2$ the *logarithmic base* of D^+ . A Reinhardt domain D is called *log-conical* if the logarithmic base of D^+ is a convex open affine cone.

We define $\mathbb{T}_\rho^2 = \{z \in \mathbb{C}^2: |z_1| = \rho_1, |z_2| = \rho_2\}$ for $\rho \in \mathbb{R}_{>0}^2$, and write $e^\Lambda \mathbb{T}^2 := \bigcup_{r \in \Lambda} \mathbb{T}_{e^r}^2$. Further let $\mathcal{E}(\Lambda) \subset \mathbb{R}^2$ denote the union of the set of *faces* of Λ , that is $\mathcal{E}(\Lambda) = \partial \Lambda \setminus \{p\}$ with p the apex of Λ . Then, every convex open affine cone $\Lambda \subset \mathbb{R}^2$ induces a log-conical Reinhardt domain in $\hat{\mathbb{C}}^2$ via

$$D = \text{cl}(e^\Lambda \mathbb{T}^2) \setminus e^{\mathcal{E}(\Lambda)} \mathbb{T}^2,$$

where the closure is taken in $\hat{\mathbb{C}}^2$ (that is, it may contain points of the form (z_1, z_2) , with either z_1 , or z_2 , or both, taking the values 0 or ∞). We shall denote by $D_{P,\delta}^\sigma$ the log-conical Reinhardt domain induced by the convex affine cone $\Lambda_{P,\delta}^\sigma$ in (4). We can calculate

$$\begin{aligned} (D_{P,\delta}^\sigma)^+ &= e^{\Lambda_{P,\delta}^\sigma} \mathbb{T}^2 = \bigcup_{r \in \Lambda_{P,\delta}^\sigma} \mathbb{T}_{e^r}^2 = \bigcup_{x \in \Lambda_P^\sigma} \mathbb{T}_{\exp(v_{P,\delta}^\sigma + x)}^2 \\ &= \{z \in (\mathbb{C} \setminus \{0\})^2: |z_1| = e^{(v_{P,\delta}^\sigma)_1 + x_1}, |z_2| = e^{(v_{P,\delta}^\sigma)_2 + x_2}, x \in \Lambda_P^\sigma\} \\ &= \{z \in (\mathbb{C} \setminus \{0\})^2: |z|^{\sigma_1 w_u} > e^{\delta_1}, |z|^{\sigma_2 w_s} > e^{\delta_2}\}, \end{aligned}$$

which yields

$$D_{P,\delta}^\sigma = \{z \in \hat{\mathbb{C}}^2: |z|^{\sigma_1 w_u} > e^{\delta_1}, |z|^{\sigma_2 w_s} > e^{\delta_2}\}.$$

The *distinguished boundary* or *Shilov boundary* of $D_{P,\delta}^\sigma$ is a torus in \mathbb{C}^2 given by

$$\mathbb{T}_{P,\delta}^\sigma := \partial^* D_{P,\delta}^\sigma := \{z \in \mathbb{C}^2: |z_1| = e^{(v_{P,\delta}^\sigma)_1}, |z_2| = e^{(v_{P,\delta}^\sigma)_2}\}.$$

2.2.3 Anosov toral diffeomorphisms with constant invariant cone fields

Definition 2.11 (Torale Anosov diffeomorphisms). Let $M = ([0, 2\pi]/\sim)^2$. A smooth diffeomorphism $\tilde{T}: M \rightarrow M$ is called *Anosov* if there exist two uniformly transversal open continuous cone fields $\mathcal{K}^u = \{K^u(x)\}$, $\mathcal{K}^s = \{K^s(x)\}$ with cones $K^u(x), K^s(x) \subset T_x M$, a norm $\|\cdot\|$ on $T_x M$ and $\lambda > 1$ such that, for all $x \in M$,

- (i) $D_x \tilde{T}(\text{cl}(K^u(x))) \subset K^u(\tilde{T}(x)) \cup \{0\}$, $D_x \tilde{T}^{-1}(\text{cl}(K^s(x))) \subset K^s(\tilde{T}^{-1}(x)) \cup \{0\}$ and
- (ii) $\|D_x \tilde{T}(v)\| > \lambda \|v\| \forall v \in \text{cl}(K^u(x))$ and $\|D_x \tilde{T}^{-1}(v)\| > \lambda \|v\| \forall v \in \text{cl}(K^s(x))$.

Without loss of generality, the cone fields can be chosen to be complementary, that is, $K^s(x) = T_x M \setminus \text{cl}(K^u(x))$ for all $x \in M$.

If the expanding and contracting cones $K^u(x)$ and $K^s(x)$ can be chosen independently of x , that is⁶ $K^u, K^s \subset T_x M$ such that $K^u(x) = K^u$ and $K^s(x) = K^s$ for all $x \in M$, then we say \tilde{T} is an *Anosov diffeomorphism with constant invariant cone fields*.

Let \tilde{T} be a toral Anosov diffeomorphism with constant complementary cone fields $\mathcal{K}^u = \{K^u\}$ and $\mathcal{K}^s = \{K^s\}$. Then K^u can be decomposed as $K^u = K^u_+ \cup -K^u_+$, where K^u_+ is a convex cone, so there exists a matrix $P \in \text{GL}_2(\mathbb{R})$ such that $K^u_+ = \Lambda_P^\sigma = PI^\sigma(\mathbb{R}_{>0}^2)$ with $\sigma = (+1, +1)$. Adopting the notation

$$\Sigma^1 = \{\sigma \in \Sigma : \sigma_1 = \sigma_2\} \quad \text{and} \quad \Sigma^{-1} = \{\sigma \in \Sigma : \sigma_1 = -\sigma_2\}$$

we can write $K^u = \bigcup_{\sigma \in \Sigma^1} \Lambda_P^\sigma$ and $K^s = \bigcup_{\tilde{\sigma} \in \Sigma^{-1}} \Lambda_P^{\tilde{\sigma}}$.

Definition 2.12. Let $\tilde{T}: M \rightarrow M$ be a smooth Anosov diffeomorphism.

- (i) We say that \tilde{T} has *P-induced constant invariant cone fields* if it has constant invariant cone fields $\mathcal{K}^u = \{K_0^u\}$ and $\mathcal{K}^s = \{K_0^s\}$ given by $K_\delta^u = \bigcup_{\sigma \in \Sigma^1} \Lambda_{P,\delta}^\sigma$ and $K_\delta^s = \bigcup_{\tilde{\sigma} \in \Sigma^{-1}} \Lambda_{P,\delta}^{\tilde{\sigma}}$ with $P \in \text{GL}_2(\mathbb{R})$ and $\delta \in \mathbb{R}^2$.
- (ii) We say that \tilde{T} has *P-induced strongly expanding constant invariant cone fields (sec)* if it has P-induced constant invariant cone fields and if there are $\delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$ and $\sigma \in \Sigma^1$, $\tilde{\sigma} \in \Sigma^{-1}$ such that, for all $x \in M$,

$$(D_x \tilde{T})v \in K_\delta^u \quad \text{and} \quad (D_x \tilde{T}^{-1})\tilde{v} \in K_{\tilde{\delta}}^s, \tag{6}$$

where $v = v_{P,\delta}^\sigma$, $\tilde{v} = v_{P,\tilde{\delta}}^{\tilde{\sigma}}$. In the special case $P = \mathbb{I}$, we refer to $K_0^u = \mathbb{R}_{>0}^2 \cup \mathbb{R}_{<0}^2$ as *positive*, and call (6) the (*p-sec*) condition.

Figure 1 illustrates the (*p-sec*) condition. Note that (i) is satisfied by all smooth toral Anosov diffeomorphisms with constant invariant cone fields, whereas (ii) is a stronger condition, which eventually will be required to prove compactness of the associated weighted composition operator (see Section 3).

Remark 2.13. The cone conditions in (6) are often easy to check. For example, the (*p-sec*) condition requires $K_0^u = \mathbb{R}_{>0}^2 \cup \mathbb{R}_{<0}^2$ and $K_0^s = \mathbb{R}^2 \setminus \text{cl}(K_0^u)$. Assuming $D_x \tilde{T} = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix}$ preserves $\mathbb{R}_{>0}^2$ for all $x \in M$, the first part of (6) is automatically satisfied if either $\inf_{x \in M} a_x \geq 1$, or $\inf_{x \in M} d_x \geq 1$, or if

$$\sup_{x \in M} \left(\frac{1 - a_x}{b_x} \right) \cdot \sup_{x \in M} \left(\frac{1 - d_x}{c_x} \right) < 1.$$

A similar condition for the first part of (6) in the case when $D_x \tilde{T}$ maps $\mathbb{R}_{\geq 0}^2$ to $\mathbb{R}_{\leq 0}^2$ can be deduced easily, as can be analogous conditions for the second part of (6).

⁶Here we slightly abuse notation, using the canonical identification $T_x M \cong \mathbb{R}^2$ for all $x \in M$.

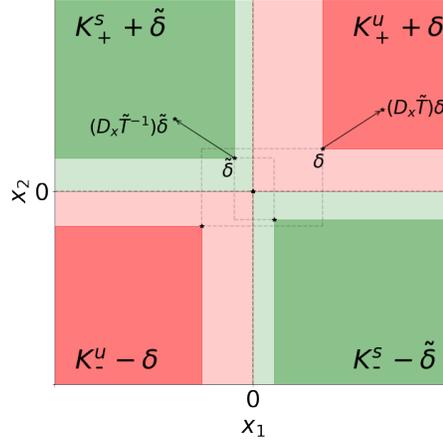


Figure 1: Illustration of the *(sec)* condition in Definition 2.12(ii) for $P = \mathbb{I}$ and $K_\delta^u = K_\pm^u \pm \delta$.

We shall next state conditions that can be easily deduced from the fact that the map \tilde{T} possesses constant invariant expanding and (co)-expanding cone fields, that is, from (i) in Definition 2.12.

Lemma 2.14. *Let $\tilde{T}: M \rightarrow M$ be a smooth Anosov diffeomorphism with P -induced constant invariant cone fields for some $P \in \text{GL}_2(\mathbb{R})$. Then, for any $\delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$, $\sigma \in \Sigma^1$ and $\tilde{\sigma} \in \Sigma^{-1}$, there exists $q \in \Lambda_{P, \tilde{\delta}}^{\tilde{\sigma}}$ with $(D_x \tilde{T})q \in \Lambda_{P, \delta}^\sigma$ for all $x \in M$.*

Proof. Fix $\delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$, $\sigma \in \Sigma^1$ and $\tilde{\sigma} \in \Sigma^{-1}$, and recall that the (constant) $(D_x \tilde{T})$ -invariant cone given by the matrix P is $K^u = \Lambda_P \cup -\Lambda_P$, with $\Lambda_P = P(\mathbb{R}_{>0}^2)$, and either $(D_x \tilde{T})(\Lambda_P) \subset \Lambda_P$ or $(D_x \tilde{T})(\Lambda_P) \subset -\Lambda_P$ (for all $x \in M$). For $x \in M$, we define $N_x = P^{-1}(D_x \tilde{T})P$, and observe that either $N_x(\mathbb{R}_{>0}^2) \subset \mathbb{R}_{>0}^2 \cup \{0\}$ or $N_x(\mathbb{R}_{>0}^2) \subset \mathbb{R}_{<0}^2 \cup \{0\}$, for all $x \in M$.

Applying Lemma A.2 to $\{N_x : x \in M\}$, there exists $q' \in R_\delta^{\tilde{\sigma}}$ such that $N_x(q') \in R_\delta^\sigma$ for all $x \in M$. Using (4), we obtain that $Pq' \in P(R_\delta^{\tilde{\sigma}}) = \Lambda_{P, \tilde{\delta}}^{\tilde{\sigma}}$, and $(D_x \tilde{T})(Pq') \in P(R_\delta^\sigma) = \Lambda_{P, \delta}^\sigma$ for all $x \in M$, finishing the proof with $q = Pq'$. \square

Let $\pi(x_1, x_2) = (e^{ix_1}, e^{ix_2})$ be the canonical diffeomorphism from M to \mathbb{T}^2 . We denote by T the diffeomorphism on \mathbb{T}^2 uniquely determined by the equation $T \circ \pi = \pi \circ \tilde{T}$. We will say that $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ satisfies the *(sec)* (or *(p-sec)*) condition, if the corresponding $\tilde{T}: M \rightarrow M$ does. As the next (key) lemma will show, for analytic toral Anosov diffeomorphisms with constant invariant cone fields, properties of the derivative $D\tilde{T}$ on the tangent bundle can be translated into properties of the map T in a small neighbourhood of \mathbb{T}^2 . With slight abuse of notation, we continue writing T for its analytic extension to such small neighbourhood of \mathbb{T}^2 .

Definition 2.15. We say that a diffeomorphism $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is *orientation-preserving* (*orientation-reversing*) if the determinant of $D_x \tilde{T}$ of the conjugated map $\tilde{T}: M \rightarrow M$ is positive (negative) for all $x \in M$. We say T is *area-preserving* if $|\det D_x \tilde{T}| = 1$ for all $x \in M$.

Lemma 2.16. *Let $\tilde{T}: M \rightarrow M$ be an analytic Anosov diffeomorphism with P -induced constant invariant cone fields for some $P \in \text{GL}_2(\mathbb{R})$, and T the conjugated diffeomorphism on \mathbb{T}^2 . If there are $\delta \in \mathbb{R}_{>0}^2$, $\sigma \in \Sigma$, and $q \in \mathbb{R}^2$ such that $(D_x \tilde{T})q \in \Lambda_{P, \delta}^\sigma$ for all $x \in M$, then there exist $\epsilon > 0$ and $\delta' > \delta$ such that for all $t \in (0, \epsilon)$ we have*

$$T(\mathbb{T}_{e^{iq}}^{2t}) \subset D_{P, t\delta'}^\sigma,$$

where $D_{P, t\delta'}^\sigma$ is the Reinhardt domain induced by the logarithmic base $\Lambda_{P, \delta'}^\sigma$.

Proof. We denote $w^{(1)} = \sigma_1 w_u$ and $w^{(2)} = \sigma_2 w_s$ (recalling $W = (w_u, w_s) = (P^T)^{-1}$). Using (5), we observe that $(D_x \tilde{T})q \in \Lambda_{P,\delta}^\sigma$ translates to the inequalities

$$\sum_{l=1,2} \sum_{k=1,2} \frac{\partial \tilde{T}_l}{\partial x_k} q_k w_l^{(j)} > \delta_j, \quad \text{for } j = 1, 2 \text{ and all } x \in M.$$

Next, for $x \in M$, $s \in \mathbb{R}$ and $a \in \mathbb{R}^2$, we define

$$f_x^a(s) = \sum_{l=1,2} a_l \log |T_l(\xi_x(s))|$$

with $\xi_x(s) = (e^{sq_1 + ix_1}, e^{sq_2 + ix_2}) \in \mathbb{T}_{e^{sq}}^2$. We note that f_x^a is continuously differentiable on $(0, \epsilon')$ for sufficiently small ϵ' , with

$$\frac{\partial}{\partial s} f_x^a(s) = \Re \left(\sum_{l=1,2} \sum_{k=1,2} \frac{e^{ix_k} \partial_k T_l(\xi_x(s))}{T_l(\xi_x(s))} q_k e^{sq_k} a_l \right),$$

where $\partial_k T_l$ denotes the (complex) derivative of the l -th component of T with respect to the k -th variable. It follows that

$$\frac{\partial}{\partial s} f_x^{w^{(j)}}(s)|_{s=0} = \Re \left(\sum_{l=1,2} \sum_{k=1,2} \frac{e^{ix_k} \partial_k T_l(e^{ix_1}, e^{ix_2})}{T_l(e^{ix_1}, e^{ix_2})} q_k w_l^{(j)} \right) = \Re \left(\sum_{l=1,2} \sum_{k=1,2} \frac{\partial \tilde{T}_l}{\partial x_k} q_k w_l^{(j)} \right) > \delta_j$$

for $j = 1, 2$ and all $x \in M$. By compactness of M and continuity, we can fix $\delta' > \delta$, so that $\frac{\partial}{\partial s} f_x^{w^{(j)}}(s) > \delta'_j$, $j = 1, 2$, holds for all $x \in M$ and all $s \in (0, \epsilon'')$ with a sufficiently small $\epsilon'' \in (0, \epsilon')$. Since $f_x^a(0) = 0$, we obtain that

$$\sum_{l=1,2} w_l^{(j)} \log |T_l(\xi_x(t))| = f_x^{w^{(j)}}(t) - f_x^{w^{(j)}}(0) = \int_0^t \frac{\partial}{\partial s} f_x^{w^{(j)}}(s) ds > \delta'_j t,$$

for $j = 1, 2$ and $t \in (0, \epsilon'')$. Exponentiating both sides, it follows that $T(\xi_x(t)) \in D_{P,t\delta'}^\sigma$ for all $x \in M$. Since $\{\xi_x(t) : x \in M\} = \mathbb{T}_{e^{tq}}^2$, this completes the proof. \square

If in addition \tilde{T} satisfies the (*sec*) condition, then one can establish all the required properties (for Theorem 1.1) for T in a neighbourhood of \mathbb{T}^2 . First, we need to introduce corresponding notions on Reinhardt domains induced by convex cones. For $\delta \in \mathbb{R}^2$ and $P \in \text{GL}_2(\mathbb{R})$, let $\mathcal{A}_{P,\delta}$ denote a two-dimensional ‘annulus’ containing \mathbb{T}^2 , given by

$$\mathcal{A}_{P,\delta} = \{z \in \mathbb{C}^2 : -|\delta| < P^{-1} \log |z| < |\delta|\}.$$

For brevity we also write $\mathcal{A}_\delta = \mathcal{A}_{\mathbb{I},\delta} = \{z \in \mathbb{C}^2 : -|\delta| < \log |z| < |\delta|\}$.

Definition 2.17. Let $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an analytic map with holomorphic extension to a neighbourhood of $\text{cl}(\mathcal{A}_{P,\delta})$ for some $\delta \in \mathbb{R}_{>0}^2$ and $P \in \text{GL}_2(\mathbb{R})$. For $\ell \in \{1, -1\}$ and $\Delta \in \mathbb{R}_{>0}^2$ the map T is said to have the $(\ell, \delta, \Delta, P)$ -strongly expanding mapping property if one of the following two alternatives holds:

(EP) $T(\mathbb{T}_{P,\delta}^\sigma) \subseteq D_{P,\Delta}^\sigma \subset D_{P,\delta}^\sigma$ for all $\sigma \in \Sigma^\ell$ (“ T preserves expanding direction”),

(ER) $T(\mathbb{T}_{P,\delta}^\sigma) \subseteq D_{P,\Delta}^{-\sigma} \subset D_{P,\delta}^{-\sigma}$ for all $\sigma \in \Sigma^\ell$ (“ T reverses expanding direction”).

Proposition 2.18. *Let $\tilde{T}: M \rightarrow M$ be an analytic Anosov diffeomorphism with constant P -induced strongly expanding invariant cone fields for some $\delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$ in (6). Then there exists $\eta \in \mathbb{R}_{>0}^2$ such that the corresponding diffeomorphisms T and T^{-1} on \mathbb{T}^2 can be analytically extended to $\mathcal{A}_{P,\eta}$, and there are $\Delta, \tilde{\Delta} \in \mathbb{R}_{>0}^2$ with $\Delta > \delta$ and $\tilde{\Delta} < \tilde{\delta}$, such that*

(i) T is $(1, t\delta, t\Delta, P)$ -strongly expanding for all sufficiently small $t > 0$,

(ii) T^{-1} is $(-1, t\tilde{\Delta}, t\tilde{\delta}, P)$ -strongly expanding for all sufficiently small $t > 0$.

Moreover, for any $\sigma \in \Sigma^1, \tilde{\sigma} \in \Sigma^{-1}$ and any $\tilde{\delta}, \Delta \in \mathbb{R}_{>0}^2$,

(iii) there exists $q \in \Lambda_{P,\tilde{\delta}}^{\tilde{\sigma}}$ such that $\mathbb{T}_{e^{tq}}^2 \subset D_{P,t\tilde{\delta}}^{\tilde{\sigma}}$ and $T(\mathbb{T}_{e^{tq}}^2) \subset D_{P,t\Delta}^{\sigma}$ for all sufficiently small t .

Proof. To prove (i), we apply Lemma 2.16 with $q = v_{P,\delta}^{\sigma}$, $\sigma \in \Sigma^1$. Since $tq = tv_{P,\delta}^{\sigma} = v_{P,t\delta}^{\sigma}$ and therefore $\mathbb{T}_{e^{tq}}^2 = \mathbb{T}_{P,t\delta}^{\sigma}$, there exist $\epsilon > 0$ and $\Delta > \delta$, such that for all $t \in (0, \epsilon)$, $T(\mathbb{T}_{P,t\delta}^{\sigma}) \subset D_{P,t\Delta}^{\sigma} \subset D_{P,t\delta}^{\sigma}$. Item (ii) follows analogously by applying Lemma 2.16 with $\tilde{q} = v_{P,\tilde{\delta}}^{\tilde{\sigma}}$, $\tilde{\sigma} \in \Sigma^{-1}$, to the map T^{-1} . Finally, for (iii), by Lemma 2.14 there exists $q \in \mathbb{R}^2$ satisfying $tq \in \Lambda_{P,t\tilde{\delta}}^{\tilde{\sigma}}$ (i.e. $\mathbb{T}_{e^{tq}}^2 \subset D_{P,t\tilde{\delta}}^{\tilde{\sigma}}$) and $D_x \tilde{T}(tq) \in \Lambda_{P,t\Delta}^{\sigma}$ for all $t > 0$ and $x \in M$. Then, by Lemma 2.16, there exists $\Delta' > \Delta$ and $\epsilon' > 0$ such that $T(\mathbb{T}_{e^{tq}}^2) \subset D_{P,t\Delta'}^{\sigma} \subset D_{P,t\Delta}^{\sigma}$ for all $t \in (0, \epsilon')$, as required. \square

Corollary 2.19. *If an analytic Anosov diffeomorphism \tilde{T} on M satisfies the (sec) condition for some $P \in \text{GL}_2(\mathbb{R})$, then there exist $\alpha, A, \gamma, \Gamma, \eta \in \mathbb{R}_{>0}^2$ with $\alpha < A < \eta$ and $\Gamma < \gamma < \eta$, such that the corresponding diffeomorphisms T and T^{-1} on \mathbb{T}^2 can be analytically extended to $\mathcal{A}_{P,\eta}$ and the following mapping properties hold:*

(i) T is $(1, \alpha, A, P)$ -strongly expanding;

(ii) T^{-1} is $(-1, \Gamma, \gamma, P)$ -strongly expanding;

(iii) For any $\sigma \in \Sigma^1, \tilde{\sigma} \in \Sigma^{-1}$, there exists q such that $\mathbb{T}_{e^q}^2 \subset D_{P,\gamma}^{\tilde{\sigma}} \cap \mathcal{A}_{P,\eta}$ and $T(\mathbb{T}_{e^q}^2) \subset D_{P,A}^{\sigma}$.

Proof. Using notation from Proposition 2.18, (i)-(iii) are satisfied with $\alpha = t\delta$, $A = t\Delta$, $\Gamma = t\tilde{\Delta}$, $\gamma = t\tilde{\delta}$, for sufficiently small $t > 0$. \square

3 Composition operator for Anosov maps with constant invariant cone fields

In this section, we shall consider (weighted) composition operators associated to analytic Anosov diffeomorphisms satisfying the (sec) condition. In this setting, using Corollary 2.19, we will show that there exist Hardy-Hilbert spaces induced by a suitable cone-wise exponential weight, such that the operator is trace-class.

3.1 Hardy-Hilbert spaces on Reinhardt domains

Before moving to the more general case of Reinhardt domains, we present a few simplified examples in which the domains are chosen to be polydisks.

Example 3.1. Let P be the identity matrix \mathbb{I} . Then $\Lambda_{P,\delta}^{\delta} = R_{\delta}^{\sigma}$, that is, it is one of the four quadrants translated by $(\pm\delta_1, \pm\delta_2)$. For simplicity we write $D_{\delta}^{\sigma} = D_{\mathbb{I},\delta}^{\sigma}$ for the disk induced by the cone R_{δ}^{σ} . Its distinguished boundary is given by

$$\partial^* D_{\delta}^{\sigma} = \mathbb{T}_{\delta}^{\sigma} = \{z \in \mathbb{C}^2 : |z_1| = e^{\sigma_1 \delta_1}, |z_2| = e^{\sigma_2 \delta_2}\}.$$

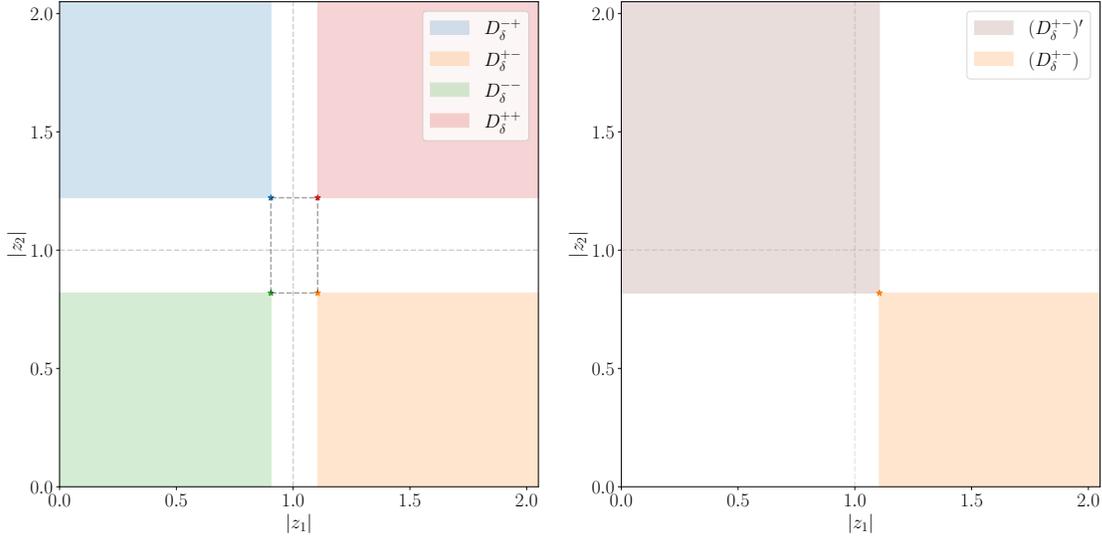


Figure 2: Absolute domains of the four polydisks D_δ^σ for $\sigma \in \Sigma$ and $\delta = (0.1, 0.2)$ (left) and depiction of the dual domain $(D_\delta^{+-})'$ of D_δ^{+-} (right).

- (i) Fix $\sigma = (-1, -1)$. Then, for $\delta = (0, 0)$ we obtain a unit bidisk, that is, $D_\delta^\sigma = \mathbb{D}^2$. If $\delta \neq 0$, then D_δ^σ is a bidisk centered at $(0, 0)$ with radii $e^{\sigma_1 \delta_1}$ and $e^{\sigma_2 \delta_2}$.
- (ii) The four domains depicted in the left panel of Figure 2 (reduced representation in the plane of absolute values $(|z_1|, |z_2|)$) corresponds to $\delta = (0.1, 0.2)$ for all $\sigma \in \Sigma$.

To proceed to the general case of Reinhardt domains, we introduce some general notation and list simple facts about toral automorphisms.

Definition 3.2. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$$

we define $\tau_A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ to be the toral automorphism given by $\tau_A(z_1, z_2) = (z_1^{a_{11}} z_2^{a_{12}}, z_1^{a_{21}} z_2^{a_{22}})$. With slight abuse of notation we also write τ_A for the extension of the map to $\hat{\mathbb{C}}^2$.

Lemma 3.3. Let $A, B \in \text{GL}_2(\mathbb{Z})$ and $f \in L^2(\mathbb{T}^2)$. Then:

- (i) $\tau_A \circ \tau_B = \tau_{A \cdot B}$, and in particular $\tau_A^{-1} = \tau_{A^{-1}}$.
- (ii) The map τ_A preserves \mathbb{T}^2 and is holomorphic in a neighbourhood.
- (iii) $f \circ \tau_A \in L^2(\mathbb{T}^2)$, and $\|f \circ \tau_A\|_{L^2(\mathbb{T}^2)} = \|f\|_{L^2(\mathbb{T}^2)}$.

We note that by Lemma A.1 in the appendix, every Reinhardt domain $D_{P,\delta}^\sigma \subset \hat{\mathbb{C}}^2$ is the image of a bounded Reinhardt domain in \mathbb{C}^2 under a holomorphic map of the form $\tau_A, A \in \text{GL}_2(\mathbb{Z})$.

Remark 3.4. In analogy to the case of the one-dimensional Riemann sphere, here we extend the notion of holomorphic functions on domains in \mathbb{C}^2 to those on domains in $\hat{\mathbb{C}}^2$ in the obvious way. For a domain $\hat{D} \subset \hat{\mathbb{C}}^2$, a function $f: \hat{D} \rightarrow \mathbb{C}$ is holomorphic if there exists a domain $D \subset \mathbb{C}^2$ and a biholomorphic mapping $\phi: D \rightarrow \hat{D}$ such that $f \circ \phi: D \rightarrow \mathbb{C}$ is holomorphic.

Definition 3.5. For $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$ and $P \in \text{GL}_2(\mathbb{R})$, the *Hardy-Hilbert space* $H_{P,\delta}^\sigma := H^2(D_{P,\delta}^\sigma)$ consists of all holomorphic functions $f: D_{P,\delta}^\sigma \rightarrow \mathbb{C}$ such that

$$\sup_{r \in \Lambda_{P,\delta}^\sigma} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |f(e^{r+it})|^2 dt < \infty.$$

In the case $P = \mathbb{I}$ (that is, $D_{P,\delta}^\sigma$ a polydisk), we will write $H_\delta^\sigma = H_{P,\delta}^\sigma$ for brevity.

Remark 3.6. As the definition indicates, the next results (Lemma 3.7 and Proposition 3.9) will establish that the space $H_{P,\delta}^\sigma$ with the inner product (7) indeed forms a Hilbert space.

We recall that $\partial^* D_{P,\delta}^\sigma = \mathbb{T}_{P,\delta}^\sigma = \mathbb{T}_{\exp(v_{P,\delta}^\sigma)}^2$, and proceed with the following generalization of a classical result for the polydisk $\mathbb{D}^2 = D_{\mathbb{I},0}^-$.

Lemma 3.7. *For any $f \in H_{P,\delta}^\sigma$ there exists $f^* \in L^2(\mathbb{T}_{P,\delta}^\sigma)$ (the “boundary value function” of f), which satisfies*

$$\lim_{r \in \Lambda_{P,\delta}^\sigma, r \rightarrow v_{P,\delta}^\sigma} \int_{[0,2\pi]^2} |f(e^{r+it}) - f^*(e^{v_{P,\delta}^\sigma+it})|^2 dt = 0.$$

Proof. Since $D_{P,\delta}^\sigma = D_{P',\delta}^{(-1,-1)}$ and for $P' = PI^\sigma \in \text{GL}_2(\mathbb{R})$, we can assume without loss of generality that $\sigma = (-1, -1)$. The case of P being a non-negative matrix and $\delta = 0$ (i.e., $D_{P,\delta}^\sigma \subseteq \mathbb{D}^2$ a bounded Reinhardt domain) is classical, see [U, Section 2.5] or [L], and the more general case of $\delta \in \mathbb{R}^2$ follows immediately. For general $P \in \text{GL}_2(\mathbb{R})$ and $f \in H_{P,\delta}^\sigma$, we write $P = A\tilde{P}$ with $A \in \text{GL}_2(\mathbb{Z})$ and $\tilde{P} \in \text{GL}_2(\mathbb{R})$ non-negative (Lemma A.1). We use $\tau_A: D_{P,\delta}^\sigma \rightarrow D_{\tilde{P},\delta}^\sigma$ and consider the function $\tilde{f} = f \circ \tau_A \in H_{\tilde{P},\delta}^\sigma$. By the previous case, there exists $\tilde{f}^* \in L^2(\mathbb{T}_{\tilde{P},\delta}^\sigma)$ such that

$$\lim_{r \in \Lambda_{\tilde{P},\delta}^\sigma, r \rightarrow v_{\tilde{P},\delta}^\sigma} \int_{[0,2\pi]^2} |\tilde{f}(e^{r+it}) - \tilde{f}^*(e^{v_{\tilde{P},\delta}^\sigma+it})|^2 dt = 0.$$

Writing $f^* = \tilde{f}^* \circ \tau_A^{-1}$, we obtain

$$\begin{aligned} & \lim_{r \in \Lambda_{P,\delta}^\sigma, r \rightarrow v_{P,\delta}^\sigma} \int_{[0,2\pi]^2} |f(e^{r+it}) - f^*(e^{v_{P,\delta}^\sigma+it})|^2 dt \\ &= \lim_{r \in \Lambda_{\tilde{P},\delta}^\sigma, r \rightarrow v_{\tilde{P},\delta}^\sigma} \int_{[0,2\pi]^2} |f \circ \tau_A(e^{r+it}) - f^* \circ \tau_A(e^{v_{\tilde{P},\delta}^\sigma+it})|^2 dt = 0. \quad \square \end{aligned}$$

We define an inner product on $H_{P,\delta}^\sigma$ by setting

$$(f, g)_{H_{P,\delta}^\sigma} := \langle f^*, \overline{g^*} \rangle_{\mathbb{T}_{P,\delta}^\sigma} := (f^*, g^*)_{L^2(\mathbb{T}_{P,\delta}^\sigma)}, \quad (7)$$

and we will omit the star notation for the boundary value function whenever there is no ambiguity. Here the L^2 inner product between f^* and g^* is defined conventionally as

$$(f^*, g^*)_{L^2(\mathbb{T}_{P,\delta}^\sigma)} = \int_{[0,2\pi]^2} f^*(e^{v_{P,\delta}^\sigma+it}) \overline{g^*(e^{v_{P,\delta}^\sigma+it})} \frac{dt}{(2\pi)^2} = \int_{\mathbb{T}_{P,\delta}^\sigma} f^*(z) \overline{g^*(z)} dm(z),$$

where $dm(z) = \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2}$ is the normalised Lebesgue measure on $\mathbb{T}_{P,\delta}^\sigma$.

Notation 3.8. We denote by $\Lambda_P^{\sigma,o}$ the polar cone of $\Lambda_P^\sigma = P(R^\sigma)$, and note that $\Lambda_P^{\sigma,o} = (P^T)^{-1}(\text{cl}(R^{-\sigma}))$. We write $\mathbb{Z}_P^\sigma = \mathbb{Z}^2 \cap \Lambda_P^{\sigma,o}$. For $P = \mathbb{I}$, we will use the shorthand $\mathbb{Z}^\sigma = \mathbb{Z}_\mathbb{I}^\sigma$.

Proposition 3.9 (Alternative characterisations). *Let $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$ and $P \in \text{GL}_2(\mathbb{R})$.*

(i) The function $p_n(z) = z^n$ is in $H_{P,\delta}^\sigma$ iff $n \in \mathbb{Z}_P^\sigma$.

(ii) Let f be given by $f(z) = \sum_{n \in \mathbb{Z}^2} f_n z^n$. Then, $f \in H_{P,\delta}^\sigma$ iff $f_n = 0$ for $n \notin \mathbb{Z}_P^\sigma$ and $\sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle} < \infty$.

(iii) $H_{P,\delta}^\sigma$ is a Hilbert space with inner product given by (7). Moreover, for $f \in H_{P,\delta}^\sigma$ with $f(z) = \sum_{n \in \mathbb{Z}_P^\sigma} f_n z^n$, we have

$$\|f\|_{H_{P,\delta}^\sigma}^2 = \sup_{r \in \Lambda_{P,\delta}^\sigma} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |f(e^{r+it})|^2 dt = \sum_{n \in \mathbb{Z}_P^\sigma} |f_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle}.$$

(iv) $H_{P,\delta}^\sigma$ is the closure of $\{p_n : n \in \mathbb{Z}_P^\sigma\}$ with respect to the norm $\|\cdot\|_{H_{P,\delta}^\sigma}$.

Proof. We begin by recalling that $\Lambda_{P,\delta}^\sigma = \Lambda_P^\sigma + v_{P,\delta}^\sigma$, and observe that

$$\sup_{r \in \Lambda_{P,\delta}^\sigma} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |p_n(e^{r+it})|^2 dt = \sup_{r \in \Lambda_{P,\delta}^\sigma} e^{2\langle n, r \rangle} = e^{2\langle n, v_{P,\delta}^\sigma \rangle} \sup_{r \in \Lambda_P^\sigma} e^{2\langle n, r \rangle}$$

for any $n \in \mathbb{Z}^2$. Since the supremum is finite if and only if $\langle n, r \rangle \leq 0$ for all $r \in \Lambda_P^\sigma$, that is, if and only if $n \in \mathbb{Z}_P^\sigma = \mathbb{Z}^2 \cap \Lambda_P^{\sigma,o}$, statement (i) follows. Next, we formally calculate for $f(z) = \sum_{n \in \mathbb{Z}^2} f_n z^n$ that

$$\begin{aligned} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |f(e^{r+it})|^2 dt &= \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \sum_{n \in \mathbb{Z}^2} f_n e^{n(r+it)} \overline{\sum_{m \in \mathbb{Z}^2} f_m e^{m(r+it)}} dt \\ &= \sum_{n \in \mathbb{Z}^2} |f_n|^2 \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |p_n(e^{r+it})|^2 dt \\ &= \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, r \rangle}. \end{aligned}$$

If $f_n \neq 0$ for some $n \notin \mathbb{Z}_P^\sigma$, by the above argument, the supremum over $r \in \Lambda_{P,\delta}^\sigma$ is infinite, and hence $f \notin H_{P,\delta}^\sigma$. On the other hand, for $n \in \mathbb{Z}_P^\sigma$ and $r \in \Lambda_{P,\delta}^\sigma$, we have $\langle n, r \rangle = \langle P^T n, P^{-1} r \rangle = \langle I^\sigma P^T n, I^\sigma P^{-1} r \rangle$ with $I^\sigma P^T n \in \mathbb{R}_{\leq 0}^2$ and $I^\sigma P^{-1} r \in \mathbb{R}_{> 0}^2 + \delta$, so that

$$\langle n, r \rangle \leq \langle I^\sigma P^T n, \delta \rangle = \langle P^T n, I^\sigma \delta \rangle = \langle n, v_{P,\delta}^\sigma \rangle.$$

Therefore, if $f_n = 0$ for all $n \notin \mathbb{Z}_P^\sigma$ and $\sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle} < \infty$, then

$$\sup_{r \in \Lambda_{P,\delta}^\sigma} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |f(e^{r+it})|^2 dt = \sup_{r \in \Lambda_{P,\delta}^\sigma} \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, r \rangle} = \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle}$$

implies $f \in H_{P,\delta}^\sigma$, proving (ii). For (iii), it remains to show the last equality, which follows immediately from the last calculation and (7), as for any $f \in H_{P,\delta}^\sigma$ we have that

$$\sup_{r \in \Lambda_{P,\delta}^\sigma} \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |f(e^{r+it})|^2 dt = \sum_{n \in \mathbb{Z}_P^\sigma} |f_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle}.$$

Finally for (iv), we assume without loss of generality that $\sigma = (-1, -1)$ (since $H_{P,\delta}^\sigma = H_{P I^\sigma, \delta}^{(-1, -1)}$). The case of P a non-negative matrix is proved in [MX, Proposition 3.6]. For general $P \in \text{GL}_2(\mathbb{R})$, we write $P = A\tilde{P}$ with $A \in \text{GL}_2(\mathbb{Z})$ and $\tilde{P} \in \text{GL}_2(\mathbb{R})$ non-negative, and use the biholomorphic mapping $\tau_A : D_{\tilde{P}, \delta}^\sigma \rightarrow D_{P, \delta}^\sigma$, noting that $\|f \circ \tau_A\|_{H_{\tilde{P}, \delta}^\sigma} = \|f\|_{H_{P, \delta}^\sigma}$ for any $f \in H_{P, \delta}^\sigma$. Since A maps $\mathbb{Z}_{\tilde{P}}^\sigma$ bijectively onto \mathbb{Z}_P^σ , we have $\{p_n : n \in \mathbb{Z}_P^\sigma\} = \{p_n \circ \tau_A : n \in \mathbb{Z}_{\tilde{P}}^\sigma\}$, and the general case follows. \square

We next show that two Hardy-Hilbert spaces $H_{P,\delta}^\sigma$ and $H_{P',\delta}^\sigma$ are isomorphic, whenever $P^{-1}P' \in \text{GL}_2(\mathbb{Z})$. For any $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$, $P \in \text{GL}_2(\mathbb{R})$ and $A \in \text{GL}_2(\mathbb{Z})$, we observe that $A(\Lambda_{P,\delta}^\sigma) = (AP)(R_\delta^\sigma) = \Lambda_{AP,\delta}^\sigma$, and $AP \in \text{GL}_2(\mathbb{R})$, from which it follows that

$$\tau_A(D_{P,\delta}^\sigma) = \tau_A(e^{\Lambda_{P,\delta}^\sigma} \mathbb{T}^2) = e^{\Lambda_{AP,\delta}^\sigma} \mathbb{T}^2 = D_{AP,\delta}^\sigma.$$

Proposition 3.10. *Let $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$, $P \in \text{GL}_2(\mathbb{R})$ and $A \in \text{GL}_2(\mathbb{Z})$. Then the operator C_{τ_A} given by $C_{\tau_A}f = f \circ \tau_A$ is an isometric isomorphism from $H_{AP,\delta}^\sigma$ to $H_{P,\delta}^\sigma$.*

Proof. Let $f = \sum_{n \in \mathbb{Z}^2} f_n p_n \in H_{AP,\delta}^\sigma$, that is, by Proposition 3.9, $f_n = 0$ for all $n \notin \mathbb{Z}_{AP}^\sigma$, and $\sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, v_{AP,\delta}^\sigma \rangle} < \infty$, and write $g = C_{\tau_A}f = f \circ \tau_A$. For $n \in \mathbb{Z}^2$ we have

$$p_n \circ \tau_A(z) = (z_1^{a_{11}} z_2^{a_{12}})^{n_1} (z_1^{a_{21}} z_2^{a_{22}})^{n_2} = z_1^{a_{11}n_1 + a_{21}n_2} z_2^{a_{12}n_1 + a_{22}n_2} = p_{A^T n}(z).$$

Therefore, it holds that

$$g = \sum_{n \in \mathbb{Z}^2} f_n p_n \circ \tau_A = \sum_{n \in \mathbb{Z}^2} f_n p_{A^T n} = \sum_{n \in \mathbb{Z}^2} f_{(A^T)^{-1}n} p_n,$$

and hence $g = \sum_{n \in \mathbb{Z}^2} g_n p_n$ with $g_n = f_{(A^T)^{-1}n}$. Further, since

$$\Lambda_{AP}^{\sigma,o} = ((AP)^T)^{-1}(\text{cl}(R^{-\sigma})) = (A^T)^{-1}(P^T)^{-1}(\text{cl}(R^{-\sigma})) = (A^T)^{-1}(\Lambda_P^{\sigma,o}),$$

it follows that $(A^T)^{-1}n \in \Lambda_{AP}^{\sigma,o}$ iff $n \in \Lambda_P^{\sigma,o}$. Therefore, $g_n = f_{(A^T)^{-1}n} = 0$ whenever $n \notin \mathbb{Z}_P^\sigma$. Moreover,

$$\begin{aligned} \|g\|_{H_{P,\delta}^\sigma}^2 &= \sum_{n \in \mathbb{Z}^2} |g_n|^2 e^{2\langle n, v_{P,\delta}^\sigma \rangle} \\ &= \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle A^T n, P v_\delta^\sigma \rangle} = \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, AP v_\delta^\sigma \rangle} = \sum_{n \in \mathbb{Z}^2} |f_n|^2 e^{2\langle n, v_{AP,\delta}^\sigma \rangle} = \|f\|_{H_{AP,\delta}^\sigma}^2, \end{aligned}$$

which by Proposition 3.9(ii) implies that $g = C_{\tau_A}f \in H_{P,\delta}^\sigma$, and $\|C_{\tau_A}f\|_{H_{P,\delta}^\sigma} = \|f\|_{H_{AP,\delta}^\sigma}$ for all $f \in H_{AP,\delta}^\sigma$, finishing the proof. \square

The next two propositions are important ingredients for proving the main result of this section.

Proposition 3.11. *Let K be a compact subset of $D_{P,\delta}^\sigma \subset \hat{\mathbb{C}}^2$ and $f \in H_{P,\delta}^\sigma = H^2(D_{P,\delta}^\sigma)$. Then, there is a $C_K > 0$ such that*

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{H_{P,\delta}^\sigma}.$$

Proof. For $P = \mathbb{I}$ and $\sigma = (-1, -1)$ (that is, $D_{P,\delta}^\sigma$ a polydisk with radii $(e^{-\delta_1}, e^{-\delta_2})$) the result follows directly from [BJ1, Lemma 2.9]. In particular, with $D' \subset\subset D_{P,\delta}^\sigma$ a domain containing K , and $U(D')$ the space of functions holomorphic on D' and continuous on $\text{cl}(D')$ endowed with the supremum norm, the natural embedding $H_{\mathbb{I},\delta}^{\sigma,-} \hookrightarrow U(D')$ is bounded, and its operator norm provides the constant C_K .

For the general case, let $\psi(z) = z^{-\sigma} = (z_1^{-\sigma_1}, z_2^{-\sigma_2})$. By Proposition 3.10, the operator $C_{\psi \circ \tau_P} : H_{\mathbb{I},\delta}^{\sigma,-} \rightarrow H_{P,\delta}^\sigma$ is an isometric isomorphism, and the result follows by observing that

$$\sup_{z \in K \subset D_{P,\delta}^\sigma} |f(z)| = \sup_{z \in (\psi \circ \tau_P)^{-1}(K)} |f \circ \psi \circ \tau_P(z)|$$

with $(\psi \circ \tau_P)^{-1}(K) \subset\subset D_{\mathbb{I},\delta}^{\sigma,-}$. \square

Definition 3.12 (Dual domains and dual space).

- (i) We denote by $(H_{P,\delta}^\sigma)'$ the dual space of $H_{P,\delta}^\sigma = H^2(D_{P,\delta}^\sigma)$, that is the space of all continuous linear functionals on $H_{P,\delta}^\sigma$.
- (ii) We denote by $(D_{P,\delta}^\sigma)'$ the dual domain of $D_{P,\delta}^\sigma$, which is given by $(D_{P,\delta}^\sigma)' = D_{P,-\delta}^{-\sigma}$ (see the right panel of Figure 2 for an illustration of the case $P = \mathbb{I}$).

Clearly, $((D_{P,\delta}^\sigma)')' = D_{P,\delta}^\sigma$ and $\mathbb{T}_{P,\delta}^\sigma = \mathbb{T}_{P,-\delta}^{-\sigma}$.

Proposition 3.13. $(H_{P,\delta}^\sigma)' = (H^2(D_{P,\delta}^\sigma))'$ is isometrically isomorphic to $H_{P,-\delta}^{-\sigma} = H^2((D_{P,\delta}^\sigma)'),$ via the isomorphism $J: H_{P,-\delta}^{-\sigma} \rightarrow (H_{P,\delta}^\sigma)'$ given by

$$g \mapsto \langle \cdot, g \rangle_{\mathbb{T}_{P,\delta}^\sigma}.$$

Proof. We begin by showing that J is injective. Assume $J(g) = 0$ for some $g \in H_{P,-\delta}^{-\sigma}$, $g(z) = \sum_{n \in \mathbb{Z}_P^{-\sigma}} g_n z^n$, that is, $\langle f, g \rangle_{\mathbb{T}_{P,\delta}^\sigma} = 0$ for all $f \in H_{P,\delta}^\sigma$. In particular, for every $m \in \mathbb{Z}_P^\sigma$ we have

$$0 = \langle p_m, g \rangle_{\mathbb{T}_{P,\delta}^\sigma} = \sum_{n \in \mathbb{Z}_P^{-\sigma}} g_n \langle p_m, p_n \rangle_{\mathbb{T}_{P,\delta}^\sigma} = \sum_{n \in \mathbb{Z}_P^{-\sigma}} g_n e^{\langle m+n, v_{P,\delta}^\sigma \rangle} \delta_{m,-n} = g_{-m}.$$

It follows that $g_n = 0$ for all $n \in \mathbb{Z}_P^{-\sigma}$ and hence $g = 0$, proving injectivity of J .

To show that J is surjective, let $\ell \in (H_{P,\delta}^\sigma)'$. By the Riesz representation theorem, there exists $g \in H_{P,\delta}^\sigma$, so that $\ell = \ell_g = (\cdot, g)_{H_{P,\delta}^\sigma}$, and we can write $g(z) = \sum_{n \in \mathbb{Z}_P^\sigma} g_n z^n$. We define $h(z) = \sum_{n \in \mathbb{Z}_P^{-\sigma}} \overline{g_{-n}} e^{-2\langle n, r \rangle} z^n$ with $r = v_{P,\delta}^\sigma$, and observe that $h \in H^2((D_{P,\delta}^\sigma)') = H_{P,-\delta}^{-\sigma}$. For $z = e^r e^{it} \in \mathbb{T}_{P,\delta}^\sigma$, $t \in [0, 2\pi)^2$, we have $e^{-2r} z = e^{-r} e^{it} = 1/\bar{z}$, and hence

$$\overline{h(z)} = \sum_{n \in \mathbb{Z}_P^{-\sigma}} g_{-n} \overline{e^{-2\langle n, r \rangle} z^n} = \sum_{n \in \mathbb{Z}_P^{-\sigma}} g_{-n} z^{-n} = \sum_{n \in \mathbb{Z}_P^\sigma} g_n z^n = g(z).$$

By (7) this implies

$$(J(h))(f) = \langle f, h \rangle_{\mathbb{T}_{P,\delta}^\sigma} = (f, g)_{H_{P,\delta}^\sigma} = \ell(f),$$

for any $f \in H_{P,\delta}^\sigma$. Hence $J(h) = \ell$, proving surjectivity of J .

Note that the Riesz representation theorem also yields $\|\ell_g\|_{(H_{P,\delta}^\sigma)'} = \|g\|_{H_{P,\delta}^\sigma}$. On the other hand, for $J(h) = \ell_g$ as above, we have

$$\|h\|_{H_{P,-\delta}^{-\sigma}}^2 = \sum_{n \in \mathbb{Z}_P^{-\sigma}} |h_n|^2 |e^{2\langle n, r \rangle}| = \sum_{n \in \mathbb{Z}_P^{-\sigma}} |g_{-n} e^{-2\langle n, r \rangle}|^2 |e^{2\langle n, r \rangle}| = \sum_{n \in \mathbb{Z}_P^\sigma} |g_n|^2 |e^{2\langle n, r \rangle}| = \|g\|_{H_{P,\delta}^\sigma}^2,$$

and so we obtain $\|J(h)\|_{(H_{P,\delta}^\sigma)'} = \|h\|_{H_{P,-\delta}^{-\sigma}}$, proving that J is an isometry. \square

Remark 3.14. Using the above proposition we have the following reformulation: if $f \in H_{P,\delta}^\sigma$ then

$$\|f\|_{H_{P,\delta}^\sigma} = \sup \left\{ \left| \langle f, g \rangle_{\mathbb{T}_{P,\delta}^\sigma} \right| : g \in H^2((D_{P,\delta}^\sigma)') \cap \mathcal{P}, \|g\|_{H^2((D_{P,\delta}^\sigma)')} \leq 1 \right\}, \quad (8)$$

where as before, \mathcal{P} denotes the space of Laurent polynomials.

3.2 Hardy-Hilbert spaces for toral diffeomorphisms

For convenience, here we introduce notation to succinctly express monomial bases for various Hilbert spaces which we will require below.

Notation 3.15. Let $R^{\sigma, \circ}$ be the (closed) polar cone of R^σ for any $\sigma \in \Sigma$. Define

$$\hat{R}^{\sigma, \circ} = \begin{cases} R^{\sigma, \circ} & \text{if } \sigma = (-1, -1), \\ R^{\sigma, \circ} \setminus \{(0, 0)\} & \text{if } \sigma = (+1, +1), \\ \text{int}(R^{\sigma, \circ}) & \text{if } \sigma \in \Sigma^{-1}, \end{cases}$$

and let $\hat{\mathbb{Z}}_P^\sigma = \mathbb{Z}^2 \cap P(\hat{R}^{\sigma, \circ})$, noting that the $\hat{R}^{\sigma, \circ}$, $\sigma \in \Sigma$, form a partition of \mathbb{R}^2 . In analogy to the characterization of $H_{P, \delta}^\sigma$ in Proposition 3.9(ii), for $\sigma \in \Sigma$, $\delta \in \mathbb{R}^2$ and $P \in \text{GL}_2(\mathbb{R})$ we define

$$\hat{H}_{P, \delta}^\sigma := \hat{H}^2(D_{P, \delta}^\sigma) := \{f \in H_{P, \delta}^\sigma : \langle f, p_n \rangle_{H_{P, \delta}^\sigma} = 0 \text{ for } n \notin \hat{\mathbb{Z}}_P^\sigma\}.$$

Writing $e_n = \frac{p_n}{\nu(n)}$ with $\nu(n) = \|p_n\|_{H_{P, \delta}^\sigma} = e^{\langle n, v_{P, \delta}^\sigma \rangle}$, we note that $\{e_n : n \in \hat{\mathbb{Z}}_P^\sigma\}$ forms an orthonormal basis for $\hat{H}_{P, \delta}^\sigma$, which is a Hilbert space with the same norm $\|\cdot\|_{H_{P, \delta}^\sigma}$. Furthermore, for conveniently handling dual spaces, we set $\check{\mathbb{Z}}_P^\sigma = -\hat{\mathbb{Z}}_P^\sigma$, and

$$\check{H}_{P, \delta}^\sigma = \{f \in H^2(D_{P, \delta}^\sigma) : \langle f, p_n \rangle = 0, n \notin \check{\mathbb{Z}}_P^\sigma\}.$$

Remark 3.16. With the above notation, the isomorphism defined in Proposition 3.13 also forms an isometric isomorphism between $\check{H}_{P, -\delta}^{-\sigma}$ and $(\hat{H}_{P, \delta}^\sigma)'$.

Notation 3.17. For $\ell \in \{1, -1\}$, we write

$$\mathcal{D}_{P, \delta}^\ell := \bigcup_{\sigma \in \Sigma^\ell} D_{P, \delta}^\sigma, \quad \text{and} \quad (\mathcal{D}_{P, \delta}^\ell)' := \bigcup_{\sigma \in \Sigma^\ell} (D_{P, \delta}^\sigma)' = \bigcup_{\sigma \in \Sigma^\ell} D_{P, -\delta}^{-\sigma},$$

and note that the distinguished boundary of $\mathcal{D}_{P, \delta}^\ell$ is $\partial^* \mathcal{D}_{P, \delta}^\ell = \bigcup_{\sigma \in \Sigma^\ell} \mathbb{T}_{P, \delta}^\sigma$.

See the left panel of Figure 2 for an illustration in the case $P = \mathbb{I}$: the green and pink rectangles represent $\mathcal{D}_{P, \delta}^1$, whereas the orange and blue rectangles represent $\mathcal{D}_{P, \delta}^{-1}$, with correspondingly coloured stars representing the respective distinguished boundaries.

Definition 3.18. For $\ell \in \{1, -1\}$, define $\mathcal{H}_{P, \delta}^\ell = \bigoplus_{\sigma \in \Sigma^\ell} \hat{H}_{P, \delta}^\sigma$, which is a Hilbert space with the inner product of $f = (f^\sigma)_{\sigma \in \Sigma^\ell}$, $g = (g^\sigma)_{\sigma \in \Sigma^\ell} \in \mathcal{H}_{P, \delta}^\ell$ given by

$$(f, g)_{\mathcal{H}_{P, \delta}^\ell} = \sum_{\sigma \in \Sigma^\ell} (f^\sigma, g^\sigma)_{H_{P, \delta}^\sigma}.$$

As before, for the case $P = \mathbb{I}$ where the domains $D_{P, \delta}^\sigma$ are polydisks, we will use the shorthands $\hat{\mathbb{Z}}^\sigma = \hat{\mathbb{Z}}_{\mathbb{I}}^\sigma$, $D_\delta^\sigma = D_{\mathbb{I}, \delta}^\sigma$, $\mathcal{D}_\delta^\ell = \mathcal{D}_{\mathbb{I}, \delta}^\ell$, $\hat{H}_\delta^\sigma = \hat{H}_{\mathbb{I}, \delta}^\sigma$, and $\mathcal{H}_\delta^\ell = \mathcal{H}_{\mathbb{I}, \delta}^\ell$.

Remark 3.19. Nominally, an $f \in \mathcal{H}_{P, \delta}^\ell$ is a tuple $f = (f^\sigma)_{\sigma \in \Sigma^\ell}$ of two holomorphic functions with distinct domains, $f^\sigma \in \hat{H}_{P, \delta}^\sigma = \hat{H}^2(D_{P, \delta}^\sigma)$. It will be useful to alternatively consider $\mathcal{H}_{P, \delta}^\ell$ as (isomorphic to) a function space, which requires us to distinguish two cases:

- (i) If $\delta \in \mathbb{R}_{<0}^2$, then $\mathcal{A}_{P, \delta} = D_{P, \delta}^\sigma \cap D_{P, \delta}^{-\sigma} \neq \emptyset$, and for any $f = (f^\sigma, f^{-\sigma}) \in \mathcal{H}_{P, \delta}^\ell$ we can define a holomorphic function \tilde{f} on $\mathcal{A}_{P, \delta}$ by $\tilde{f}(z) = f^\sigma(z) + f^{-\sigma}(z)$, yielding an isometric isomorphism between $\mathcal{H}_{P, \delta}^\ell = \hat{H}^2(D_{P, \delta}^\sigma) \oplus \hat{H}^2(D_{P, \delta}^{-\sigma})$ and (a subspace of) $H^2(\mathcal{A}_{P, \delta})$.

- (ii) If $\delta_k \geq 0$ for some $k \in \{1, 2\}$, then $D_{P,\delta}^\sigma \cap D_{P,\delta}^{-\sigma} = \emptyset$, and for any $f = (f^\sigma, f^{-\sigma}) \in \mathcal{H}_{P,\delta}^\ell$ we can define a holomorphic function \tilde{f} on $\mathcal{D}_{P,\delta}^\ell = D_{P,\delta}^\sigma \cup D_{P,\delta}^{-\sigma}$ by $\tilde{f}(z) = f^\sigma(z)$ for $z \in D_{P,\delta}^\sigma$, $\sigma \in \Sigma^\ell$, yielding an isomorphism between $\mathcal{H}_{P,\delta}^\ell$, and (a subspace of) the space of holomorphic functions on $\mathcal{D}_{P,\delta}^\ell$ which extend to square-integrable functions on $\partial^* \mathcal{D}_{P,\delta}^\ell = \partial^* D_{P,\delta}^\sigma \cup \partial^* D_{P,\delta}^{-\sigma}$.

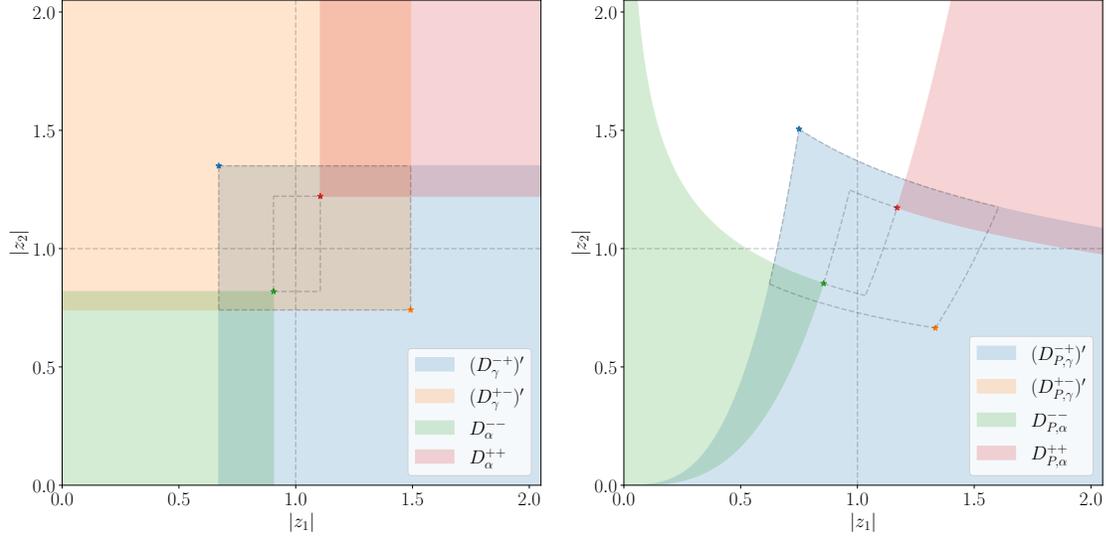


Figure 3: Absolute domains for the Reinhardt domains underlying $H_\nu = H_{P,\alpha,-\gamma}$ with $\alpha = (0.1, 0.2)$, $\gamma = (0.4, 0.3)$, for $P = \mathbb{I}$ (left) and the rotation matrix with rotation angle $\pi/10$ (right).

We can now express the weighted function spaces with respect to quadrant-wise exponential weights considered in Section 2.1 as direct sums of Hardy-Hilbert spaces on certain disks, see Figure 3 (left). For this, let $\alpha, \gamma \in \mathbb{R}^2$, $\nu = \nu_{\alpha,-\gamma}$ the associated quadrant-wise exponential weight (see Definition 2.5) and $H_\nu = H_{\alpha,-\gamma}$ the resulting Hilbert space, given as the completion of \mathcal{P} with respect to the norm $\|\cdot\|_\nu$ (see Definition 2.2). Then

$$\begin{aligned} H_{\alpha,-\gamma} &\cong \mathcal{H}_\alpha^1 \oplus \mathcal{H}_{-\gamma}^{-1} = \hat{H}_\alpha^{--} \oplus \hat{H}_\alpha^{++} \oplus \hat{H}_{-\gamma}^{+-} \oplus \hat{H}_{-\gamma}^{+-} \\ &= \hat{H}^2(D_\alpha^{--}) \oplus \hat{H}^2(D_\alpha^{++}) \oplus \hat{H}^2((D_\gamma^{+-})') \oplus \hat{H}^2((D_\gamma^{+-})') \end{aligned}$$

with the isometric isomorphism given by

$$f = \sum_{n \in \mathbb{Z}^2} f_n p_n \mapsto (f^\sigma)_{\sigma \in \Sigma}, \quad f^\sigma = \sum_{n \in \mathbb{Z}^\sigma} f_n p_n.$$

With the obvious generalization of the weight function $\nu = \nu_{P,\alpha,-\gamma}$ one can also define this for $P \neq \mathbb{I}$, obtaining the more general space $H_\nu = H_{P,\alpha,-\gamma} \cong \mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1}$, see Figure 3 (right).

Remark 3.20. The above isomorphic representation of H_ν reveals an intuitive structure of this Hilbert space. For $\alpha, \gamma \in \mathbb{R}_{>0}^2$, by Remark 3.19, the first part $\mathcal{H}_{P,\alpha}^1$ can be viewed as (isomorphic to) “ $H^2(\mathcal{D}_{P,\alpha}^1)$ ”, the space of functions holomorphic on $\mathcal{D}_{P,\alpha}^1$ extending to square-integrable functions on $\partial^* \mathcal{D}_{P,\alpha}^1$. The second part $\mathcal{H}_{P,-\gamma}^{-1}$, on the other hand, can be seen to be isomorphic to the dual $(\mathcal{H}_{P,\gamma}^{-1})'$ (see Lemma 3.29), with $\mathcal{H}_{P,\gamma}^{-1}$ isomorphic to a space “ $H^2(\mathcal{D}_{P,\gamma}^{-1})$ ”: the space of holomorphic functions on $\mathcal{D}_{P,\gamma}^{-1}$ extending to square-integrable functions on $\partial^* \mathcal{D}_{P,\gamma}^{-1}$.

Notation 3.21. For $f \in \mathcal{H}_{P,\delta}^\ell$ for some $\ell \in \{1, -1\}$, or $f \in \mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1}$, we denote the canonical projection onto one of the components as $\hat{\Pi}^\sigma$ (omitting the dependence on P), given by

$$\hat{\Pi}^\sigma p_n = \begin{cases} p_n & \text{if } n \in \hat{\mathbb{Z}}_P^\sigma, \\ 0 & \text{otherwise,} \end{cases}$$

so that if $f = \sum_{n \in \mathbb{Z}^2} f_n p_n$ (in the sense of Remark 3.19), then $\hat{\Pi}^\sigma f = \sum_{n \in \hat{\mathbb{Z}}_P^\sigma} f_n p_n$, where the operator's domain and range can be inferred from context. Similarly, we write $\check{\Pi}^\sigma f = \sum_{n \in \check{\mathbb{Z}}_P^\sigma} f_n p_n$.

Remark 3.22. With the isometric isomorphism $\Phi: \mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1} \rightarrow H_{P,\alpha,-\gamma}$, we can now use any well-defined operator L on $H_{P,\alpha,-\gamma}$ to define a conjugated operator on the respective space of function tuples. With slight abuse of notation, in such cases we will continue denoting the respective operator on $\mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1}$ by the same symbol L . Furthermore, every toral automorphism τ_Q , $Q \in \text{GL}_2(\mathbb{Z})$, yields an isometric isomorphism $C_{\tau_Q}: H_{QP,\delta}^\sigma \rightarrow H_{P,\delta}^\sigma$ for any $P \in \text{GL}_2(\mathbb{R})$, $\delta \in \mathbb{R}^2$, $\sigma \in \Sigma$ (Proposition 3.10), which can be used to define an operator $\mathcal{H}_{QP,\alpha}^1 \oplus \mathcal{H}_{QP,-\gamma}^{-1} \rightarrow \mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1}$ given by $(f^\sigma)_{\sigma \in \Sigma} \mapsto (\hat{\Pi}^\sigma(f^\sigma \circ \tau_Q))_{\sigma \in \Sigma}$, conjugated to the composition operator $f \mapsto f \circ \tau_Q$ viewed as an operator from $H_{QP,\alpha,-\gamma}$ to $H_{P,\alpha,-\gamma}$. We will refer to all three of these operators as C_{τ_Q} .

The above decomposition of the space H_ν will allow us to prove the main result of this section (Theorem 3.24) for holomorphic maps on the torus satisfying the strongly expanding mapping property from Definition 2.17, by adapting a method previously used in the one-dimensional setting of analytic expanding circle maps $\tau: \mathbb{T} \rightarrow \mathbb{T}$, see [BJS]. To summarize, writing $U_r = \{z \in \mathbb{C}: |z| < r\}$ and $\mathcal{U}_r = U_r \cup (\hat{\mathbb{C}} \setminus \text{cl}(U_{1/r}))$ with $r \in (0, 1)$, analyticity and expansivity of τ imply that there exists $r \in (0, 1)$ such that τ extends holomorphically to a suitable neighbourhood of \mathbb{T} , and $\tau(\partial\mathcal{U}_r) \subset \mathcal{U}_r$. This in turn guarantees compactness of C_τ on $H^2(U_r) \oplus H_0^2(\hat{\mathbb{C}} \setminus \text{cl}(U_{1/r}))$. In the same spirit, for $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ an analytic Anosov map, if T^ℓ is $(\ell, \delta_\ell, \Delta_\ell, P)$ -strongly expanding for $\ell \in \{1, -1\}$ and suitable $\delta_\ell, \Delta_\ell \in \mathbb{R}_{>0}^2$, then

$$T(\partial^* \mathcal{D}_{P,\delta_\ell}^\ell) \subset \mathcal{D}_{P,\Delta_\ell}^\ell \subset \mathcal{D}_{P,\delta_\ell}^\ell,$$

which will be used to prove compactness of C_T on $\mathcal{H}_{\delta_1}^1 \oplus (\mathcal{H}_{\delta_{-1}}^{-1})'$ with similar techniques as in [BJS].

Lemma 3.23. *Let T be a map with $(\ell, \delta, \Delta, P)$ -strongly expanding mapping property, then for every $\sigma, \tilde{\sigma} \in \Sigma^\ell$ there is $\hat{\sigma} \in \Sigma^\ell$ such that*

$$\mathbb{T}_{P,\delta}^{\hat{\sigma}} \subset \text{cl}((D_{P,\delta}^\sigma)') \text{ and } T(\mathbb{T}_{P,\delta}^{\hat{\sigma}}) \subset D_{P,\Delta}^{\tilde{\sigma}} \subset D_{P,\delta}^{\tilde{\sigma}}.$$

Proof. The first property is obvious as $\mathbb{T}_{P,\delta}^{\hat{\sigma}} \subset \text{cl}((D_{P,\delta}^\sigma)')$ for all $\hat{\sigma} \in \Sigma^\ell$. Next, if T satisfies the (EP) property, pick $\hat{\sigma} = \tilde{\sigma}$, whereas if T satisfies (ER) pick $\hat{\sigma} = -\tilde{\sigma}$. \square

Theorem 3.24. *Let T be an analytic diffeomorphism of \mathbb{T}^2 . Further assume that there are $\alpha, \gamma, A, \Gamma, \eta \in \mathbb{R}_{>0}^2$ with $\alpha, \gamma < \eta$, and $P \in \text{GL}_2(\mathbb{R})$, such that T and T^{-1} can be analytically extended to $\mathcal{A}_{P,\eta}$ and the following mapping properties hold:*

- (i) T is $(1, \alpha, A, P)$ -strongly expanding;
- (ii) T^{-1} is $(-1, \Gamma, \gamma, P)$ -strongly expanding;
- (iii) For any $\tilde{\sigma} \in \Sigma^1, \sigma \in \Sigma^{-1}$, there exist $\mathbb{T}_q^2 \subset D_{P,\gamma}^\sigma \cap \mathcal{A}_{P,\eta}$ with $T(\mathbb{T}_q^2) \subset D_{P,A}^{\tilde{\sigma}}$.

Then, the composition operator C_T given by

$$f \mapsto f \circ T$$

maps $H_{P,A,-\Gamma}$ continuously to $H_{P,\alpha,-\gamma}$.

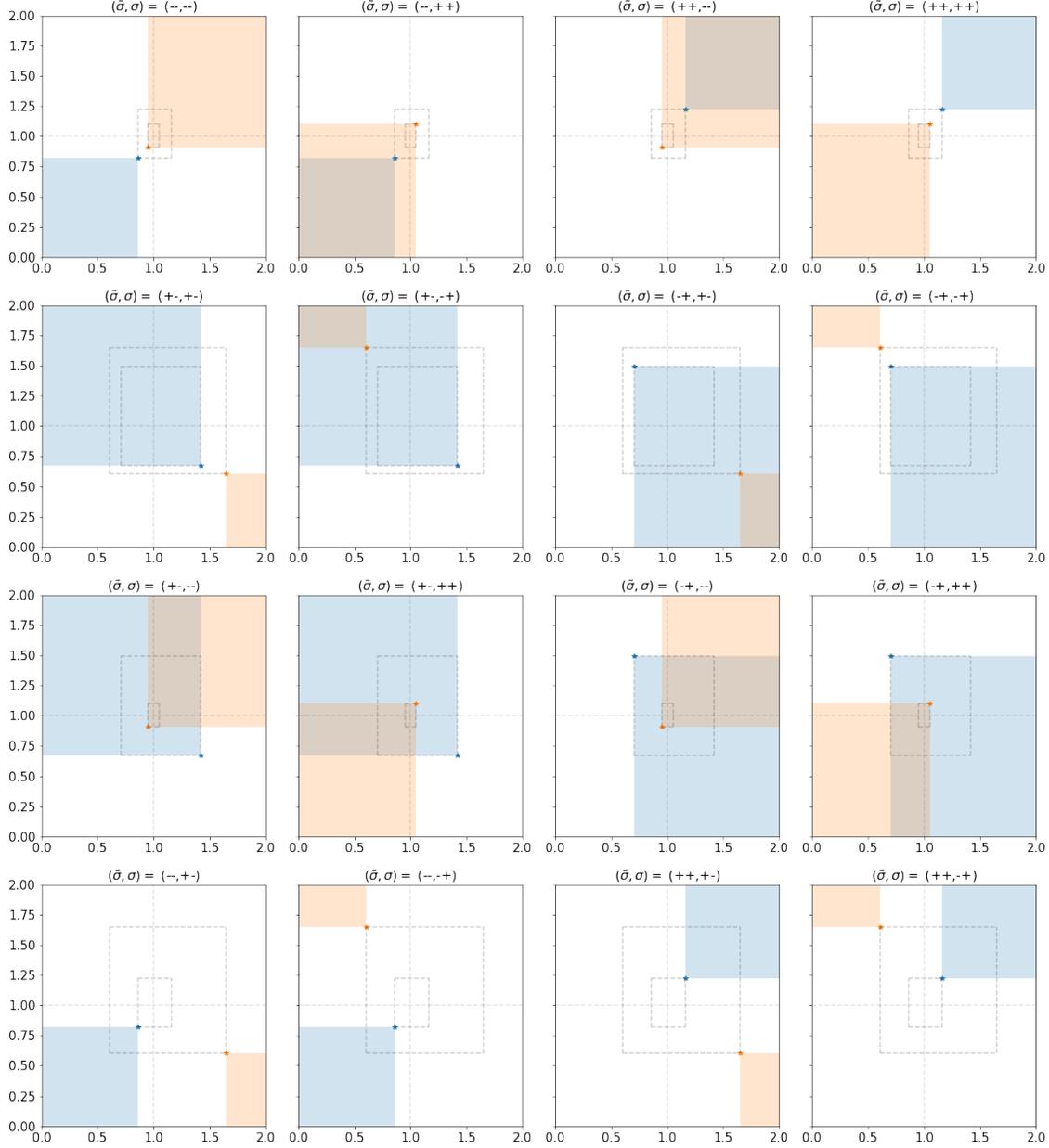


Figure 4: Depiction of the 16 different cases for the absolute domains in the proof of Theorem 3.24. Each row corresponds to $\tilde{\sigma} \in \Sigma^k$ and $\sigma \in \Sigma^l$, for $k, l \in \{1, -1\}$. The blue rectangle represents the domain of holomorphicity for the function f , and the orange rectangle the domain of holomorphicity for the function g . We chose $P = \mathbb{I}$, $\alpha = (0.05, 0.1)$, $A = (0.15, 0.2)$, $\Gamma = (0.35, 0.4)$ and $\gamma = (0.5, 0.5)$.

Proof. Let $\mathcal{S}_{\delta, \delta'} = \{\hat{H}_{\delta}^{\sigma} : \sigma \in \Sigma^1\} \cup \{\hat{H}_{\delta'}^{\sigma} : \sigma \in \Sigma^{-1}\}$ be the collection of four spaces such that $H_{A, -\Gamma} = \bigoplus \mathcal{S}_{A, -\Gamma}$ and $H_{\alpha, -\gamma} = \bigoplus \mathcal{S}_{\alpha, -\gamma}$. By definition of the norm on $H_{A, -\Gamma}$ it is enough to

prove that there is a constant $K > 0$ such that

$$\|\Pi C_T \tilde{\Pi} f\|_H \leq K \|f\|_{\tilde{H}}, \quad f \in \tilde{H}, \quad (9)$$

for any $H \in \mathcal{S}_{\alpha, -\gamma}$ and $\tilde{H} \in \mathcal{S}_{A, -\Gamma}$, where $\Pi: H_{\alpha, -\gamma} \rightarrow H$ and $\tilde{\Pi}: H_{A, -\Gamma} \rightarrow \tilde{H}$ are the respective projection operators. We will denote by D, \tilde{D} the domains of holomorphicity of H and \tilde{H} , respectively, and we will write D' for the dual domain of D . Using (8), for any $f \in \tilde{H}$ we have

$$\|\Pi C_T f\|_H = \sup \{ |\langle C_T f, g \rangle_{\partial^* D}| : g \in H^2(D') \cap \mathcal{P}, \|g\|_{H^2(D')} \leq 1 \}.$$

(Note that this holds for all $H \in \mathcal{S}_{A, -\Gamma}$ without the need to adapt the function space for g .) By the density of Laurent polynomials \mathcal{P} in $H_{A, -\Gamma}$, see Proposition 3.9(iv), it is enough to establish

$$|\langle C_T f, g \rangle_{\partial^* D}| \leq \tilde{C} \|f\|_{\tilde{H}} \|g\|_{H^2(D')}$$

for some $\tilde{C} > 0$, for all $f \in \tilde{H} \cap \mathcal{P}$ and $g \in H^2(D') \cap \mathcal{P}$. We shall prove this by breaking the 16 possible different configurations (see Figure 4) into several cases.

Fix $H \in \mathcal{S}_{\alpha, -\gamma}$ and $\tilde{H} \in \mathcal{S}_{A, -\Gamma}$ (with corresponding domains D and \tilde{D}) and let $g \in H^2(D')$.

- (i) We first consider the case $D = D_{P, \alpha}^\sigma$ with $\sigma \in \Sigma^1$ (so that $\partial^* D = \mathbb{T}_{P, \alpha}^\sigma$) and $\tilde{D} = D_{P, A}^{\tilde{\sigma}}$ with $\tilde{\sigma} \in \Sigma^1$. Since T is $(1, \alpha, A, P)$ -strongly expanding, Lemma 3.23 yields that there is a $\hat{\sigma} \in \Sigma^1$ such that $\mathbb{T}_{P, \alpha}^{\hat{\sigma}} \subset \text{cl}(D')$ and $T(\mathbb{T}_{P, \alpha}^{\hat{\sigma}})$ is a compact subset of \tilde{D} .

We obtain

$$\begin{aligned} |\langle C_T f, g \rangle_{\mathbb{T}_{P, \alpha}^\sigma}| &= \left| \int_{\mathbb{T}_{P, \alpha}^\sigma} (f \circ T) g \, dm \right| = \left| \int_{\mathbb{T}_{P, \alpha}^{\hat{\sigma}}} (f \circ T) g \, dm \right| \\ &\leq \left(\int_{\mathbb{T}_{P, \alpha}^{\hat{\sigma}}} |f \circ T|^2 \, dm \right)^{1/2} \left(\int_{\mathbb{T}_{P, \alpha}^{\hat{\sigma}}} |g|^2 \, dm \right)^{1/2}, \end{aligned}$$

where the integral equality (in the case $\hat{\sigma} \neq \sigma$) follows by Cauchy's Theorem and the holomorphicity of T on $\text{cl}(\mathcal{A}_{P, \alpha})$, and the last step is the Cauchy-Schwarz inequality.

As $T(\mathbb{T}_{P, \alpha}^{\hat{\sigma}}) \subset \tilde{D}$ is compact, Proposition 3.11 yields a $C_1 > 0$ such that $\sup_{z \in T(\mathbb{T}_{P, \alpha}^{\hat{\sigma}})} |f(z)| \leq C_1 \|f\|_{\tilde{H}}$. Since $\mathbb{T}_{P, \alpha}^{\hat{\sigma}} \in \text{cl}(D')$ and $g \in H^2(D')$, we obtain

$$|\langle C_T f, g \rangle_{\mathbb{T}_{P, \alpha}^\sigma}| \leq C_1 \|f\|_{\tilde{H}} \|g\|_{H^2(D')}.$$

- (ii) The case $D = D_{P, \alpha}^\sigma$ with $\sigma \in \Sigma^1$ and $\tilde{D} = (D_{P, \Gamma}^{\tilde{\sigma}})'$ with $\tilde{\sigma} \in \Sigma^{-1}$ is similar to the previous one. In this case, it holds that $T(\mathbb{T}^2) = \mathbb{T}^2 \subset\subset D' \cap \tilde{D}$, and using holomorphicity of T on $\text{cl}(\mathcal{A}_{P, \alpha})$ and the Cauchy-Schwarz inequality, as before we obtain

$$|\langle C_T f, g \rangle_{\mathbb{T}_{P, \alpha}^\sigma}| = \left| \int_{\mathbb{T}^2} (f \circ T) g \, dm \right| \leq \left(\int_{\mathbb{T}^2} |f \circ T|^2 \, dm \right)^{1/2} \left(\int_{\mathbb{T}^2} |g|^2 \, dm \right)^{1/2}.$$

By Proposition 3.11, we again have that $|\langle C_T f, g \rangle_{\mathbb{T}_{P, \alpha}^\sigma}| \leq C_2 \|f\|_{\tilde{H}} \|g\|_{H^2(D')}$ for some $C_2 > 0$.

- (iii) Next we consider the case $D = (D_{P, \gamma}^\sigma)'$, $\sigma \in \Sigma^{-1}$ and $\tilde{D} = D_{P, A}^{\tilde{\sigma}}$, $\tilde{\sigma} \in \Sigma^1$. From assumption (iii) we have that there exists a torus $\mathbb{T}_q^2 \subset D_{P, \gamma}^\sigma \cap \mathcal{A}_{P, \eta}$ such that $T(\mathbb{T}_q^2)$ is a compact subset of \tilde{D} . As in case (i), we obtain

$$|\langle C_T f, g \rangle_{\mathbb{T}_{P, \gamma}^\sigma}| \leq \left(\int_{\mathbb{T}_q^2} |f \circ T|^2 \, dm \right)^{1/2} \left(\int_{\mathbb{T}_q^2} |g|^2 \, dm \right)^{1/2} \leq C_3 \|f\|_{\tilde{H}} \|g\|_{H^2(D')}$$

for some $C_3 > 0$.

(iv) Finally we consider the case $D = (D_{P,\gamma}^\sigma)'$, $\tilde{D} = (D_{P,\Gamma}^{\tilde{\sigma}})'$ with $\sigma, \tilde{\sigma} \in \Sigma^{-1}$. As T and T^{-1} are holomorphic on a neighbourhood of $\text{cl}(\mathcal{A}_{P,\gamma})$, by Cauchy's Theorem we have

$$\begin{aligned} \langle C_T f, g \rangle_{\mathbb{T}_{P,\gamma}^\sigma} &= \int_{\mathbb{T}_{P,\gamma}^\sigma} (f \circ T)g \, dm = \int_{\mathbb{T}^2} (f \circ T)g \, dm = \int_{\mathbb{T}^2} f(g \circ T^{-1})w \, dm \\ &= \int_{\mathbb{T}_{P,\Gamma}^\sigma} f(g \circ T^{-1})w \, dm, \end{aligned}$$

where $w(z) = \omega_T \det(DT^{-1}(z)) \cdot z/T^{-1}(z)$ with $\omega_T = 1$ if T is orientation-preserving and $\omega_T = -1$ otherwise. As T^{-1} is $(-1, \Gamma, \gamma, P)$ -strongly expanding, by Lemma 3.23, there is $\hat{\sigma} \in \Sigma^{-1}$ such that $\mathbb{T}_{\Gamma}^{\hat{\sigma}} \in \text{cl}((D_{P,\Gamma}^{\tilde{\sigma}})') = \text{cl}(\tilde{D})$ and $T^{-1}(\mathbb{T}_{P,\Gamma}^{\hat{\sigma}})$ is a compact set in $D_{P,\gamma}^\sigma = D'$. By the same argument as before we obtain a $C_4 > 0$ such that

$$\begin{aligned} |\langle C_T f, g \rangle_{\mathbb{T}_{P,\gamma}^\sigma}| &\leq \sup_{z \in \mathbb{T}_{P,\Gamma}^{\hat{\sigma}}} |w(z)| \left(\int_{\mathbb{T}_{P,\Gamma}^{\hat{\sigma}}} |f|^2 \, dm \right)^{1/2} \left(\int_{\mathbb{T}_{P,\Gamma}^{\hat{\sigma}}} |g \circ T^{-1}|^2 \, dm \right)^{1/2} \\ &\leq C_4 \|f\|_{\tilde{H}} \|g\|_{H^2(D')}. \end{aligned}$$

Setting $\tilde{C} = \max\{C_1, C_2, C_3, C_4\}$, we obtain the required inequality. \square

Corollary 3.25. *Let T be an analytic Anosov diffeomorphism of \mathbb{T}^2 satisfying the (sec) condition for some $P \in \text{GL}_2(\mathbb{R})$. Then there are $\alpha, A, \gamma, \Gamma \in \mathbb{R}_{>0}^2$ with $\alpha < A$ and $\Gamma < \gamma$ such that the associated composition operator C_T is a well-defined and bounded operator from $H_{P,A,-\Gamma}$ to $H_{P,\alpha,-\gamma}$.*

Remark 3.26. Note that the (sec) condition is sufficient, but not necessary for the above corollary. For example, take $P = \mathbb{I}$ and $\tilde{T}(x, y) = (x + y, x)$, then the union of the first and third quadrants of \mathbb{R}^2 is not an expanding invariant cone, as $D_x \tilde{T}(0, 1)^T = (1, 0)^T$ for all $x \in M$. However, it is not difficult to find $\alpha, A, \gamma, \Gamma$ satisfying the assumptions of Theorem 3.24 (for $P = \mathbb{I}$) for the corresponding map $T: (z_1, z_2) \mapsto (z_1 z_2, z_1)$ on \mathbb{T}^2 .

Using a standard factorisation argument we can now deduce that the composition operator from the above theorem is trace-class when considered as an operator on $H_{A,-\Gamma}$. We defer the proof of the following lemma to the appendix.

Lemma 3.27. *For $\alpha, A, \gamma, \Gamma \in \mathbb{R}_{>0}^2$ with $\alpha < A$ and $\Gamma < \gamma$ let $J: H_{P,\alpha,-\gamma} \rightarrow H_{P,A,-\Gamma}$ be the canonical embedding operator. For $n \in \mathbb{N}$, denote by $s_n(J)$ the n -th singular value of J . Then*

$$\lim_{n \rightarrow \infty} \frac{-\log s_n(J)}{n^{1/2}} = \eta,$$

where $\eta = \left(\frac{1}{\log(A_1 - \alpha_1) \log(A_2 - \alpha_2)} + \frac{1}{\log(\gamma_1 - \Gamma_1) \log(\gamma_2 - \Gamma_2)} \right)^{-1/2}$.

Corollary 3.28. *Let T be as in Theorem 3.24. Then there are $\alpha, \gamma \in \mathbb{R}_{>0}^2$ such that the composition operator C_T associated to T is well-defined as an operator from $H_{P,\alpha,-\gamma}$ to $H_{P,\alpha,-\gamma}$. Moreover, there are constants $\tilde{c}_1, \tilde{c}_2, \hat{c}_1, \hat{c}_2 > 0$ such that*

$$s_n(C_T) \leq \tilde{c}_1 e^{-\tilde{c}_2 n^{1/2}} \quad (n \in \mathbb{N}),$$

and

$$|\lambda_n(C_T)| \leq \hat{c}_1 e^{-\hat{c}_2 n^{1/2}} \quad (n \in \mathbb{N}),$$

where $s_n(C_T)$ and $\lambda_n(C_T)$ are the n -th largest (counted with multiplicity) singular values and eigenvalues of C_T , respectively. In particular, $C_T: H_{P,\alpha,-\gamma} \rightarrow H_{P,\alpha,-\gamma}$ is trace-class.

Proof. By Theorem 3.24 the composition operator lifts to a continuous operator $\tilde{C}_T: H_{P,A,-\Gamma} \rightarrow H_{P,\alpha,-\gamma}$. Let J be the embedding operator from Lemma 3.27, then $C_T = \tilde{C}_T J$ is a well-defined trace-class operator from $H_{P,\alpha,-\gamma}$ to $H_{P,\alpha,-\gamma}$, as \tilde{C}_T is bounded and J is trace-class. By [Pi, 2.2] we have $s_n(C_T) \leq C s_n(J)$ with $C = \|\tilde{C}_T\|_{H_{P,A,-\Gamma} \rightarrow H_{P,\alpha,-\gamma}}$, and by Lemma 3.27 we have $s_n(J) \leq c_1 e^{-\eta n^{1/2}}$ for some $c_1 > 0$. Using the multiplicative Weyl inequality [Pi, 3.5.1] we obtain $|\lambda_n(C_T)| \leq \hat{c}_1 e^{-\hat{c}_2 n^{1/2}}$ with $\hat{c}_1 = C c_1, \hat{c}_2 = 2/3\eta$ (see, for example, [BJ2, Lemma 5.11]). \square

We are now ready to precisely state and prove our first main theorem.

Theorem 1.1. *Let T be an analytic Anosov diffeomorphism of \mathbb{T}^2 satisfying the (sec) condition for some $P \in \text{GL}_2(\mathbb{R})$, and let $w: \mathbb{T}^2 \rightarrow \mathbb{C}$ be analytic. Then there exist $\alpha, \gamma \in \mathbb{R}_{>0}^2$ such that the weighted composition operator*

$$f \mapsto w \cdot f \circ T$$

is a well-defined trace-class operator on $H_{P,\alpha,-\gamma}$.

Proof. Corollary 3.28 proves the theorem for $w \equiv 1$, and remains valid if the operator C_T is replaced by $M_w C_T$, with M_w the multiplication operator with a weight function that is holomorphic on $\mathcal{A}_{P,\eta}$ for some $\eta > \alpha, \gamma$. Since w is analytic on \mathbb{T}^2 , for η sufficiently small we can assume without loss of generality that w holomorphically extends to $\mathcal{A}_{P,\eta}$, proving the general case. \square

3.3 Relation to transfer operator

The following lemma is analogous to Proposition 3.13, replacing the Hilbert space $H_{P,\delta}^\sigma$ by $\mathcal{H}_{P,\delta}^1$, $\mathcal{H}_{P,\delta}^{-1}$ or $\mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1}$, with the respective inner products as in Definition 3.18.

Lemma 3.29.

(i) *For $\ell \in \{1, -1\}$ and $\delta \in \mathbb{R}^2$, the dual $(\mathcal{H}_{P,\delta}^\ell)^\vee = (\bigoplus_{\sigma \in \Sigma^\ell} \hat{H}_{P,\delta}^\sigma)^\vee$ is isometrically isomorphic to $\bigoplus_{\sigma \in \Sigma^\ell} \check{H}_{P,-\delta}^{-\sigma}$ via the isomorphism $\mathcal{J}_{P,\delta}^\ell: \bigoplus_{\sigma \in \Sigma^\ell} \check{H}_{P,-\delta}^{-\sigma} \rightarrow (\mathcal{H}_{P,\delta}^\ell)^\vee$, $g \mapsto l_g$, given by*

$$l_g(f) = \sum_{\sigma \in \Sigma^\ell} \langle \hat{\Pi}^\sigma f, \check{\Pi}^{-\sigma} g \rangle_{\mathbb{T}_{P,\delta}^\sigma}.$$

(ii) *For $\alpha, \gamma \in \mathbb{R}^2$, the dual $(\mathcal{H}_{P,\alpha}^1 \oplus \mathcal{H}_{P,-\gamma}^{-1})^\vee = (\hat{H}_{P,\alpha}^{--} \oplus \hat{H}_{P,\alpha}^{++} \oplus \hat{H}_{P,-\gamma}^{+-} \oplus \hat{H}_{P,-\gamma}^{-+})^\vee$ is isometrically isomorphic to $\check{H}_{P,-\alpha}^{++} \oplus \check{H}_{P,-\alpha}^{--} \oplus \check{H}_{P,\gamma}^{+-} \oplus \check{H}_{P,\gamma}^{-+}$ via the isomorphism $\mathcal{J}_{P,\alpha,\gamma}: g \mapsto l_g$, given by*

$$l_g(f) = \sum_{\sigma \in \Sigma^1} \langle \hat{\Pi}^\sigma f, \check{\Pi}^{-\sigma} g \rangle_{\mathbb{T}_{P,\alpha}^\sigma} + \sum_{\sigma \in \Sigma^{-1}} \langle \hat{\Pi}^\sigma f, \check{\Pi}^{-\sigma} g \rangle_{\mathbb{T}_{P,-\gamma}^\sigma}.$$

Remark 3.30. As before we can identify the Hardy-Hilbert space $H_{P,-\alpha,\gamma}$ associated to a cone-wise exponential weight (similar to Definition 2.5), with a topological direct sum of Hardy-Hilbert spaces on corresponding Reinhardt domains, that is,

$$H_{P,-\alpha,\gamma} \cong \check{H}_{P,-\alpha}^{++} \oplus \check{H}_{P,-\alpha}^{--} \oplus \check{H}_{P,\gamma}^{+-} \oplus \check{H}_{P,\gamma}^{-+}$$

Note that in particular this implies $(H_{P,\alpha,-\gamma})^\vee \cong H_{P,-\alpha,\gamma}$.

Let $\tilde{T}: M \rightarrow M$ be a smooth diffeomorphism of M , then the associated Perron-Frobenius operator $\mathcal{L}_{\tilde{T}}$ given by $g \mapsto |\det D\tilde{T}^{-1}| \cdot (g \circ \tilde{T}^{-1})$ is a well-defined operator on $L^2(M)$. The respective operator on $L^2(\mathbb{T}^2)$ is given by⁷

$$(\mathcal{L}_T g)(z) = w(z) \cdot (g \circ T^{-1})(z), \quad (10)$$

⁷Use the relation $\pi \circ \tilde{T} = T \circ \pi$ with $\pi(x, y) = (\exp(ix), \exp(iy))$. Also observe that $\det D\tilde{T}^{-1}(x, y) = (\det DT^{-1}(z)) \cdot \frac{z_1 z_2}{(T^{-1}(z))_1 (T^{-1}(z))_2} = w(z)$ for $z = (e^{ix}, e^{iy})$.

with $w(z) = \omega_T \cdot (\det DT^{-1}(z)) \cdot \frac{p_{1,1}(z)}{p_{1,1}(T^{-1}(z))}$, where $p_{1,1}(z) = z_1 \cdot z_2$ and $\omega_T = 1$ if T is orientation-preserving and $\omega_T = -1$ otherwise.

Proposition 3.31. *Let T satisfy the assumptions of Theorem 3.24 and let C_T be the respective operator on a suitable space $H_{P,\alpha,-\gamma}$. Then the isomorphism $\mathcal{J} = \mathcal{J}_{P,\alpha,\gamma}$ from Lemma 3.29 conjugates the adjoint $(C_T)^*$ of C_T to the operator \mathcal{L}_T given by (10), which is well defined and bounded as an operator on $H_{P,-\alpha,\gamma}$.*

Proof. For notational simplicity, we assume $P = \mathbb{I}$, the proof for general $P \in \text{GL}_2(\mathbb{R})$ being identical. We want to show that $C_T^* \mathcal{J} = \mathcal{J} \mathcal{L}_T$. By the density of Laurent polynomials in $H_{P,-\alpha,\gamma}$ and $H_{P,\alpha,-\gamma}$, it suffices to show this for monomials, i.e. $(C_T^* \mathcal{J}(p_n))(p_m) = (\mathcal{J} \mathcal{L}_T(p_n))(p_m)$ for all $n, m \in \mathbb{Z}^2$. For any $n, m \in \mathbb{Z}^2$ we have that

$$(C_T^* \mathcal{J}(p_n))(p_m) = (\mathcal{J}(p_n))(C_T p_m) = \sum_{\sigma \in \Sigma} \langle \hat{\Pi}^\sigma(C_T p_m), \check{\Pi}^{-\sigma} p_n \rangle_{\mathbb{T}_{\delta(\sigma)}^\sigma}$$

with $\delta(\sigma) = \alpha$ for $\sigma \in \Sigma^1$, and $\delta(\sigma) = -\gamma$ otherwise. We note that $\mathbb{Z}^2 = \bigcup_{\sigma \in \Sigma} \check{\mathbb{Z}}^\sigma$ is a disjoint union, so that for every $n \in \mathbb{Z}^2$ there exists exactly one $\sigma' \in \Sigma$ such that $\check{\Pi}^{-\sigma'} p_n = p_n$ and $\check{\Pi}^{-\sigma} p_n = 0$ for all $\sigma \neq \sigma'$. Moreover we have $\langle \hat{\Pi}^\sigma f, \check{\Pi}^{\sigma'} g \rangle_{\mathbb{T}_\delta^\sigma} = 0$ whenever $\sigma' \neq -\sigma$. It follows that

$$\sum_{\sigma \in \Sigma} \langle \hat{\Pi}^\sigma(C_T p_m), \check{\Pi}^{-\sigma} p_n \rangle_{\mathbb{T}_{\delta(\sigma)}^\sigma} = \langle \hat{\Pi}^{\sigma'}(C_T p_m), \check{\Pi}^{-\sigma'} p_n \rangle_{\mathbb{T}_{\delta(\sigma')}^{\sigma'}} = \langle C_T p_m, p_n \rangle_{\mathbb{T}_{\delta(\sigma')}^{\sigma'}},$$

where the second step follows from $\sum_{\sigma \in \Sigma} \hat{\Pi}^\sigma(C_T p_m) = C_T p_m$. Analogously,

$$(\mathcal{J} \mathcal{L}_T(p_n))(p_m) = \sum_{\sigma \in \Sigma} \langle \hat{\Pi}^\sigma p_m, \check{\Pi}^{-\sigma}(\mathcal{L}_T(p_n)) \rangle_{\mathbb{T}_{\delta(\sigma)}^\sigma} = \langle p_m, \mathcal{L}_T(p_n) \rangle_{\mathbb{T}_{\delta(\sigma'')}^{\sigma''}},$$

with suitable σ'' . Finally, we have that

$$\begin{aligned} \langle C_T p_m, p_n \rangle_{\mathbb{T}_{\delta(\sigma')}^{\sigma'}} &= \int_{\mathbb{T}_{\delta(\sigma')}^{\sigma'}} (p_m \circ T) p_n \, dm = \int_{\mathbb{T}^2} (p_m \circ T) p_n \, dm \\ &= \int_{\mathbb{T}^2} p_m(p_n \circ T^{-1}) w \, dm = \int_{\mathbb{T}_{\delta(\sigma'')}^{\sigma''}} p_m(p_n \circ T^{-1}) w \, dm = \langle p_m, \mathcal{L}_T(p_n) \rangle_{\mathbb{T}_{\delta(\sigma'')}^{\sigma''}}, \end{aligned}$$

where $w(z) = \omega_T \det(DT^{-1}(z)) \cdot z/T^{-1}(z)$ and we have used that the integrands are holomorphic on a neighbourhood of $\text{cl}(A_\gamma)$. Combining the above, we obtain the claim of the proposition. \square

Remark 3.32. Using Theorem 1.1 with the weight function being (the complex version of) the determinant of DT gives rise to a transfer operator corresponding to the map T^{-1} , which is well defined and trace-class on a suitable $H_{P,\alpha,-\gamma}$. Now, using the previous proposition, it follows that the operator $f \mapsto f \circ T^{-1}$ is well-defined and trace-class on $H_{P,-\alpha,\gamma}$.

4 Resonances for certain rational Anosov maps

This section is devoted to proving Theorem 1.2, that is, determining the explicit form of eigenvalues of composition operators associated to analytic maps with holomorphic extensions to certain domains of $\hat{\mathbb{C}}^2$. In order to capture all the intricacies of the resonances in this result, we will use a fundamental result by Rudin and Stout [Rud, Theorem 5.2.5] characterising inner functions on polydisks. An inner function on \mathbb{D}^n is a function $f \in H^\infty(\mathbb{D}^n)$ whose radial boundary values satisfy $|f^*(z)| = 1$ almost everywhere⁸ on $\mathbb{T}^n = \partial\mathbb{D}^n$. We denote by $U(\mathbb{D}^n)$ the class of all continuous complex functions on $\text{cl}(\mathbb{D}^n)$ whose restriction to \mathbb{D}^n is holomorphic.

⁸ Knese [Kn, Cor. 14.6] proved that the radial limit exists and is unimodular at every point $z \in \mathbb{T}^n$. However, in general f^* need not be continuous on \mathbb{T}^n .

Theorem 4.1. *Every rational inner function f on \mathbb{D}^n , $n \in \mathbb{N}$, has the form*

$$f(z) = \frac{M(z)\tilde{Q}(1/z)}{Q(z)}, \quad (11)$$

where M is a monomial, Q a polynomial with no zeros in \mathbb{D}^n , and \tilde{Q} is the polynomial whose coefficients are the complex conjugates of the coefficients of Q . Moreover, every function $f \in U(\mathbb{D}^n)$ which is inner is rational, and in this case Q has no zeros in $\text{cl}(\mathbb{D}^n)$.

A direct consequence of this theorem is a characterization of the form that analytic Anosov diffeomorphisms on \mathbb{T}^2 from Theorem 1.2 can take, namely that each component is necessarily a rational function satisfying a certain set of properties. For $k, l \in \{0, 1\}$, we denote by $I_{kl}: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ the map

$$I_{kl}(z_1, z_2) = (z_1^{1-2k}, z_2^{1-2l}),$$

noting that $I_{kl} = \tau_{I\sigma}$ with $\sigma = (1 - 2k, 1 - 2l)$.

Corollary 4.2. *Let T be an analytic diffeomorphism of \mathbb{T}^2 with holomorphic extension to a neighbourhood of \mathbb{T}^2 . Assume there exist $\sigma, \sigma' \in \Sigma$ so that T holomorphically extends to D^σ with $T(D^\sigma) \subset D^{\sigma'}$. Then each component of T can be written as a rational function. Moreover, writing $T = T_A$ with $A = (a_1, \dots, a_n)$ the collection of all coefficients occurring in T (in any order), and denoting $\bar{A} = (\bar{a}_1, \dots, \bar{a}_n)$, we have the following properties:*

(i) $I_{11} \circ T_A \circ I_{11} = T_{\bar{A}}$,

(ii) $T_{\bar{A}}(\bar{z}) = \overline{T_A(z)}$ for any $z \in \hat{\mathbb{C}}^2$.

Proof. We define $\phi^\sigma: \mathbb{D}^2 \rightarrow D^\sigma$ by $\phi^\sigma(z) = z^{-\sigma} = (z_1^{-\sigma_1}, z_2^{-\sigma_2})$, then each component of $\hat{T} = (\phi^{\sigma'})^{-1} \circ T \circ \phi^\sigma: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is a rational inner function in \mathbb{D}^2 continuous on all of $\text{cl}(\mathbb{D}^2)$. By Theorem 4.1, each component is a rational function of the form (11), and this property is preserved under composition with ϕ^σ . Furthermore, property (i) holds for maps whose components are of the form (11). Since I_{11} and ϕ^σ commute for any $\sigma \in \Sigma$, we have that

$$I_{11} \circ T_A \circ I_{11} = \phi^{\sigma'} \circ I_{11} \circ \hat{T}_A \circ I_{11} \circ (\phi^\sigma)^{-1} = \phi^{\sigma'} \circ \hat{T}_{\bar{A}} \circ (\phi^\sigma)^{-1} = T_{\bar{A}},$$

proving that property (i) is preserved under composition with ϕ^σ . Lastly, (ii) holds for all maps whose components are rational functions, and hence for the given map $T = T_A$. \square

We shall require the following lemma, a direct consequence of the maximum modulus principle.

Lemma 4.3. *Fix $\sigma, \sigma' \in \Sigma$, $a \in \mathbb{R}_{>0}^2$, and let $T: D^\sigma \rightarrow D^{\sigma'}$ be holomorphic with $T(\mathbb{T}_a^\sigma) \subset D_a^{\sigma'}$. Then $T(D_a^\sigma) \subset D_a^{\sigma'}$.*

Proof. We write $\hat{\sigma} = (-1, -1)$, and begin with the case $\sigma = \sigma' = \hat{\sigma}$ (i.e. $D^\sigma = D^{\sigma'} = \mathbb{D}^2$), whose proof is a direct application of the (multivariate) maximum modulus principle. By compactness, there exists $z^* = (z_1^*, z_2^*) \in \text{cl}(D_a^\sigma)$ such that $|T_1(z^*)| = \max_{z \in \text{cl}(D_a^\sigma)} |T_1(z)|$. Defining $f: \mathbb{D} \rightarrow \mathbb{D}$ by $f(z_1) = T_1(z_1, z_2^*)$, the maximum modulus principle implies that

$$|T_1(z^*)| = \max_{|z_1| \leq e^{-a_1}} |f(z_1)| = \max_{|z_1| = e^{-a_1}} |f(z_1)|,$$

and hence without loss of generality we can assume $|z_1^*| = e^{-a_1}$. Analogously, z_2^* can be assumed to have modulus e^{-a_2} . It follows that $\max_{z \in \text{cl}(D_a^\sigma)} |T_1(z)| = \max_{z \in \mathbb{T}_a^\sigma} |T_1(z)| < e^{-a_1}$ (using $T(\mathbb{T}_a^\sigma) \subset D_a^{\sigma'}$). Repeating the argument for T_2 yields $\max_{z \in \text{cl}(D_a^\sigma)} |T_2(z)| < e^{-a_2}$, and hence $T(D_a^\sigma) \subset D_a^{\sigma'}$.

For general $\sigma, \sigma' \in \Sigma$, let $\phi^\sigma: \mathbb{D}^2 \rightarrow D^\sigma$ be given by $\phi^\sigma(z) = z^{-\sigma} = (z_1^{-\sigma_1}, z_2^{-\sigma_2})$, so that $\phi(D_a^{\hat{\sigma}}) = D_a^\sigma$ and $\phi(\mathbb{T}_a^{\hat{\sigma}}) = \mathbb{T}_a^\sigma$. Then $\tilde{T} = (\phi^{\sigma'})^{-1} \circ T \circ \phi^\sigma: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is holomorphic and satisfies $\tilde{T}(\mathbb{T}_a^{\hat{\sigma}}) \subset D_a^{\hat{\sigma}}$. It follows that $\tilde{T}(D_a^{\hat{\sigma}}) \subset D_a^{\hat{\sigma}}$, and the assertion $T(D_a^\sigma) \subset D_a^{\sigma'}$ follows. \square

The explicit form of the spectral determinant in Theorem 1.2 will follow by computing traces of certain trace-class operators. If $L: H \rightarrow H$ is a trace-class operator on a separable Hilbert space $(H, (\cdot, \cdot))$ and $\{e_n\}_{n \in \mathcal{I}}$ is an orthonormal basis of H with some index set \mathcal{I} , then the trace of L is

$$\mathrm{Tr}(L) = \sum_{n \in \mathcal{I}} (Le_n, e_n), \quad (12)$$

and its determinant is given by

$$\det(\mathrm{Id} - zL) = \exp\left(-\sum_{k=1}^{\infty} \frac{z^k}{k} \mathrm{Tr}(L^k)\right), \quad (13)$$

for all $z \in \mathbb{C}$ in a suitable neighbourhood of 0. Moreover, both Tr and \det are spectral, that is, $\mathrm{Tr}(L) = \sum_{k=1}^{\infty} \lambda_k(L)$, and counting multiplicities, the zeros of the entire function $z \mapsto \det(\mathrm{Id} - zL)$ are precisely the reciprocals of the eigenvalues $\lambda_k(L)$ of L (see [Pi, 4.62, 4.7.14, 4.7.15]).

Notation 4.4. The multipliers at a fixed point $z^* \in \hat{\mathbb{C}}^2$ of a rational map $T: \hat{\mathbb{C}}^2 \rightarrow \hat{\mathbb{C}}^2$ are given by the eigenvalues of $DT(z^*)$ if $z^* \in \mathbb{C}^2$, and by the eigenvalues of $D\hat{T}(I_{kl}(z^*))$ with $(k, l) = (1, 1), (1, 0)$ or $(0, 1)$ for $z^* = (\infty, \infty), (\infty, w)$ or (w, ∞) respectively, where $w \in \mathbb{C}$ and $\hat{T} = I_{kl} \circ T \circ I_{kl}$.

Lemma 4.5. For $\sigma \in \Sigma$ and $\delta \in \mathbb{R}^2$ let $\varphi: D_\delta^\sigma \rightarrow D_\delta^\sigma$ be a holomorphic map such that $\varphi(D_\delta^\sigma) \subset D_\delta^\sigma$. Let C_φ denote the corresponding composition operator and M_w the multiplication operator with w a holomorphic function on a neighbourhood of $\mathrm{cl}(D_\delta^\sigma)$. Then $M_w C_\varphi$ is trace-class on H_δ^σ and

$$\mathrm{Tr}((M_w C_\varphi)^k) = \frac{w(z^*)^k}{\det(\mathbb{I} - D\varphi^k(z^*))} = \frac{w(z^*)^k}{(1 - \mu_1^k)(1 - \mu_2^k)},$$

with μ_1, μ_2 the (not necessarily distinct) multipliers at the unique attracting fixed point $z^* \in D_\delta^\sigma$ of φ .

Proof. Let $\hat{\sigma} = (-1, -1)$ and consider the case $\sigma = \hat{\sigma}$. As $H_\delta^{\hat{\sigma}}$ is a ‘favourable Hilbert space’ (see [BJ1, Definition 2.7]), the result follows by [BJ1, Proposition 2.10 and Theorem 4.2].

For general $\sigma \in \Sigma$, D_δ^σ is biholomorphically equivalent to $D_\delta^{\hat{\sigma}}$ under the map $\phi^\sigma: D_\delta^{\hat{\sigma}} \rightarrow D_\delta^\sigma$ given by $\phi^\sigma(z) = z^{-\sigma}$, and $\hat{\varphi} = (\phi^\sigma)^{-1} \circ \varphi \circ \phi^\sigma: D_\delta^{\hat{\sigma}} \rightarrow D_\delta^{\hat{\sigma}}$ satisfies $\hat{\varphi}(D_\delta^{\hat{\sigma}}) \subset D_\delta^{\hat{\sigma}}$. Defining $\hat{w} = w \circ \phi^\sigma$ on a neighbourhood of $\mathrm{cl}(D_\delta^{\hat{\sigma}})$, the first case implies the statement of the lemma for the operator $M_{\hat{w}} C_{\hat{\varphi}}$ on $H_\delta^{\hat{\sigma}}$. By Proposition 3.10 the operator $C_{\phi^\sigma}: H_\delta^\sigma \rightarrow H_\delta^{\hat{\sigma}}$ is an isometric isomorphism, which conjugates $M_w C_\varphi$ to $M_{\hat{w}} C_{\hat{\varphi}}$. The statement for $M_w C_\varphi$ follows, as the multipliers of $z^* \in D_\delta^\sigma$ for φ coincide with those of the unique attracting fixed point $(\phi^\sigma)^{-1}(z^*) \in D_\delta^{\hat{\sigma}}$ for $\hat{\varphi}$. \square

Lemma 4.6. Let T be a smooth diffeomorphism of \mathbb{T}^2 , and let $\omega_T = 1$ if T is orientation-preserving, and $\omega_T = -1$ otherwise. Let $r, s \in \{0, 1\}$, and write $\hat{T} = I_{rs} \circ T^{-1} \circ I_{rs}$ and $\hat{w} = \omega_T \det D\hat{T}$. Then for any $n \in \mathbb{Z}^2$ and $k \in \mathbb{N}$ we have

$$(C_T^k p_n, p_n)_{L^2(\mathbb{T}^2)} = ((M_{\hat{w}} C_{\hat{T}})^k p_m, p_m)_{L^2(\mathbb{T}^2)} \text{ with } m = ((-1)^{1-r} n_1 - 1, (-1)^{1-s} n_2 - 1).$$

Proof. All the steps follow by change of variables. For any $n \in \mathbb{Z}^2$, we have

$$\begin{aligned} (C_T^k p_n, p_n)_{L^2(\mathbb{T}^2)} &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} p_n(T^k(z)) p_{-n}(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{\omega_T^k}{(2\pi)^2} \int_{\mathbb{T}^2} p_n(w) p_{-n}(T^{-k}(w)) \det DT^{-k}(w) \frac{dw_1}{(T^{-k}(w))_1} \frac{dw_2}{(T^{-k}(w))_2} \\ &= \frac{\omega_T^k}{(2\pi)^2} \int_{\mathbb{T}^2} p_{n+1}(w) p_{-(n+1)}(T^{-k}(w)) \det DT^{-k}(w) \frac{dw_1}{w_1} \frac{dw_2}{w_2}, \end{aligned}$$

using the shorthand $n + 1 = (n_1 + 1, n_2 + 1)$. Next, observe that

$$\det D(T^{-k} \circ I_{rs})(z) = \det D(I_{rs} \circ \hat{T}^k)(z) = (-1)^{r+s} p_{-2r, -2s}(\hat{T}^k(z)) \cdot \det D\hat{T}^k(z).$$

Substituting $w = I_{rs}(z)$ and using that I_{rs} is orientation-preserving iff $r + s$ is even, we obtain

$$\begin{aligned} (C_T^k p_n, p_n)_{L^2(\mathbb{T}^2)} &= \frac{(-1)^{r+s} \omega_T^k}{(2\pi)^2} \int_{\mathbb{T}^2} p_{n+1}(I_{rs}(z)) p_{-(n+1)}(T^{-k}(I_{rs}(z))) \det D(T^{-k} \circ I_{rs})(z) \frac{dz_1}{z_1^{1-2r}} \frac{dz_2}{z_2^{1-2s}}, \\ &= \frac{\omega_T^k}{(2\pi)^2} \int_{\mathbb{T}^2} p_{-m}(z) p_m(\hat{T}^k(z)) \det D\hat{T}^k(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= ((\omega_T^k \det D\hat{T}^k) \cdot C_{\hat{T}}^k p_m, p_m)_{L^2(\mathbb{T}^2)}, \end{aligned}$$

with $m = (-(-1)^r n_1 - 1, -(-1)^s n_2 - 1)$, as claimed. \square

Remark 4.7. If $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ has an analytic extension to a neighbourhood of $\text{cl}(\mathcal{A}_\delta)$ for some δ , then one can check that $(C_T p_n, p_n)_{L^2(\mathbb{T}^2)} = \langle C_T e_n, \bar{e}_n \rangle_{\mathbb{T}_\delta^\sigma} = \langle C_T e_n, e_n \rangle_{H_\delta^\sigma}$ for any $n \in \mathbb{Z}^2, \sigma \in \Sigma$.

We recall that $\mathcal{H}_\delta^1 = \hat{H}_\delta^{--} \oplus \hat{H}_\delta^{++}$ and $\mathcal{H}_\delta^{-1} = \hat{H}_\delta^{+-} \oplus \hat{H}_\delta^{-+}$, and that $\{e_n : n \in \hat{\mathbb{Z}}^\sigma\}$ forms an orthonormal basis for \hat{H}_δ^σ for $\sigma \in \Sigma$.

Lemma 4.8. *Let $\delta \in \mathbb{R}^2$ and $T: \hat{\mathbb{C}}^2 \rightarrow \hat{\mathbb{C}}^2$ be holomorphic on a neighbourhood of $\text{cl}(\mathcal{A}_\delta)$. Assume additionally that T is holomorphic on D_δ^σ and $T(D_\delta^\sigma) \subset D_\delta^{-\sigma}$ for every $\sigma \in \Sigma^1$. Let w be a holomorphic function on a neighbourhood of $\text{cl}(\mathcal{A}_\delta)$. Then $M_w C_T$ is a well-defined and trace-class operator on \mathcal{H}_δ^1 with trace $\text{Tr}(M_w C_T) = \sum_{\sigma \in \Sigma^1} \sum_{n \in \hat{\mathbb{Z}}^\sigma} \langle M_w C_T e_n, e_n \rangle_{\mathbb{T}_\delta^\sigma}$. Moreover,*

(i) if $w \equiv 1$, then $\text{Tr}(M_w C_T) = \text{Tr}(C_T) = 1$,

(ii) if $w = \det DT$, then $\text{Tr}(M_w C_T) = 0$.

Proof. We define $\tilde{T} = I_{11} \circ T$, so that \tilde{T} is holomorphic on a neighbourhood of $\text{cl}(\mathcal{A}_\delta)$, and moreover, for every $\sigma \in \Sigma^1$ it is holomorphic on D_δ^σ with $\tilde{T}(D_\delta^\sigma) \subset D_\delta^\sigma$. By Lemma 4.5 the composition operator $C_{\tilde{T}}$ is trace-class on \hat{H}_δ^σ , and hence C_T is well-defined and trace-class viewed as an operator on \mathcal{H}_δ^1 . Since M_w is well-defined and bounded as an operator on \mathcal{H}_δ^1 , it follows that $M_w C_T$ is also trace-class. Since $\bar{e}_n(z) = p_{-n}(z) \nu(n)$ for $z \in \mathbb{T}_\delta^\sigma$ and $n \in \mathbb{Z}^2$, we have that

$$(M_w C_T e_n, e_n)_{\hat{H}_\delta^\sigma} = \langle M_w C_T e_n, \bar{e}_n \rangle_{\mathbb{T}_\delta^\sigma} = \langle w \cdot p_n \circ T, p_{-n} \rangle_{\mathbb{T}_\delta^\sigma} = \langle w \cdot p_{-n} \circ \tilde{T}, p_{-n} \rangle_{\mathbb{T}_\delta^\sigma}.$$

Moreover, $p_{-n} \circ \tilde{T} \in H_\delta^\sigma$ for any $\sigma \in \Sigma^1$ and $-n \in \hat{\mathbb{Z}}^\sigma$, that is, $p_{-n} \circ \tilde{T} = \sum_{m \in \mathbb{Z}^\sigma} c_m p_m$. For $w \equiv 1$, recalling that $\langle p_k, p_l \rangle_{\mathbb{T}_\delta^\sigma} = 0$ whenever $k \neq -l$, and noting $(-\hat{\mathbb{Z}}^\sigma) \cap \mathbb{Z}^\sigma \subset \{(0, 0)\}$, it follows that

$$(M_w C_T e_n, e_n)_{\hat{H}_\delta^\sigma} = \delta_{n,0} \cdot \langle p_0 \circ \tilde{T}, p_0 \rangle_{\mathbb{T}_\delta^\sigma} = \delta_{n,0},$$

which yields $\text{Tr}(C_T) = 1$.

For $w = w_T = \det DT$, it is easy to see that $w = \sum_{m \in \mathbb{Z}_{<0}^2} w_m p_m$ for suitable $w_m \in \mathbb{C}$. We consider first the case $n \in \hat{\mathbb{Z}}^{--}$. In this case we have $p_n \circ T = \sum_{m \in \mathbb{Z}_{\leq 0}^2} c_m p_m$, and hence $w \cdot p_n \circ T = \sum_{m \in \mathbb{Z}_{<0}^2} d_m p_m$, for suitable coefficients $c_m, d_m \in \mathbb{C}$. It follows that $\langle w_T \cdot p_n \circ T, p_{-n} \rangle_{\mathbb{T}_\delta^\sigma} = 0$. For $n \in \hat{\mathbb{Z}}^{++}$, let $\tilde{T} = I_{11} \circ T \circ I_{11}$. Direct calculation using a change of variables $y = I_{11}(z)$ yields

$$\langle w_T \cdot p_n \circ T, p_{-n} \rangle_{\mathbb{T}_\delta^\sigma} = \langle w_{\tilde{T}} \cdot p_{-(n+2)} \circ \tilde{T}, p_{n+2} \rangle_{\mathbb{T}_\delta^\sigma}.$$

Noting that $-(n+2) \in \hat{\mathbb{Z}}^{--}$ and that \tilde{T} and $w_{\tilde{T}}$ also satisfy the assumptions of the lemma, we can apply the previous case and obtain that again $\langle w_T \cdot p_n \circ T, p_{-n} \rangle_{\mathbb{T}_\delta^\sigma} = 0$. The conclusion $\text{Tr}(M_w C_T) = 0$ follows. \square

Lemma 4.9. *Let $\ell \in \{\pm 1\}$, $\delta \in \mathbb{R}^2$, and let T be an analytic diffeomorphism of \mathbb{T}^2 , holomorphic on a neighbourhood of $\text{cl}(\mathcal{A}_\delta)$. Assume additionally that T extends holomorphically to D_δ^σ with $T(D_\delta^\sigma) \subset D_\delta^{-\sigma}$ for every $\sigma \in \Sigma^\ell$. Then for any $\sigma \in \Sigma^\ell$, $T \circ T$ has a unique fixed point $z^\sigma \in D_\delta^\sigma$, and*

$$D(T \circ T)(z^\sigma) = \overline{D\hat{T}(z^\sigma)} D\hat{T}(z^\sigma),$$

where $\hat{T} = I_{11} \circ T$.

Proof. We first note that $(T \circ T)(D_\delta^\sigma) \subset D_\delta^\sigma$ for $\sigma \in \Sigma^\ell$, implying the existence of a unique fixed point $z^\sigma \in D_\delta^\sigma$ for $\sigma \in \Sigma^\ell$. The map $\hat{T} = I_{11} \circ T$ also satisfies $\hat{T}(D_\delta^\sigma) \subset D_\delta^\sigma$ for $\sigma \in \Sigma^\ell$, so by Corollary 4.2 each component of \hat{T} is a rational function, and we write $\hat{T} = \hat{T}_A$ with $A = (a_1, \dots, a_n)$ the collection of all coefficients occurring in T (in any order). Using Corollary 4.2(i) we obtain

$$T \circ T = I_{11} \circ \hat{T}_A \circ I_{11} \circ \hat{T}_A = \hat{T}_{\bar{A}} \circ \hat{T}_A.$$

Next, observe that on the one hand, we have

$$(\hat{T}_A \circ \hat{T}_{\bar{A}})(\hat{T}_A(z^\sigma)) = \hat{T}_A(\hat{T}_{\bar{A}} \circ \hat{T}_A(z^\sigma)) = \hat{T}_A(z^\sigma),$$

and on the other hand, using Corollary 4.2(ii), we have

$$(\hat{T}_A \circ \hat{T}_{\bar{A}})(\overline{z^\sigma}) = \hat{T}_A(\overline{\hat{T}_A(z^\sigma)}) = \overline{(\hat{T}_{\bar{A}} \circ \hat{T}_A)(z^\sigma)} = \overline{z^\sigma},$$

so that $\hat{T}_A(z^\sigma) = \overline{z^\sigma}$. It follows that

$$D(T \circ T)(z^\sigma) = D(\hat{T}_{\bar{A}} \circ \hat{T}_A)(z^\sigma) = D\hat{T}_{\bar{A}}(\overline{z^\sigma}) D\hat{T}_A(z^\sigma) = \overline{D\hat{T}_A(z^\sigma)} D\hat{T}_A(z^\sigma). \quad \square.$$

Remark 4.10. The above lemma implies that the two multipliers of a fixed point z^σ of $T \circ T$ are either real or complex conjugates of each other, as

$$\det(D(T \circ T)(z^\sigma)) = \overline{\det(D\hat{T}_A(z^\sigma))} \cdot \det(D\hat{T}_A(z^\sigma)) = |\det(D\hat{T}_A(z^\sigma))|^2$$

and $\text{Tr}(D(T \circ T)(z^\sigma)) = \text{Tr}(\overline{D\hat{T}_A(z^\sigma)} D\hat{T}_A(z^\sigma)) \in \mathbb{R}$. In contrast to the one-dimensional setting of anti-Blaschke products [BN], examples of orientation-reversing circle maps allowing for explicit determination of resonances, the multipliers are no longer necessarily real. Note also that under the assumptions of the lemma for $\ell = 1$ or $\ell = -1$ the two attracting fixed points of $T \circ T$ in $\mathcal{D}_\delta^\ell(z^\sigma, \sigma \in \Sigma^\ell)$ have identical sets of multipliers.

We are now ready to prove our second main theorem.

Proof of Theorem 1.2. By Theorem 1.1, C_T is trace-class on $H_{\alpha, -\gamma}$ for suitable $\alpha, \gamma \in \mathbb{R}_{>0}^2$, and its trace is given by $\text{Tr} C_T = \sum_{n \in \mathbb{Z}^2} \langle C_T e_n, e_n \rangle_{\nu_{\alpha, -\gamma}}$. Using the isometric isomorphism between $H_{\alpha, -\gamma}$ and $\mathcal{H}_\alpha^1 \oplus \mathcal{H}_{-\gamma}^{-1}$ and the fact that $C_T^k = C_{T^k}$, for every $k \in \mathbb{N}$ we have

$$\text{Tr}(C_T^k) = \sum_{\sigma \in \Sigma^1} \sum_{n \in \hat{\mathbb{Z}}^\sigma} \langle C_T^k e_n, e_n \rangle_{\hat{H}_\alpha^\sigma} + \sum_{\sigma \in \Sigma^{-1}} \sum_{n \in \hat{\mathbb{Z}}^{-\sigma}} \langle C_T^k e_n, e_n \rangle_{\hat{H}_{-\gamma}^{-\sigma}} =: S_1(k) + S_{-1}(k), \quad (14)$$

as well as

$$\log \det(\text{Id} - z C_T) = - \sum_{k=1}^{\infty} \frac{z^k}{k} S_1(k) - \sum_{k=1}^{\infty} \frac{z^k}{k} S_{-1}(k). \quad (15)$$

We note that the assumptions and Lemma 4.3 imply that for every $\ell \in \{\pm 1\}$ and $\sigma \in \Sigma^\ell$, T^ℓ is holomorphic in a neighbourhood of $\text{cl}(D_\delta^\sigma)$ and $T^\ell(D_\delta^\sigma) \subset D_\delta^{\pm\sigma}$, where $\delta = \alpha$ for $\ell = 1$ and

$\delta = -\gamma$ for $\ell = -1$. We will calculate (15) by handling the two sums $S_\ell, \ell \in \{\pm 1\}$, separately, considering for each the two possible cases $T^\ell(D_\delta^\sigma) \subseteq D_\delta^\sigma$ and $T^\ell(D_\delta^\sigma) \subseteq D_\delta^{-\sigma}$ for all $\sigma \in \Sigma^\ell$. The claim of the theorem will follow with $(1-z)\chi_T^1(z) = \exp(-\sum_{k=1}^\infty \frac{z^k}{k} S_1(k))$ and $\chi_T^{-1}(z) = \exp(-\sum_{k=1}^\infty \frac{z^k}{k} S_{-1}(k))$. We first calculate $S_1(k)$:

- (1) Consider first the case $T(D_\alpha^\sigma) \subset D_\alpha^\sigma$ for $\sigma \in \Sigma^1$. For $\sigma = (-1, -1) \in \Sigma^1$, the composition operator \tilde{C}_T associated to T on $H_\alpha^\sigma = \hat{H}_\alpha^\sigma$ is trace-class, and its trace, computed using (12), coincides with the term in (14) corresponding to σ (note that $\tilde{C}_T^k = \tilde{C}_{T^k}$). By Lemma 4.5, we obtain the value $((1-\lambda_1^k)(1-\lambda_2^k))^{-1}$, where λ_1, λ_2 are the multipliers of the unique fixed point $z^\sigma \in D_\alpha^\sigma$. Similarly, for $\sigma = (+1, +1) \in \Sigma^1$, the associated composition operator \tilde{C}_T is trace-class on H_α^σ . Writing $T = T_A$ for some $A \in \mathbb{C}^m$, $m \in \mathbb{N}_0$, by Corollary 4.2(i) we have $T_A = I_{11} \circ T_{\bar{A}} \circ I_{11}$, where $T_{\bar{A}}$ is a holomorphic map on $D_\alpha^{-\sigma}$ with unique attracting fixed point \bar{z}^σ (see Corollary 4.2(ii)). Moreover, it follows that $DT_{\bar{A}}(z^\sigma) = \overline{DT_A(z^\sigma)}$, thus by Lemma 4.5 we have $\text{Tr } \tilde{C}_T^k = ((1-\bar{\lambda}_1^k)(1-\bar{\lambda}_2^k))^{-1}$. Since

$$\text{Tr } \tilde{C}_T^k = (\tilde{C}_T^k e_0, e_0)_{H_\alpha^\sigma} + \sum_{n \in \mathbb{Z}^{++}} (\tilde{C}_T^k e_n, e_n)_{H_\alpha^\sigma} = 1 + \sum_{n \in \mathbb{Z}^{++}} (C_T^k e_n, e_n)_{\hat{H}_\alpha^\sigma},$$

we obtain

$$S_1(k) = 1 + D(\lambda_1^k, \lambda_2^k) + D(\bar{\lambda}_1^k, \bar{\lambda}_2^k),$$

where $D(a, b) := \frac{1}{(1-a)(1-b)} - 1 = \sum_{n \in \mathcal{N}^1} a^{n_1} b^{n_2}$ for $a, b \in \mathbb{D}$, $\mathcal{N}^1 = \mathbb{N}_0^2 \setminus \{(0, 0)\}$. Calculating

$$-\sum_{k=1}^\infty \frac{z^k}{k} D(a^k, b^k) = -\sum_{n \in \mathcal{N}^1} \sum_{k=1}^\infty \frac{z^k a^{kn_1} b^{kn_2}}{k} = \sum_{n \in \mathcal{N}^1} \log(1 - za^{n_1} b^{n_2}),$$

we finally obtain

$$-\sum_{k=1}^\infty \frac{z^k}{k} S_1(k) = \log(1-z) + \sum_{\sigma \in \Sigma^1} \sum_{n \in \mathcal{N}^1} \log(1 - z\lambda_\sigma^n).$$

- (2) Next we consider the case $T(D_\alpha^\sigma) \subset D_\alpha^{-\sigma}$ for $\sigma \in \Sigma^1$. For $k \in \mathbb{N}$ odd, T^k satisfies the assumptions of Lemma 4.8, and the trace of the composition operator associated to T^k on \mathcal{H}_α^1 exactly corresponds to the first sum in (14), yielding $S_1(k) = 1$. For k even, T^k satisfies $T^k(D_\alpha^\sigma) \subset D_\alpha^\sigma$ for $\sigma \in \Sigma^1$, and so we can apply case (1). Moreover, by Remark 4.10 in this case the fixed point multipliers λ_1, λ_2 of T^2 are either real or complex conjugates of each other, and hence

$$S_1(k) = \begin{cases} 1 & \text{for } k \text{ odd,} \\ 1 + 2D(\lambda_1^{k/2}, \lambda_2^{k/2}) & \text{for } k \text{ even.} \end{cases}$$

A straightforward calculation using the fact that $\lambda_\sigma = \lambda_{-\sigma}$ now yields

$$-\sum_{k=1}^\infty \frac{z^k}{k} S_1(k) = \log(1-z) + \frac{1}{2} \sum_{\sigma \in \Sigma^1} \sum_{n \in \mathcal{N}^1} \log(1 - z^2 \lambda_\sigma^n).$$

Next, we proceed to calculate $S_{-1}(k)$. The approach to calculating $S_1(k)$ does not immediately translate to this case, as the bidisks $D_\delta^\sigma, \sigma \in \Sigma^{-1}$, are not invariant under the map T , and so do not directly give rise to trace-class composition operators on the respective spaces H_δ^σ . Instead, we will show that the sums in $S_{-1}(k)$ correspond to the traces of certain weighted composition

operators $M_{\hat{w}}C_{\hat{T}}$ on H_{δ}^{-} , where \hat{T} will be a map conjugated to T^{-1} from Lemma 4.6. Combining Remark 4.7 with Lemma 4.6, we calculate for $\sigma = (-1, +1)$ that

$$\sum_{n \in \hat{\mathbb{Z}}^{-\sigma}} (C_T^k e_n, e_n)_{\hat{H}_{-\gamma}^{-\sigma}} = \sum_{n \in \hat{\mathbb{Z}}^{-\sigma}} (C_T^k p_n, p_n)_{L^2(\mathbb{T}^2)} = \sum_{n \in \hat{\mathbb{Z}}^{-\sigma}} ((M_{\hat{w}_\sigma} C_{\hat{T}_\sigma})^k p_n, p_n)_{L^2(\mathbb{T}^2)},$$

with $\hat{w}_\sigma = \omega_T \det D\hat{T}_\sigma$ and $\hat{T}_\sigma = I_{01} \circ T^{-1} \circ I_{01}$. For $\sigma = (+1, -1)$, Lemma 4.6 yields the exact same equality with $\hat{T}_\sigma = I_{10} \circ T^{-1} \circ I_{10}$. We now consider two cases again.

(1') If $T^{-1}(D_\gamma^\sigma) \subset D_\gamma^\sigma$ for $\sigma \in \Sigma^{-1}$, then $\hat{T}_\sigma(D_\gamma^{-\sigma}) \subset D_\gamma^{-\sigma}$. We can then apply the same argument as in the above case (1), using that $\hat{w}_\sigma(\zeta^*) = \omega_T \mu_1 \mu_2$ for μ_1, μ_2 the multipliers of the unique fixed point $\zeta^* \in D_\gamma^{-\sigma}$ of \hat{T}_σ , which corresponds to the unique fixed point $z^\sigma \in D_\gamma^\sigma$ of T^{-1} . By the same argument as before, the multipliers of the respective fixed points in D_γ^σ and $D_\gamma^{-\sigma}$ are complex conjugates of each other, and we obtain

$$S_{-1}(k) = \frac{(\omega_T \mu_1 \mu_2)^k}{(1 - \mu_1^k)(1 - \mu_2^k)} + \frac{(\omega_T \overline{\mu_1 \mu_2})^k}{(1 - \overline{\mu_1^k})(1 - \overline{\mu_2^k})} = (\omega_T)^k \sum_{n \in \mathcal{N}^{-1}} ((\mu_1^{n_1} \mu_2^{n_1})^k + (\overline{\mu_1^{n_1} \mu_2^{n_2}})^k),$$

where $\mathcal{N}^{-1} = \mathbb{N}^2$. A similar calculation to above yields

$$-\sum_{k=1}^{\infty} \frac{z^k}{k} S_{-1}(k) = \sum_{\sigma \in \Sigma^{-1}} \sum_{n \in \mathcal{N}^{-1}} \log(1 - z \omega_T \lambda_\sigma^n).$$

(2') Finally, we consider the case $T^{-1}(D_\gamma^\sigma) \subset D_\gamma^{-\sigma}$ for $\sigma \in \Sigma^{-1}$, which implies $\hat{T}_\sigma(D_\gamma^\sigma) \subset D_\gamma^{-\sigma}$ for $\sigma \in \Sigma^1$. If $k \in \mathbb{N}$ is odd, we can apply Lemma 4.8 to \hat{T}_σ^k and the weight function \hat{w}_σ . The trace of $(M_{\hat{w}_\sigma} C_{\hat{T}_\sigma})^k$ on \mathcal{H}_γ^1 in the lemma exactly coincides with $S_{-1}(k)$, yielding $S_{-1}(k) = 0$. For k even, we can apply case (1') to T^{-2} instead of T^{-1} , again using the fact that the fixed point multipliers μ_1, μ_2 are either both real or complex conjugates of each other, which yields

$$S_{-1}(k) = \begin{cases} 0 & \text{for } k \text{ odd,} \\ \frac{2(\mu_1 \mu_2)^{k/2}}{(1 - \mu_1^{k/2})(1 - \mu_2^{k/2})} & \text{for } k \text{ even,} \end{cases}$$

and again using $\lambda_\sigma = \lambda_{-\sigma}$ we obtain

$$-\sum_{k=1}^{\infty} \frac{z^k}{k} S_{-1}(k) = \frac{1}{2} \sum_{\sigma \in \Sigma^{-1}} \sum_{n \in \mathcal{N}^{-1}} \log(1 - z^2 \lambda_\sigma^n).$$

Claims (i) and (ii) of the theorem now follow by combining the cases (1)-(2) and (1')-(2') for $\ell = 1$ and $\ell = -1$, respectively. \square

Using the explicit form of the zeros of $\det(\text{Id} - zC_T)$ obtained in Theorem 1.2, we can calculate the decay rate of their reciprocals, the eigenvalues of C_T . For a map T satisfying the assumptions of Theorem 1.2, for any $\ell \in \{\pm 1\}$ and $\sigma \in \Sigma^\ell$, we denote by $\lambda_\sigma = (\lambda_{\sigma,1}, \lambda_{\sigma,2})$ the multipliers of the unique attracting fixed point in D^σ of T^ℓ if $T^\ell(D^\sigma) \subseteq D^\sigma$, and of $T^{2\ell}$ otherwise.

Corollary 4.11. *Let T satisfy the assumptions of Theorem 1.2, and let $\lambda = \lambda_{(-1,-1)}$ and $\mu = \lambda_{(-1,+1)}$. Let $\omega_T = 1$ if T is orientation-preserving⁹, and $\omega_T = -1$ otherwise. Then the nonzero*

⁹ T is orientation-preserving exactly if either both, or neither of T and T^{-1} satisfy the case (i) in Theorem 1.2.

eigenvalues of C_T on $H_{\alpha, -\gamma}$ are $\{1\} \cup \mathcal{E}_1 \cup \mathcal{E}_{-1}$, where

$$\mathcal{E}_1 = \begin{cases} \{\lambda^n, \bar{\lambda}^n : n \in \mathcal{N}^1\}, & \text{if } T(D^\sigma) \subseteq D^\sigma, \sigma \in \Sigma^1, \\ \{\pm \lambda^{n/2} : n \in \mathcal{N}^1\}, & \text{if } T(D^\sigma) \subseteq D^{-\sigma}, \sigma \in \Sigma^1, \end{cases}$$

$$\mathcal{E}_{-1} = \begin{cases} \{\omega_T \cdot \mu^n, \omega_T \cdot \bar{\mu}^n : n \in \mathcal{N}^{-1}\}, & \text{if } T^{-1}(D^\sigma) \subseteq D^\sigma, \sigma \in \Sigma^{-1}, \\ \{\pm \mu^{n/2} : n \in \mathcal{N}^{-1}\}, & \text{if } T^{-1}(D^\sigma) \subseteq D^{-\sigma}, \sigma \in \Sigma^{-1}. \end{cases}$$

Moreover, the algebraic multiplicity of each nonzero eigenvalue is exactly the number of its occurrences in the above sets.

Corollary 4.12. *Let T satisfy the assumptions of Theorem 1.2, $(\lambda_n)_{n \in \mathbb{N}}$ be the sequence of eigenvalues of C_T sorted in order of decreasing modulus, and $N_T(r) = \#\{n \in \mathbb{N} : |\lambda_n| \geq r\}$. Then*

$$\lim_{r \rightarrow 0} \frac{\log N_T(r)}{\log |\log r|} = d,$$

where

(i) if $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0$ for some $\sigma \in \Sigma$, then $d = 2$ (stretched-exponential decay), and

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n^{1/2}} = \eta_2$$

$$\text{with } \eta_2 = \left(\frac{1}{2} \sum_{\sigma \in \Sigma: \lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0} (\log |\lambda_{\sigma,1}| \cdot \log |\lambda_{\sigma,2}|)^{-1} \right)^{-1/2}.$$

(ii) if $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} = 0$ for all $\sigma \in \Sigma$, and $\lambda_{\sigma,k} \neq 0$ for some $\sigma \in \Sigma^1$ and $k \in \{1, 2\}$, then $d = 1$ (exponential decay), and

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n} = \eta_1,$$

$$\text{with } \eta_1 = \left(\sum_{\sigma \in \Sigma^1} \sum_{k: \lambda_{\sigma,k} \neq 0} (\log |\lambda_{\sigma,k}|)^{-1} \right)^{-1}.$$

(iii) if $\lambda_\sigma = 0$ for all $\sigma \in \Sigma^1$, and $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} = 0$ for all $\sigma \in \Sigma^{-1}$, then $d = 0$ (super-exponential decay). In this case the set of eigenvalues of C_T is trivial, and $\text{spec}(C_T) = \{0, 1\}$.

Proof. This follows directly from Lemma A.3 applied to the eigenvalues of C_T written as the values of a cone-wise exponential function $f: \mathbb{Z}^2 \rightarrow \mathbb{C}$. In the case when both T and T^{-1} satisfy the case (i) in Theorem 1.2, this function is given by $f(n) = \lambda_\sigma^{|n|}$ with $\sigma = \sigma(n) \in \Sigma$ such that either $n \in \mathbb{Z}^2 \cap R^\sigma$, $\sigma \in \Sigma^1$, or $n \in (\mathbb{Z} \setminus \{0\})^2 \cap R^\sigma$, $\sigma \in \Sigma^{-1}$. The other cases are similar. \square

5 Anosov maps with different decay rates for resonances

Based on the results of the previous section, in this section we shall prove our last main result, Theorem 1.3. The proof will use the classical result that every toral Anosov diffeomorphism is homotopic to a toral automorphism, and exploit the algebraic structure of $\text{GL}_2(\mathbb{Z})$, which is isomorphic to the group of toral automorphisms $\text{Aut}(\mathbb{T}^2)$. To explicitly construct diffeomorphisms whose corresponding composition operators have resonances exhibiting a desired decay rate, we shall introduce a special group \mathcal{F} of toral diffeomorphisms, containing the automorphisms as a subgroup. The extension will consist of diffeomorphisms each of which is homotopic to an automorphism in an explicit way.

Beyond its immediate usefulness for our proof, the group \mathcal{F} provides a rich source of explicit examples of toral diffeomorphisms whose resonances can often be explicitly computed, and which includes both area-preserving and non-area-preserving, orientation-preserving and -reversing examples, as well as examples satisfying various symmetries. As we believe this might be of broader interest, we include in Appendix B a more comprehensive discussion and illustrative set of examples, while constraining ourselves to a minimal introduction in this section.

5.1 A special group of toral diffeomorphisms

The group $\text{Aut}(\mathbb{T}^2)$ of linear diffeomorphisms of \mathbb{T}^2 is isomorphic to $\text{GL}_2(\mathbb{Z})$, with any $A = (a_{ij}) \in \text{GL}_2(\mathbb{Z})$ giving rise to the toral automorphism $\tau_A(z_1, z_2) = (z_1^{a_{11}} z_2^{a_{12}}, z_1^{a_{21}} z_2^{a_{22}})$. For our purposes it will be convenient to view $\text{Aut}(\mathbb{T}^2)$ as generated by the following automorphisms, which can also be viewed as rational maps of $\hat{\mathbb{C}}^2$:

- (i) a map F given by $F(z_1, z_2) = (z_1 z_2, z_2)$, with $F^{-1}(z_1, z_2) = (z_1/z_2, z_2)$,
- (ii) an involution R given by $R(z_1, z_2) = (z_2, z_1)$,
- (iii) involutions I_{kl} for $k, l \in \{0, 1\}$ given by $I_{kl}(z_1, z_2) = (z_1^{1-2k}, z_2^{1-2l})$.

The set $\Gamma = \{F, R, I_{01}\}$ generates $\text{Aut}(\mathbb{T}^2)$. To create a richer group of toral diffeomorphisms, we extend the above by a continuous family of maps. Utilising automorphisms of \mathbb{D} , the so-called Moebius maps $b_a: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}, a \in \mathbb{D}$, given by

$$b_a(z) = \frac{z - a}{1 - \bar{a}z},$$

we define the additional set of generators as

- (iv) a family $\mathcal{G} = \{G_{a,b} : a, b \in \mathbb{D}\}$ of maps given by

$$G_{a,b}(z_1, z_2) = (b_a(z_1), b_b(z_2)),$$

$$\text{satisfying } G_{a,b}^{-1} = G_{-a,-b}.$$

Definition 5.1. Denote by \mathcal{F} the group of diffeomorphisms of \mathbb{T}^2 generated by the set $\Gamma \cup \mathcal{G}$.

Remark 5.2. The proof of Theorem 1.3 will be based on Theorem 1.2, in particular we will require all constructed maps T to analytically extend to a neighbourhood of \mathbb{T}^2 , and to satisfy that

$$T^\ell \text{ extends holomorphically to } D^\sigma \text{ and } T^\ell(D^\sigma) \subseteq D^{\pm\sigma} \quad (\sigma \in \Sigma^\ell, \ell \in \{\pm 1\}). \quad (16)$$

A convenient class of maps satisfying (16) is the semigroup of finite compositions of $\{F, R, I_{11}\} \cup \mathcal{G}$.

We remark that the choice of generators \mathcal{G} is not the only possible, though arguably the simplest choice of non-linear maps satisfying (16). More generally, this property is satisfied by a certain class of rational inner skew products, see [ST], taking the form

$$(z_1, z_2) \mapsto \left(e^{i\theta} \frac{\tilde{p}(z_1, z_2)}{p(z_1, z_2)}, z_2 \right),$$

with $\theta \in \mathbb{R}$ and p a polynomial of bidegree $(1, k)$ for $k \in \mathbb{N}_0$, that is, of degree 1 in z_1 and degree k in z_2 . Here, \tilde{p} is the reflection of p defined as $\tilde{p}(z_1, z_2) = z_1 z_2^k p(1/\bar{z}_1, 1/\bar{z}_2)$. The map $G_{a,0}(z) = (b_a(z_1), z_2)$ corresponds to the polynomial $p(z) = 1 - \bar{a}z_1$ of bidegree $(1, 0)$, and a general $G_{a,b}$ can be written as $G_{a,b} = G_{a,0} \circ R \circ G_{b,0}$.

5.2 Explicit homotopies of Anosov diffeomorphisms

We proceed by stating an algebraic fact about conjugacy classes of $\mathrm{GL}_2(\mathbb{Z})$, which will allow us to establish an analytic conjugacy between an arbitrary hyperbolic automorphism of \mathbb{T}^2 and an element of \mathcal{F} . While our proof is based on results from [He], for variants of this result see, e.g., [Ka, BR] and references therein. We defer the proof of the lemma to Appendix A.

Lemma 5.3. *Every hyperbolic matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ is similar (in $\mathrm{GL}_2(\mathbb{Z})$) to a matrix of the form*

$$\pm \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_n & 1 \\ 1 & 0 \end{pmatrix}, \quad k_1, \dots, k_n \in \mathbb{N}, n \geq 1. \quad (17)$$

Corollary 5.4. *Every hyperbolic automorphism of \mathbb{T}^2 is conjugated via an (analytic) toral automorphism to an automorphism of the form*

$$I_{11}^s \circ (F^{k_1} \circ R) \circ (F^{k_2} \circ R) \circ \cdots \circ (F^{k_n} \circ R), \quad k_1, \dots, k_n \in \mathbb{N}, n \geq 1, s \in \{0, 1\}. \quad (18)$$

Proof. Let τ_A be a hyperbolic automorphism of \mathbb{T}^2 associated to a hyperbolic matrix $A \in \mathrm{GL}_2(\mathbb{Z})$. By the previous lemma, A is similar to a matrix B of the form (17), that is, there exists $Q \in \mathrm{GL}_2(\mathbb{Z})$, such that $A = Q^{-1}BQ$ with B decomposing into a product of matrices

$$-\mathbb{I} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k \in \mathbb{N}.$$

We note that for every $k \in \mathbb{N}$, the latter is equal to $(M_F)^k M_R$, where M_F, M_R correspond to the toral automorphisms F and R (i.e., $\tau_{M_F} = F$ and $\tau_{M_R} = R$), and $\tau_{-\mathbb{I}} = I_{11}$. It follows that τ_B has the desired form, and $\tau_A = \tau_Q^{-1} \circ \tau_B \circ \tau_Q$. \square

Next, for any hyperbolic automorphism of the form (18) we construct non-linear area-preserving Anosov diffeomorphisms from \mathcal{F} in the same homotopy class with easily computable resonances. For this, we derive from the linear map $F^k \circ R, k \in \mathbb{N}$, the one-parameter family of non-linear toral diffeomorphisms

$$U_{k,a}(z) = (G_{0,a} \circ F^k \circ R \circ G_{-a,0})(z) = (b_{-a}(z_1)^k z_2, z_1), \quad a \in \mathbb{D}.$$

Applying this to (18), we define

$$\Psi_{K,A} = I_{11}^s \circ U_{k_1,a_1} \circ \cdots \circ U_{k_n,a_n}, \quad (19)$$

where $s \in \{0, 1\}, n \in \mathbb{N}, A = (a_1, \dots, a_n) \in \mathbb{D}^n, K = (k_1, \dots, k_n) \in \mathbb{N}^n$. We write $A_o = (a_1, a_3, \dots)$ and $A_e = (a_2, a_4, \dots)$ for the respective tuples only involving odd or even indices (analogously for K_o, K_e). For convenience, we shall use the multiindex notation $A^K := a_1^{k_1} \cdots a_n^{k_n}$ for arbitrary n -tuples A and $K, n \in \mathbb{N}_0$, with the convention $A^K = 1$ when A and K are of length 0.

Proposition 5.5 (Area-preserving maps homotopic to (18)). *For $s \in \{0, 1\}, A = (a_1, \dots, a_n) \in \mathbb{D}^n$ and $K = (k_1, \dots, k_n) \in \mathbb{N}^n, n \in \mathbb{N}$, the map $\Psi_{K,A}$ is an area-preserving hyperbolic toral diffeomorphism satisfying the conclusions of Theorem 1.2. In particular, $\Psi_{K,A}$ satisfies Theorem 1.2(i) if $s = 0$, and (ii) if $s = 1$, and $\Psi_{K,A}^{-1}$ satisfies (i) if $n + s$ is even, and (ii) if $n + s$ is odd. Moreover, denoting λ_σ the multipliers of the unique attracting fixed point of $\Psi_{K,A}^\ell$ in D^σ for $\sigma \in \Sigma^\ell, \ell \in \{\pm 1\}$ if $\Psi_{K,A}^\ell(D^\sigma) \subset D^\sigma$ and of $\Psi_{K,A}^{2\ell}$ otherwise, we have:*

(i) *If $s = 0$ and n is odd, then*

$$\lambda_{--} = \overline{\lambda_{++}} = ((A^K)^{1/2}, -(A^K)^{1/2}) \quad \text{and} \quad \lambda_{+-} = \lambda_{-+} = (A_o^{K_o} \overline{A_e^{K_e}}, \overline{A_o^{K_o}} A_e^{K_e}).$$

(ii) If $s = 0$ and n is even, then

$$\lambda_{--} = \overline{\lambda_{++}} = (A_o^{K_o}, A_e^{K_e}) \quad \text{and} \quad \lambda_{-+} = \overline{\lambda_{+-}} = (\overline{A_o^{K_o}}, \overline{A_e^{K_e}}).$$

(iii) If $s = 1$ and n is odd, then

$$\lambda_{--} = \lambda_{++} = (\overline{A_o^{K_o}} A_e^{K_e}, A_o^{K_o} \overline{A_e^{K_e}}) \quad \text{and} \quad \lambda_{-+} = \overline{\lambda_{+-}} = ((A_o^{K_o} \overline{A_e^{K_e}})^{1/2}, -(A_o^{K_o} \overline{A_e^{K_e}})^{1/2}).$$

(iv) If $s = 1$ and n is even, then

$$\lambda_\sigma = (|A_o^{K_o}|^2, |A_e^{K_e}|^2) \quad \text{for all } \sigma \in \Sigma.$$

Proof. We begin by showing that $C_{\Psi_{K,A}}$ is trace-class on a suitable Hilbert space $H_{\alpha,-\gamma}$. In the case $n > 1$ this will follow from Theorem 1.2 by proving that $\Psi_{K,A}$ satisfies the (p -sec) condition, while the case $n = 1$ will be handled separately.

For any map $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ and $M = ([0, 2\pi]/\sim)^2$, we denote by $\tilde{T}: M \rightarrow M$ the map determined by $\pi \circ \tilde{T} = T \circ \pi$ with $\pi(x) = e^{ix}$ for $x \in M$, and analogously for maps on \mathbb{T} and $([0, 2\pi]/\sim)$. By Lemma B.1 in the appendix, we have $\tilde{b}_{-a}(e^{i\theta}) = \theta + g_{-a}(\theta)$ with $g'_{-a}(\theta) > -1$ for all $a \in \mathbb{D}$, and $\tilde{U}_{k,a}(x_1, x_2) = (k(x_1 + g_{-a}(x_1)) + x_2, x_1)$. Thus we obtain

$$D\tilde{U}_{k,a}(x) = \begin{pmatrix} s_{k,a}(x) & 1 \\ 1 & 0 \end{pmatrix}, \quad (20)$$

where $s_{k,a}(x) = k(1 + g'_{-a}(x_1)) > 0$ for all $x = (x_1, x_2) \in M$.

Consider first the case $\Psi_{K,A} = I_{11}^s \circ U_{k_1, a_1} \circ \cdots \circ U_{k_n, a_n}$ with $s = 0$ and $n > 1$. Then $D\tilde{\Psi}_{K,A}$ is positive and hence $(D\tilde{\Psi}_{K,A}(x))(\mathbb{R}_{\geq 0}^2) \subset \mathbb{R}_{> 0}^2 \cup \{0\}$ for every $x \in M$. Using (20) we obtain

$$D\tilde{\Psi}_{K,A}(x) = M_x + \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } n \text{ is even,} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } n \text{ is odd,} \end{cases} \quad (21)$$

with $M_x \geq 0$. By the criterion in Remark 2.13, this implies that the first half of the (p -sec) condition (6) is satisfied. Since $\det D\tilde{\Psi}_{K,A}(x) = (-1)^n$, we also have that $D\tilde{\Psi}_{K,A}^{-1}(x) = \begin{pmatrix} 1 + a_x & -b_x \\ -c_x & 1 + d_x \end{pmatrix}$

if n is even, and $D\tilde{\Psi}_{K,A}^{-1}(x) = \begin{pmatrix} -a_x & 1 + b_x \\ 1 + c_x & -d_x \end{pmatrix}$ for n odd, with $a_x, b_x, c_x, d_x > 0$. It is easy to

see that these are conjugated to matrices of the form (21) via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which via the criterion in Remark 2.13 implies the second half of the (p -sec) condition (6). The case of $s = 1$ follows immediately, since condition (6) holds for a map \tilde{T} if and only if it holds for $-\tilde{T}$, finishing the proof of the (p -sec) condition for $\Psi_{K,A}$ with $n > 1$.

In the case $n = 1$ the (p -sec) condition does not hold, however one can verify that the assumptions of Theorem 3.24 still apply (similar to the case in Remark 3.26), so that by Corollary 3.28, $C_{\Psi_{K,A}}$ is trace-class in this case also.

Next, we observe that for any $k \in \mathbb{N}$, $a \in \mathbb{D}$ and $\sigma \in \Sigma^1$, the map $U_{k,a}$ extends holomorphically to D^σ with $U_{k,a}(D^\sigma) \subseteq D^\sigma$, and fixes $z^* = (0, 0)$ with

$$DU_{k,a}(z^*) = \begin{pmatrix} 0 & a^k \\ 1 & 0 \end{pmatrix},$$

while for $\sigma \in \Sigma^{-1}$ the map $U_{k,a}^{-1}$ extends holomorphically to D^σ with $U_{k,a}^{-1}(D^\sigma) \subseteq D^{-\sigma}$, which implies the claim about how the cases Theorem 1.2(i)-(ii) apply to $\Psi_{K,A}$ and $\Psi_{K,A}^{-1}$.

We now proceed to show the assertions (i)-(iv). Denoting $U_{K,A} = U_{k_1,a_1} \circ \cdots \circ U_{k_n,a_n}$ with $A = (a_1, \dots, a_n)$ and $K = (k_1, \dots, k_n)$, we note that $DU_{K,A}(z^*) = DU_{k_1,a_1}(z^*) \cdots DU_{k_n,a_n}(z^*)$, allowing us to compute the relevant multipliers for $\Psi_{K,A} = I_{11}^s \circ U_{K,A}$, starting with the case of $s = 0$.

(i) If $s = 0$ and n is odd, then

$$D\Psi_{K,A}(z^*) = DU_{K,A}(z^*) = \begin{pmatrix} 0 & A_o^{K_o} \\ A_e^{K_e} & 0 \end{pmatrix}.$$

Thus, the multipliers of $\Psi_{K,A}$ at z^* are $\lambda = (v, -v)$ with $v = (A^K)^{1/2}$.

(ii) If $s = 0$ and n is even, then

$$D\Psi_{K,A}(z^*) = DU_{K,A}(z^*) = \begin{pmatrix} A_o^{K_o} & 0 \\ 0 & A_e^{K_e} \end{pmatrix},$$

and, the multipliers of $\Psi_{K,A}$ at z^* are $\lambda = (A_o^{K_o}, A_e^{K_e})$.

For $s = 1$ we shall use Lemma 4.9 with $\hat{T} = U_{K,A}$ and $T = I_{11} \circ U_{K,A} = \Psi_{K,A}$, which yields $D(\Psi_{K,A} \circ \overline{\Psi_{K,A}})(z^*) = \overline{DU_{K,A}(z^*)} DU_{K,A}(z^*)$.

(iii) If $s = 1$ and n is odd, then the multipliers of $\Psi_{K,A}^2$ at z^* are $\lambda = (v, \bar{v})$ with $v = \overline{A_o^{K_o}} A_e^{K_e}$.

(iv) If $s = 1$ and n is even, then the multipliers of $\Psi_{K,A}^2$ at z^* are $\lambda = (|A_o^{K_o}|^2, |A_e^{K_e}|^2)$.

Now, it remains to compute the multipliers of the inverse of $I_{11}^s \circ U_{K,A}$. We set $S = I_{01} \circ R$ (so that $S^2 = S^{-2} = I_{11}$), and observe using Lemma B.2 in the appendix that the inverse $U_{k,a}^{-1} = G_{a,0} \circ R \circ F^{-1} \circ G_{0,-a}$ obeys the relations

$$\begin{aligned} U_{k,a}^{-1} \circ S &= S^{-1} \circ U_{k,\bar{a}}, \\ U_{k,b}^{-1} \circ S^{-1} &= S \circ U_{k,b}, \end{aligned}$$

and hence we have $S^{-1} \circ U_{k,a}^{-1} \circ S = I_{11} U_{k,\bar{a}}$ and $S^{-1} \circ (U_{k,a} \circ U_{k,b})^{-1} \circ S = U_{k,b} \circ U_{k,\bar{a}}$. Iterating, we obtain the conjugation

$$S^{-1} \circ U_{K,A}^{-1} \circ S = I_{11}^n \circ U_{k_n,\bar{a}_n} \circ \cdots \circ U_{k_4,a_4} \circ U_{k_3,\bar{a}_3} \circ U_{k_2,a_2} \circ U_{k_1,\bar{a}_1} =: \tilde{U}_{K,A}, \quad (22)$$

where \bar{a}_n is a_n if n is even or \bar{a}_n if n is odd. With the same conjugacy S , we obtain a conjugation

$$S^{-1} \circ (I_{11} \circ U_{K,A})^{-1} \circ S = I_{11}^{n-1} \circ U_{k_n,\bar{a}_n} \circ \cdots \circ U_{k_4,\bar{a}_4} \circ U_{k_3,a_3} \circ U_{k_2,\bar{a}_2} \circ U_{k_1,a_1} =: \check{U}_{K,A}, \quad (23)$$

where \check{a}^n is \bar{a}_n if n is even or a_n if n is odd. We can now compute the relevant multipliers for $\Psi_{K,A}^{-1} = (I_{11}^s \circ U_{K,A})^{-1}$. For $s = 0$, by (22) these are given by the fixed-point multipliers of $\tilde{U}_{K,A}$.

(i) If $s = 0$ and n is odd, by Lemma 4.9 the multipliers of $\tilde{U}_{K,A}$ at z^* are $\lambda = (v, \bar{v})$, $v = \overline{A_o^{K_o}} A_e^{K_e}$.

(ii) If $s = 0$ and n is even, then the multipliers of $\tilde{U}_{K,A}$ at z^* are $\lambda = (\overline{A_o^{K_o}}, A_e^{K_e})$.

For $s = 1$, the fixed-point multipliers of $\Psi_{K,A}^{-1}$ can be computed via those of $\check{U}_{K,A}$ by (23).

(iii) If $s = 1$ and n is odd, the multipliers of $\check{U}_{K,A}$ at z^* are $\lambda = (v, -v)$ with $v = (A_o^{K_o} \overline{A_e^{K_e}})^{1/2}$.

(iv) If $s = 1$ and n is even, by Lemma 4.9 the multipliers of $\check{U}_{K,A}^2$ at z^* are $\lambda = (|A_o^{K_o}|^2, |A_e^{K_e}|^2)$.

Claims (i)-(iv) follow by combining the respective cases for the multipliers of $\Psi_{K,A}$ and $\Psi_{K,A}^{-1}$. \square

Corollary 5.6. *Let $\Psi_{K,A}$ be as in Proposition 5.5. Then:*

(i) *If $a_i \neq 0$ for all i , then the fixed point multipliers λ_σ in Theorem 1.2 satisfy $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0$ for all $\sigma \in \Sigma$. Moreover, $|\lambda_{\sigma,1}|$ and $|\lambda_{\sigma,2}|$ can (independently) be chosen to take any value in $(0, 1)$ via suitable choice of $A \in \mathbb{D}^n$.*

(ii) *If n is even and exactly one of the a_i is zero, then either case (i) or (ii) of Theorem 1.2 applies to both $\Psi_{K,A}$ and $\Psi_{K,A}^{-1}$, and all fixed point multipliers are of the form $(\lambda_1, 0)$ with $\lambda_1 \neq 0$. Moreover, $|\lambda_1|$ can be chosen to take any value in $(0, 1)$ via suitable choice of $A \in \mathbb{D}^n$, and in the case (ii) of Theorem 1.2, $\lambda_1 \in \mathbb{R}$.*

(iii) *If $n > 2$ and at most one of the a_i is nonzero, then all multipliers are $(0, 0)$.*

We shall next construct non-linear non-area-preserving maps in \mathcal{F} homotopic to maps of the form (18) with $n > 1$, yielding trivial resonances. For this we define the toral diffeomorphisms

$$W_{k,a}(z) = (F^k \circ R \circ G_{0,a})(z) = (z_1^k b_a(z_2), z_1) \quad (k \in \mathbb{N}, a \in \mathbb{D}, z \in \mathbb{T})$$

and

$$\Xi_{K,a} = I_{11}^s \circ W_{k_1,0} \circ W_{k_2,0} \circ \cdots \circ W_{k_{n-1},0} \circ W_{k_n,a},$$

where $s \in \{0, 1\}$, $n > 1$, $a \in \mathbb{D}$ and $K = (k_1, \dots, k_n) \in \mathbb{N}^n$.

Lemma 5.7 (Non-area-preserving maps homotopic to (18)). *For any $s \in \{0, 1\}$, $a \in \mathbb{D}$, $K \in \mathbb{N}^n$ with $n > 1$, the map $\Xi_{K,a}$ satisfies the assumptions of Theorem 1.2. Moreover $\chi_{\Xi_{K,a}} = 1$.*

Proof. Following similar calculations and notations as in the proof of Proposition 5.5 we first show that $\Xi_{K,a}$ satisfies the (p -sec) condition. We have that $D\check{W}_{k,a}(x) = \begin{pmatrix} k & s_a(x) \\ 1 & 0 \end{pmatrix}$ with $s_a(x) > 0$

for all $x \in M$. Thus, for $s = 0$ and $n > 1$ it follows that $D\check{\Xi}_{K,a}(x)$ can be written as $M_x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with $M_x \geq 0$ and thus satisfies the first part of the (p -sec) condition (6) by Remark 2.13. The case $s = 1$ and the second part of the (p -sec) condition follow similarly.

To show that $\Xi_{K,a}$ does not yield any non-trivial resonances, first note that for $\ell \in \{1, -1\}$ the map $W_{k,a}^\ell$ extends holomorphically to D^σ with $W_{k,a}^\ell(D^\sigma) \subseteq D^{\ell\sigma}$ for all $\sigma \in \Sigma^\ell$. As the forward map $W_{k,a}$ fixes $z^* = (0, 0)$ and

$$DW_{k,a}(z^*) = \begin{pmatrix} -a\delta_{k,1} & 0 \\ 1 & 0 \end{pmatrix},$$

we see that the multipliers of $\Xi_{K,a}$ at z^* are trivial for any $n > 1$ and $s \in \{0, 1\}$. For the inverse map we use the conjugation $S = I_{01} \circ R$, obtaining the relations

$$\begin{aligned} W_{k,a}^{-1} \circ S &= S^{-1} \circ G_{-a,0} \circ W_{k,0}, \\ W_{k,a}^{-1} \circ S^{-1} &= S \circ G_{-\bar{a},0} \circ W_{k,0}, \end{aligned}$$

yielding $S^{-1} \circ \Xi_{K,a}^{-1} \circ S = G_{-\bar{a},0} \circ I_{11}^{n+s} \circ W_{k_n,0} \circ \cdots \circ W_{k_1,0} := E_{K,a}$. One can calculate that $(\bar{a}, 0)$ is a fixed point of $E_{K,a}$ for $n + s$ is even, and of $(E_{K,a})^2$ if $n + s$ is odd. In both cases, its multipliers are $\lambda = (c, 0)$ for some $c \in \mathbb{D}$. Thus, by Corollary 4.12(iii) all resonances are trivial. \square

Proposition 5.8. *The homotopy class of every map of the form (18) contains non-linear Anosov diffeomorphisms $T \in \mathcal{F}$, such that the corresponding operator C_T is well defined and trace-class on $H_{\alpha,-\gamma}$ for some $\alpha, \gamma \in \mathbb{R}_{>0}^2$. Moreover, T can be chosen such that the eigenvalue sequence of C_T satisfies either of the cases (i)-(iii) from the conclusions of Theorem 1.3.*

Proof. To prove cases (i) and (ii) from Theorem 1.3, we choose T to be of the form $\Psi_{K,A}$ from (19), noting that for any fixed $s \in \{0, 1\}$ and $K \in \mathbb{N}^n$, $n \in \mathbb{N}$, the map $\Psi_{K,0}$ is an area-preserving hyperbolic automorphism of the form (18), homotopic to any $\Psi_{K,A}$, $A \in \mathbb{D}^n$. The claim follows directly from Proposition 5.5 together with Corollaries 5.6 and 4.12.

The case (iii) follows from Lemma 5.7, noting that the maps $\Xi_{K,a}$ are not area-preserving and hence not C^1 -conjugated to toral automorphisms for $a \neq 0$, but have trivial resonances. \square

We will also need the following well-known result (see, e.g., [F, Lemma 1.1] and [M, Theorem A]):

Lemma 5.9. *For any Anosov diffeomorphism $f: \mathbb{T}^n \rightarrow \mathbb{T}^n$ there exists a hyperbolic automorphism $g: \mathbb{T}^n \rightarrow \mathbb{T}^n$ which is homotopic to f . That is, there exists a continuous one-parameter family of maps $h: [0, 1] \times \mathbb{T}^n \rightarrow \mathbb{T}^n$, such that $h(0, \cdot) = f$ and $h(1, \cdot) = g$.*

We are now ready to prove our last main theorem, concluding that every homotopy class of toral Anosov diffeomorphisms contains elements with resonances exhibiting any of stretched-exponential, exponential, or trivially super-exponential decay rate.

Proof of Theorem 1.3. Let \mathcal{H} be any homotopy class of toral Anosov diffeomorphisms. By Lemma 5.9, there exists a hyperbolic matrix $B \in \text{GL}_2(\mathbb{Z})$ with $\tau_B \in \mathcal{H}$. Corollary 5.4 yields that there are $A, Q \in \text{GL}_2(\mathbb{Z})$ such that τ_A is of the form (18), and τ_B is analytically conjugated to τ_A , via $\tau_A = \tau_Q \circ \tau_B \circ \tau_Q^{-1}$. We call \mathcal{H}' the homotopy class containing τ_A . We note that because of its special form (18) and using Remark 3.22, the operator C_{τ_A} given by $f \mapsto f \circ \tau_A$ yields an isomorphism from $H_{\alpha,-\gamma}$ to itself, while τ_Q gives rise to the isometric isomorphism $C_{\tau_Q}: H_{Q,\alpha,-\gamma} \rightarrow H_{\alpha,-\gamma}$, conjugating $C_{\tau_A}: H_{\alpha,-\gamma} \rightarrow H_{\alpha,-\gamma}$ and $C_{\tau_B}: H_{Q,\alpha,-\gamma} \rightarrow H_{Q,\alpha,-\gamma}$, for any $\alpha, \gamma \in \mathbb{R}^2$.

Finally, by Proposition 5.8, there exists an Anosov diffeomorphism $\tau' \in \mathcal{H}'$ whose corresponding composition operator on $H_{\alpha',-\gamma'}$ for suitable $\alpha', \gamma' \in \mathbb{R}_{>0}^2$ has an eigenvalue sequence with any one of the desired decay rates. Writing out the homotopy explicitly, we have a one-parameter family of maps $h': [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $h'_t = h'(t, \cdot) \in \mathcal{H}'$, such that $h'_0 = \tau_A$ and $h'_1 = \tau'$. Conjugating with τ_Q we obtain a homotopy $h_t = \tau_Q^{-1} \circ h'_t \circ \tau_Q \in \mathcal{H}$ with $h_0 = \tau_B$ and $h_1 = \tau_Q^{-1} \circ \tau' \circ \tau_Q$. Since $C_{\tau_Q}: H_{Q,\alpha',-\gamma'} \rightarrow H_{\alpha',-\gamma'}$ is an isometric isomorphism, the spectra of $C_{\tau'}$ and $C_{h_1} = C_{\tau_Q^{-1}} \circ C_{\tau'} \circ C_{\tau_Q}$ coincide, and so the composition operator associated to $T = h_1$ satisfies the assertion of the proposition for $\nu = \nu_{P,\alpha,-\gamma}$ with $P = Q$, $\alpha = \alpha'$ and $\gamma = \gamma'$. \square

A Auxiliary results

Here we list a number of auxiliary results and proofs omitted but used in the main text.

Lemma A.1. *For every $P \in \text{GL}_2(\mathbb{R})$, there exist $A \in \text{GL}_2(\mathbb{Z})$ and $\tilde{P} \in \text{GL}_2(\mathbb{R})$ with \tilde{P} having only non-negative entries, such that $P = A\tilde{P}$.*

Proof. It suffices to show that for $P \in \text{GL}_2(\mathbb{R})$, there exists $B \in \text{GL}_2(\mathbb{Z})$ such that $(BP)_{ij} > 0$ for $i, j = 1, 2$. Writing the rows of B as b^u, b^s , and the columns of P as p_u, p_s and denoting the cone $\mathcal{C} = \{v \in \mathbb{R}^2 : \langle v, p_u \rangle > 0, \langle v, p_s \rangle > 0\}$, this is equivalent to there existing $b^u, b^s \in \mathbb{Z}^2 \cap \mathcal{C}$, such that

$$b_1^u b_2^s - b_2^u b_1^s = 1. \quad (24)$$

We fix any $b^u \in \mathbb{Z}^2 \cap \mathcal{C}$ (non-empty since \mathcal{C} is an open convex cone), without loss of generality satisfying $\gcd(b_1^u, b_2^u) = 1$. By Bezout's identity, there exist $\hat{b}^s = (\hat{b}_1^s, \hat{b}_2^s) \in \mathbb{Z}^2$ such that $b_1^s(k) = \hat{b}_1^s + kb_1^u, b_2^s(k) = \hat{b}_2^s + kb_2^u$ are solutions to (24) for every $k \in \mathbb{Z}$. Moreover, it is easy to see that for sufficiently large $k \in \mathbb{Z}$, $b^s = (b_1^s(k), b_2^s(k)) = \hat{b}^s + k \cdot b^u$ lies in the cone \mathcal{C} , finishing the proof. \square

Lemma A.2. *Let $\hat{\sigma} \in \Sigma^1$, and let $\{D_c : c \in C\} \subset \text{GL}_2(\mathbb{R})$ be a continuous family of matrices indexed by some compact set C , satisfying $D_c(\mathbb{R}_{\geq 0}^2) \subset R^{\hat{\sigma}} \cup \{0\}$ for all $c \in C$. Then, for any $\sigma \in \Sigma^1, \tilde{\sigma} \in \Sigma^{-1}, \delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$, there exists $q \in R_{\tilde{\delta}}^{\tilde{\sigma}}$ such that $D_c(q) \in R_{\delta}^{\sigma}$ for all $c \in C$.*

Proof. Let us first assume that $\hat{\sigma} = (1, 1)$, that is, $D_c(\mathbb{R}_{\geq 0}^2) \subset \mathbb{R}_{>0}^2 \cup \{0\}$. This implies $(D_c)_{kl} > 0$ for all $k, l \in \{1, 2\}$ and all $c \in C$. By compactness of C it follows that there are $\underline{D}, \overline{D} > 0$ such that $\underline{D} < (D_c)_{kl} < \overline{D}$ for all $k, l \in \{1, 2\}$ and all $c \in C$. We fix $\delta, \tilde{\delta} \in \mathbb{R}_{>0}^2$, and set $\bar{\delta} = \max\{\delta_1, \delta_2, \tilde{\delta}_1, \tilde{\delta}_2\}$.

We first consider the case $\sigma = (1, 1), \tilde{\sigma} = (-1, 1)$. Setting $q_1 = -\bar{\delta} < -\tilde{\delta}_1$ and $q_2 = \bar{\delta} \max\{1, \frac{1+\overline{D}}{\underline{D}}\} > \tilde{\delta}_2$, we have $q = (q_1, q_2) \in R_{\tilde{\delta}}^{\tilde{\sigma}}$, and since $(D_c)_{k,1}q_1 + (D_c)_{k,2}q_2 \geq -\overline{D}\bar{\delta} + \underline{D}\frac{1+\overline{D}}{\underline{D}}\bar{\delta} = \bar{\delta} > \delta_k$ for $k = 1, 2$ and all $c \in C$, we obtain $D_c(q) \in R_{\delta}^{\sigma} + \delta = R_{\delta}^{\sigma}$, for all $c \in C$, as required.

The case $\sigma = (-1, -1), \tilde{\sigma} = (1, -1)$ is similar, with $q' = -q \in R_{\tilde{\delta}}^{\tilde{\sigma}}$ and $D_c(q') = -D_c(q) \in R_{\delta}^{\sigma} - \delta = R_{\delta}^{\sigma}$ for all $c \in C$. The other two cases ($\sigma = (1, 1), \tilde{\sigma} = (1, -1)$ and $\sigma = (-1, -1), \tilde{\sigma} = (-1, 1)$) can be shown analogously by swapping the roles of q_1 and q_2 in the above construction.

Finally, for $\hat{\sigma} = (-1, -1)$, we note that the claim holds for $-D_c$ by the above, and hence it follows for D_c by replacing q by $-q$. \square

Lemma A.3. *Let $f: \mathbb{Z}^2 \rightarrow \mathbb{C}$ be a cone-wise exponential function with cones being the quadrants $\hat{R}^{\sigma, \circ}$, $\sigma \in \Sigma$; that is, for all $\sigma \in \Sigma$ there exist $\lambda_{\sigma} \in \mathbb{D}^2$, such that $f(n) = \lambda_{\sigma}^n$ whenever $n \in \mathbb{Z}^2 \cap \hat{R}^{\sigma, \circ}$. Denote by $(\lambda_n)_{n \in \mathbb{N}}$ be an enumeration of $\{f(n) : n \in \mathbb{Z}^2\}$ sorted by decreasing modulus, and $N(r) = \#\{n \in \mathbb{N} : |\lambda_n| \geq r\}$ for $r \in (0, 1)$. Then $(\lambda_n)_{n \in \mathbb{N}}$ satisfies*

$$\lim_{r \rightarrow 0} \frac{\log N(r)}{\log |\log r|} = d,$$

where:

(i) if $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0$ for some $\sigma \in \Sigma$, then $d = 2$ (stretched-exponential decay) and

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n^{1/2}} = \eta_2$$

$$\text{with } \eta_2 = \left(1/2 \sum_{\sigma \in \Sigma: \lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0} (\log |\lambda_{\sigma,1}| \cdot \log |\lambda_{\sigma,2}|)^{-1}\right)^{-1/2}.$$

(ii) if $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} = 0$ for all $\sigma \in \Sigma$, and $\lambda_{\sigma,k} \neq 0$ for some $\sigma \in \Sigma^1$ and $k \in \{1, 2\}$, then $d = 1$ (exponential decay), and

$$\lim_{n \rightarrow \infty} \frac{-\log |\lambda_n|}{n} = \eta_1,$$

$$\text{with } \eta_1 = \left(\sum_{\sigma \in \Sigma^1} \sum_{k: \lambda_{\sigma,k} \neq 0} (\log |\lambda_{\sigma,k}|)^{-1}\right)^{-1}.$$

(iii) if $\lambda_{\sigma} = 0$ for all $\sigma \in \Sigma^1$, and $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} = 0$ for all $\sigma \in \Sigma^{-1}$, then $d = 0$ (super-exponential decay). In this case $(\lambda_n)_{n \in \mathbb{N}}$ is the trivial sequence with $\lambda_n = \delta_{n,1}$.

Proof. We write $\ell_s(r) = \log r / \log |s|$ for $s \in \mathbb{D}$ and $r \in (0, 1)$. For $p, q \in \mathbb{D}$, $r \in (0, 1)$ we have

$$\begin{aligned} N_p^r &:= \#\{n \geq 1 : |p|^n \geq r\} = \lfloor \ell_p(r) \rfloor, \\ N_{p,q}^r &:= \#\{n, m \geq 1 : |p|^n |q|^m \geq r\} \\ &= \#\{n, m \geq 1 : n(-\log |p|) + m(-\log |q|) \leq -\log r\} \\ &= \frac{\lfloor \ell_p(r) \rfloor \lfloor \ell_q(r) \rfloor}{2} + \delta, \quad \text{with } |\delta| \leq 1 + \lfloor \ell_p(r) \rfloor + \lfloor \ell_q(r) \rfloor, \end{aligned}$$

where the last equality is obtained from counting the number of integer lattice points in the triangle spanned by $(0, 0)$, $(\ell_p(r), 0)$ and $(0, \ell_q(r))$, and subtracting those lying on one of the axes. From the definition of f we obtain

$$N(r) = 1 + \sum_{\sigma \in \Sigma} \left(\frac{\lfloor \ell_{\lambda_{\sigma,1}}(r) \rfloor \lfloor \ell_{\lambda_{\sigma,2}}(r) \rfloor}{2} + \delta_\sigma \right) + \sum_{\sigma \in \Sigma^1} (\lfloor \ell_{\lambda_{\sigma,1}}(r) \rfloor + \lfloor \ell_{\lambda_{\sigma,2}}(r) \rfloor),$$

with $|\delta_\sigma| \leq 1 + \lfloor \ell_{\lambda_{\sigma,1}}(r) \rfloor + \lfloor \ell_{\lambda_{\sigma,2}}(r) \rfloor$.

We now prove (i). Since $\ell_s(r) \rightarrow \infty$ as $r \rightarrow 0$ for any $s \in \mathbb{D}$ and $\lambda_{\sigma,1} \cdot \lambda_{\sigma,2} \neq 0$ for some $\sigma \in \Sigma$, we have that for every $\epsilon > 0$ there exists $r_\epsilon > 0$ such that for $r \in (0, r_\epsilon)$ it holds that

$$(1 - \epsilon) \cdot \left(\frac{\log r}{\eta_2} \right)^2 \leq N(r) \leq (1 + \epsilon) \cdot \left(\frac{\log r}{\eta_2} \right)^2. \quad (25)$$

The assertion $d = 2$ immediately follows. Moreover, since $n > (1 + \epsilon) \cdot (\log r / \eta_2)^2 \geq N(r)$ implies $|\lambda_n| < r$, a short calculation yields that $\log |\lambda_n| < \log r$ holds for any sufficiently large n and $\log r \in (-(1 + \epsilon)^{-1/2} \eta_2 \sqrt{n}, \log r_\epsilon)$, which implies

$$\frac{-\log |\lambda_n|}{\sqrt{n}} \geq (1 + \epsilon)^{-1/2} \eta_2.$$

Conversely, $n < (1 - \epsilon) \cdot (\log r / \eta_2)^2 \leq N(r)$ implies $|\lambda_n| \geq r$, and hence for large n we obtain

$$\frac{-\log |\lambda_n|}{\sqrt{n}} \leq (1 - \epsilon)^{-1/2} \eta_2.$$

Since the choice of $\epsilon > 0$ was arbitrary, assertion (i) follows.

The proof of (ii) is very similar; replacing (25) by

$$(1 - \epsilon) \cdot \frac{|\log r|}{\eta_1} \leq N(r) \leq (1 + \epsilon) \cdot \frac{|\log r|}{\eta_1} \quad (26)$$

yields $d = 1$, as well as $\eta_1 / (1 + \epsilon) < -\log |\lambda_n| / n < \eta_1 / (1 - \epsilon)$ for sufficiently large n .

Finally, (iii) follows directly by observing that in this case $f(n) = \delta_{n_1,0} \delta_{n_2,0}$. \square

Proof of Lemma 3.27. The proof follows the same steps as [BJ2, Propositions 3.4 & 3.5]. Let $J^*: H_{P,A,-\Gamma} \rightarrow H_{P,\alpha,-\gamma}$ denote the Hilbert space adjoint of J , then J^*J is diagonal in the orthogonal basis of monomials, as $(J^*J p_n, p_m)_{H_{P,\alpha,-\gamma}} = (J p_n, J p_m)_{H_{P,A,-\Gamma}} = (p_n, p_m)_{H_{P,A,-\Gamma}} = \omega_n (p_n, p_m)_{H_{P,\alpha,-\gamma}}$ with $\omega_n = \nu_{P,A,-\Gamma}(n) / \nu_{P,\alpha,-\gamma}(n)$. Therefore the eigenvalues of J^*J , which are square roots of the singular values of J , are given by $\{\sqrt{\omega_n} : n \in \mathbb{Z}^2\}$, which can be written as the set of all λ^n , $n \in \mathbb{N}_0^2$, and μ^n , $n \in \mathbb{N}^2$, with $\lambda = e^{\frac{A-\alpha}{2}}$ and $\mu = e^{\frac{\gamma-\Gamma}{2}}$. Since $A - \alpha, \gamma - \Gamma \in \mathbb{R}_{>0}^2$, Lemma A.3(i) applies, with

$$\eta = \eta_2 = \left(\frac{1}{\log(A_1 - \alpha_1) \log(A_2 - \alpha_2)} + \frac{1}{\log(\gamma_1 - \Gamma_1) \log(\gamma_2 - \Gamma_2)} \right)^{-1/2}. \quad \square$$

Proof of Lemma 5.3. We first note that it is sufficient to consider the case $\text{Tr } M \geq 0$, as the general case easily follows by considering $-M$ in the opposite case. By [He, Theorem 3], every $M \in \text{GL}_2(\mathbb{Z})$ with $\text{Tr } M \geq 0$ is similar to a so-called ‘standard matrix’ (see [He, Definition 1]), which in the case of a hyperbolic matrix (implying $\text{Tr } M \neq 0$ and real eigenvalues $\neq -1, 1$) reduces to two (alternative) cases:

- (i) M is similar to $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ in the case $\text{Tr } M = 1$,
- (ii) M is similar to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $0 \leq d \leq b, c < a$ in the case $\text{Tr } M \geq 2$.

We are left to show that every matrix in case (ii) is similar to one of (17). We note that every matrix of this form satisfies $a > 0$, since $c < a \leq 0$ would imply $d = \text{Tr } M - a \geq 2$, and hence $1 \geq \det M = ad - bc \geq (a - c)d \geq 2$, a contradiction. We also note that $c = 0$ yields a non-hyperbolic matrix, so we can assume $c \neq 0$.

For $c > 0$ the claim follows immediately from [He, Theorem 4]. For $c < 0$, we note that $0 \leq d \leq b$ implies $b > 0$ (as otherwise $\det M = 0$), and hence $-bc \geq 1$. Since $ad - bc \leq 1$, it follows that $ad = 0$ and hence $d = 0$, as well as $b = 1$ and $c = -1$. We obtain that M is similar to a matrix of the form $M_a = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$, which is only hyperbolic for $a \geq 3$. The claim follows by observing that $M_a, a \geq 3$, is similar to

$$N_a = \begin{pmatrix} a-1 & 1 \\ a-2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a-2 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$$

via $M_a = C^{-1}N_aC$ with $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. □

B Notes on the special group of toral diffeomorphisms

Here we briefly restate the definition of the group of toral diffeomorphisms \mathcal{F} from Section 5, before expanding on the various types of maps that can be constructed within this group via a set of examples. For $a \in \mathbb{D}$, let $b_a: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be an automorphism of \mathbb{D} (or Moebius map) given by

$$b_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Every such map satisfies $b_a(\mathbb{T}) = \mathbb{T}$, and a straightforward calculation yields the following lemma.

Lemma B.1. *Fix $a = |a|e^{i\alpha} \in \mathbb{D}$ and let b_a be as above. For $M = ([0, 2\pi]/\sim)$ let $\tilde{b}_a: M \rightarrow M$ be the map determined by $\pi \circ \tilde{b}_a = b_a \circ \pi$ with $\pi(\theta) = e^{i\theta}$ for $\theta \in M$. Then*

$$\tilde{b}_a(\theta) = \theta + g_a(\theta),$$

with $g_a(\theta) = 2 \arctan\left(\frac{|a|\sin(\theta-\alpha)}{1-|a|\cos(\theta-\alpha)}\right)$ and $g'_a(\theta) = 2\left(\frac{|a|\cos(\theta-\alpha)-|a|^2}{1-2|a|\cos(\theta-\alpha)+|a|^2}\right) > -1$.

We recall the definition of the maps $F: (z_1, z_2) \mapsto (z_1z_2, z_2)$, $R: (z_1, z_2) \mapsto (z_2, z_1)$, $I_{kl}: (z_1, z_2) \mapsto (z_1^{1-2k}, z_2^{1-2l})$ for $k, l \in \{0, 1\}$, and $\mathcal{G} = \{G_{a,b}: (z_1, z_2) \mapsto (b_a(z_1), b_a(z_2)) : a, b \in \mathbb{D}\}$ from Section 5, as well as the definition of \mathcal{F} as the group of toral diffeomorphisms generated by these maps. The group of toral automorphisms $\text{Aut}(\mathbb{T}^2)$ is generated by the set $\Gamma = \{F, R, I_{01}\}$. Any $T \in \{F, I_{00}, I_{11}\} \cup \mathcal{G}$ yields an orientation-preserving diffeomorphism of \mathbb{T}^2 , while all of $\{I_{01}, I_{10}, R\}$ are orientation-reversing. The next lemma summarises some basic properties of these maps.

Lemma B.2. *Let $a, b \in \mathbb{D}$ and $k, l, m, n \in \{0, 1\}$. The following commutation relations hold.*

- (i) $F \circ I_{kl} = I_{kl} \circ F^{-1}$ and $F^{-1} \circ I_{kl} = I_{kl} \circ F$ for $k \neq l$,
- (ii) $F \circ I_{11} = I_{11} \circ F$ and $F^{-1} \circ I_{11} = I_{11} \circ F^{-1}$,
- (iii) $R \circ I_{kl} = I_{lk} \circ R$,
- (iv) $I_{kl} \circ I_{mn} = I_{1-\delta_{km}, 1-\delta_{ln}}$,
- (v) $G_{a,b} \circ I_{kl} = I_{kl} \circ G_{(1-k)a+k\bar{a}, (1-l)b+l\bar{b}}$,
- (vi) $G_{a,b} \circ R = R \circ G_{b,a}$.

Proof. All statements follow by direct computation, with part (v) using $b_a(z^{-1}) = b_{\bar{a}}(z)^{-1}$. \square

We now provide examples from various interesting sub-classes of diffeomorphisms in \mathcal{F} .

Anosov diffeomorphisms in \mathcal{F} . For $a \in \mathbb{D}$ let $T_a: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be given by

$$T_a = G_{0,-a} \circ F \circ R \circ G_{a,0}.$$

Using Lemma B.1, we can see that the derivative of the respective map $\tilde{T}_a: M \rightarrow M$ is given by

$$D\tilde{T}_a(x) = \begin{pmatrix} s_a(x) & 1 \\ 1 & 0 \end{pmatrix},$$

with $s_a(x) = (1 + g'_a(x_1)) > 0$ for all $x = (x_1, x_2) \in M$.

Example B.3.

- (i) For $a = 0$ the map $T_0(z) = (z_1 z_2, z_1)$ is an Anosov automorphism induced by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalues $\lambda_{u/s} = \varphi^{\pm 1}$ where $\varphi = (1 + \sqrt{5})/2$ is the golden mean.
- (ii) One can check that $T_a(z) = (b_a(z_1) z_2, z_1)$ is Anosov for all $a \in \mathbb{D}$ by finding suitable cone fields (see Definition 2.11).
- (iii) The maps $T_b \circ T_a$ given by

$$(T_b \circ T_a)(z) = (b_b(b_a(z_1) z_2) z_1, b_a(z_1) z_2)$$

are Anosov for all $a, b \in \mathbb{D}$, and in fact $\mathbb{R}_{>0}^2 \cup \mathbb{R}_{<0}^2$ can be chosen as the invariant expanding cone, and its complementary cone as the invariant contracting one. These are the maps considered in [SBJ] and [PoS].

The maps in (iii) are orientation-preserving while the ones in (i) and (ii) are orientation-reversing.

Area-preserving diffeomorphisms in \mathcal{F} .

Lemma B.4. *Let $F_a = G_{0,-a} \circ F \circ G_{0,a}$ for $a \in \mathbb{D}$. Then any finite composition of the elements of $\Gamma_{ap} = \Gamma \cup \{F_a : a \in \mathbb{D}, k \in \mathbb{N}\}$ is area-preserving.*

Proof. For $z \in \mathbb{T}^2$ we have $F_a(z_1, z_2) = (z_1 b_a(z_2), z_2)$. As b_a preserves \mathbb{T} we have $|\det DF_a(z)| = |b_a(z_2)| = 1$ for all $z \in \mathbb{T}^2$. As all elements in Γ_{ap} are area-preserving, so is their composition. \square

Example B.5.

(i) The maps in Example B.3 are area-preserving Anosov diffeomorphisms as $T_a = G_{0,-a} \circ F \circ G_{0,a} \circ R = F_a \circ R$ with F_a as in Lemma B.4.

(ii) The map $T = F \circ R \circ G_{0,a} \circ F \circ R$ given by

$$T(z_1, z_2) = (z_1 b_a(z_1) z_2, z_1 z_2)$$

is not area-preserving for $a \in \mathbb{D} \setminus \{0\}$ as $|\det DT(z)| = |z_1^2 b'_a(z_1) z_2| = |b'_a(z_1)|$ for $z \in \mathbb{T}^2$.

(iii) The map $T = F \circ R \circ G_{0,-a} \circ F \circ R \circ G_{a,b}$ given by

$$T(z_1, z_2) = (z_1 b_a(z_1) b_b(z_2), b_a(z_1) b_b(z_2)), \quad (a \in \mathbb{D}),$$

is area-preserving for $b = 0$ (see Example B.3(ii)), but is not area-preserving for $b \neq 0$.

Maps with symmetries. An automorphism T of some topological space is said to have a *symmetry* if there exists an automorphism H so that

$$H^{-1} \circ T \circ H = T,$$

and to have a *reversing symmetry* if there exists an automorphism H so that

$$H^{-1} \circ T \circ H = T^{-1}.$$

Clearly, for any rational map T with only real coefficients, Corollary 4.2(i) implies that I_{11} is a symmetry of T , which by Theorem 1.2 and its proof induces symmetry relations on the resonances.

On the other hand, presence of reversing symmetries is of considerable interest in classical and quantum mechanics. For systems with *time-reversal symmetry* the reverse motion satisfies the same laws of motion as the forward motion. Usually this time-reversal symmetry corresponds to a particular involution map H . However, the notion of reversible systems was extended to include all involutions and even non-involutive reversing symmetries, see for example the survey [LR] adapted to dynamical systems or [BR] specifically for toral automorphisms. We will next present some Anosov diffeomorphisms in \mathcal{F} which have reversing symmetries.

Lemma B.6. *Let $k \in \mathbb{N}$ and $a \in (0, 1)$, and define the maps $T_{k,a} = F^k \circ R \circ G_{a,-a} \circ F^k \circ R$ and $U_{k,a} = G_{0,-a} \circ T_{k,a} \circ G_{a,0}$.*

(i) *The map $T_{k,a}$ is a non-area-preserving Anosov diffeomorphism with a reversing symmetry.*

(ii) *The map $U_{k,a}$ is an area-preserving Anosov diffeomorphism with a reversing symmetry.*

Proof. Using Lemma B.1 it is not difficult to see that both $T_{k,a}$ and $U_{k,a}$ are Anosov with the first and third quadrant of \mathbb{R}^2 forming an unstable, and the second and fourth quadrant forming a stable invariant cone. Area preservation of $U_{k,a}$ and non-preservation for $T_{k,a}$ can be computed directly, noting that $U_{k,a} = (G_{0,-a} \circ F^k \circ R \circ G_{a,0})^2$. For the symmetries, a calculation with $H = I_{01} \circ R$ using Lemma B.2 and the fact that $a \in \mathbb{R}$ reveals that $H^{-1} \circ T_{k,a}^{-1} \circ H = T_{k,a}$. Since $G_{0,a} \circ H = H \circ G_{a,0}$ we also have $H^{-1} \circ U_{k,a}^{-1} \circ H = U_{k,a}$. \square

Comparison to Blaschke product diffeomorphisms. In [PS] the authors coined the notion of Blaschke product diffeomorphisms, which are maps of the form

$$T(z_1, z_2) = (A(z_1)B(z_2), C(z_1)D(z_2)),$$

where A, B, C, D are Blaschke products in one variable. They state that these are precisely the analytic maps on a neighbourhood of the open bidisk \mathbb{D}^2 , mapping \mathbb{D}^2 to itself and \mathbb{T}^2 diffeomorphically to itself, and provide an explanation of this in [PS, Remark 5.2]. Here we observe that this claim is inaccurate: while Examples B.5(ii)-(iii) are instances of Blaschke product diffeomorphisms, Example B.3(iii) is a hyperbolic diffeomorphism in \mathcal{F} containing Blaschke factors of a product of two variables, and cannot be written as a Blaschke product diffeomorphism.

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References

- [A] A. Adam, *Generic non-trivial resonances for Anosov diffeomorphisms*, *Nonlinearity* **30** (2017) 1146–1164.
- [B] V. Baladi, *The quest for the ultimate anisotropic Banach space*, *J. Stat. Phys.* **166** (2017) 525–557.
- [BJ1] O.F. Bandtlow and O. Jenkinson, *On the Ruelle eigenvalue sequence*. *Ergod. Th. & Dynam. Sys.* **28** (2008) 1701–1711.
- [BJ2] O.F. Bandtlow and O. Jenkinson, *Explicit estimates for transfer operators acting on spaces of holomorphic functions* *Adv. Math.* **218** (2008) 902–925.
- [BJS] O.F. Bandtlow, W. Just and J. Slipantschuk, *Spectral structure of transfer operators for expanding circle maps*, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017) 31–43.
- [BKL] M. Blank, G. Keller and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, *Nonlinearity* **15** (2002) 1905–1973.
- [BN] O.F. Bandtlow and F. Naud, *Lower bounds for the Ruelle spectrum of analytic expanding circle maps*, *Ergod. Th. & Dynam. Sys.* **39** (2019) 289–310.
- [BR] M. Baake and A.G. Roberts, *Reversing symmetry group of $GL(2, \mathbb{Z})$ and $PGL(2, \mathbb{Z})$ matrices with connections to cat maps and trace maps*. *J. Phys. A: Math. Gen.* **30** (1997) 1549–1573.
- [BT] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*; in K. Burns, D. Dolgopyat and Ya. Pesin (eds.), *Geometric and Probabilistic Structures in Dynamics* (Contemp. Math. **469**, American Mathematical Society, Providence, 2008), 29–68.
- [DFG] S. Dyatlov, F. Faure and C. Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds*, *Analysis & PDE* **8** (2015) 923–1000.
- [F] J. Franks, *Anosov Diffeomorphisms on Tori*, *Trans. Am. Math. Soc.* **145** (1969) 117–124.
- [FG] F.T. Farrell, A. Gogolev, *The space of Anosov diffeomorphisms*, *J. Lond. Math. Soc.* **89** (2014) 383–396.
- [FGL] F. Faure, S. Gouëzel and E. Lanneau, *Ruelle spectrum for linear pseudo-Anosov maps*, *J. Éc. Polytech. Math.* **6** (2019) 811–877.
- [FR] F. Faure and N. Roy, *Ruelle-Pollicott resonances for real analytic hyperbolic maps*, *Nonlinearity* **19** (2006) 1233–1252.
- [FRS] F. Faure, N. Roy and J. Sjöstrand, *Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances*, *Open Math. J.* **1** (2008) 35–81.
- [GL] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, *Ergod. Th. & Dynam. Sys.* **26** (2006) 189–217.

- [He] J. P. Henniger, *Factorization and Similarity in $GL(2, \mathbb{Z})$* , Linear Algebra Appl. **251** (1997) 223–237.
- [Hi] K. Hiraide, *A simple proof of the Franks-Newhouse theorem on codimension-one Anosov diffeomorphisms*, Ergod. Th. & Dynam. Sys. **21** (2001) 801–806.
- [J] M. Jezéquel, *Local and global trace formula for smooth hyperbolic diffeomorphisms*, J. Spectr. Theory **10** (2020) 185–249.
- [Ka] O. N. Karpenkov, *Geometry of Continued Fractions*, Springer, Berlin, Heidelberg, 2022.
- [Kn] G. Knese, *Integrability and regularity of rational functions*, Proc. London Math. Soc **111** (2015) 1261–1306.
- [L] M. Lassalle, *L'espace de Hardy d'un domaine de Reinhardt généralisé*, J. Funct. Anal. **60** (1985) 309–340.
- [LR] J.S.W. Lamb and J.A.G. Roberts, *Time-reversal symmetry in dynamical systems: A survey*, Physica D **112** (1998) 1–39.
- [M] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math. **96** (1974) 422–429.
- [MX] J.D. McNeal and J. Xiong, *Norm convergence of partial sums of H^1 functions*, Int. J. Math. **29** (2018).
- [Pi] A. Pietsch, *Eigenvalue and s -numbers*, CUP, Cambridge, 1986.
- [PS] E. Pujals and M. Shub, *Dynamics of two-dimensional Blaschke products*, Ergod. Th. & Dyn. Sys. **20** (2008) 575–585.
- [PoS] M. Pollicott and B. Sewell, *Explicit examples of resonances for Anosov maps of the torus*, (2022) Preprint: arXiv:2204.01511.
- [Rud] W. Rudin, *Function Theory in Polydiscs*, W.A. Benjamin, Inc., New York, 1969.
- [Rug] H.-H. Rugh, *The correlation spectrum for hyperbolic analytic maps*, Nonlinearity **5** (1992) 1237–1263.
- [S] K.R. Shrestha, *Hardy Spaces on the Polydisk*, Eur. J. Pure Appl. Math. **20** (2016) 292–304.
- [SBJ] J. Slipantschuk, O.F. Bandtlow and W. Just, *Complete spectral data for analytic Anosov maps of the torus*, Nonlinearity **30** (2017) 2667–2686.
- [ST] A. Sola and R. Tully-Doyle, *Dynamics of low-degree rational inner skew products on \mathbb{T}^2* , (2021) Preprint: arXiv:2109.03437.
- [U] H. Upmeyer, *Toeplitz operators and index theory in several complex variables*, Birkhäuser, Basel (2012).