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INVARIANT MEASURES FOR REAL ANALYTIC EXPANDING MAPS

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Abstract

Let X be a compact connected subset of $\mathbb{R}^d$ with non-empty interior, and $T : X \to X$ a real analytic full branch expanding map with countably many branches. Elements of a thermodynamic formalism for such systems are developed, including criteria for compactness of transfer operators acting on spaces of bounded holomorphic functions. In particular a new sufficient condition for the existence of a T-invariant probability measure equivalent to Lebesgue measure is obtained.

1. *Introduction*

Let X be a compact connected subset of $\mathbb{R}^d$ with non-empty interior, and suppose that $T : X \to X$ is a real analytic full branch expanding map (see Definition 4.1). Let $(T_i)_{i\in\mathcal{I}}$ be the (countable) collection of inverse branches of T, and suppose the sets $X_i = T_i(int(X))$ form a partition of X up to a set of zero Lebesgue measure.

By an *acip* (absolutely continuous invariant probability) we mean a T-invariant Borel probability measure on X which is absolutely continuous with respect to Lebesgue measure *Leb*. If T has only finitely many branches then it is well known that there exists a unique acip μ, that the associated density function $d\mu/dLeb$ is real analytic, and that the corresponding dynamical system (T, μ) is exact. The main purpose of this article is to give a sufficient condition for the same facts to hold in the case where $\mathcal{I} = \mathbb{N}$ is countably infinite.

We say that T has *uniformly summable derivatives* if there exists a complex neighbourhood $D \subset \mathbb{C}^d$ of X such that

$$\sup_{z\in D} \sum_{i=n}^{\infty} \|T_i'(z)\| \to 0 \quad \text{as } n \to \infty \,,$$

where $T_i'(z)$ denotes the derivative of T_i at z, and $\| \cdot \|$ is any norm on $\mathbb{C}^{d\times d}$, for example the operator norm induced by the Euclidean norm on $\mathbb{C}^d$.

Our main result (see Theorem 11.4) is:

Theorem. *Let $T : X \to X$ be a real analytic full branch expanding map with uniformly summable derivatives, such that $Leb(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$.*

Then T has a unique acip μ. The corresponding density function is real analytic and strictly positive on X. The dynamical system (T, μ) is exact.

The uniformly summable derivatives condition turns out to be independent from a well known alternative sufficient condition for the existence of an acip, the bounded distortion condition (see e.g. [**1**], [**9**], [**11**]): in [**3**] we construct maps with uniformly summable derivatives but unbounded distortion, as well as maps with bounded distortion but without uniformly summable derivatives.

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The uniformly summable derivatives condition is often easy to check: for example it is implied by the condition

$$\sum_{i=1}^{\infty} \sup_{z\in D} \|T_i'(z)\| < \infty \,. \tag{1.1}$$

Another advantage of the uniformly summable derivatives condition is that the proof of the above Theorem can be carried out wholly within a complex analytic framework, so that the real analyticity of the density function $d\mu/dLeb$ follows immediately from the existence of μ.

In fact the hypotheses of this Theorem can be weakened (see Theorem 11.2): it suffices to assume that $\limsup_{i\to\infty} \sup_{z\in D} \|T_i'(z)\| < 1$ and

$$\sup_{z\in D} \sum_{i=1}^{\infty} |\mathrm{Jac}(T_i)(z)| < \infty \,, \tag{1.2}$$

where $\mathrm{Jac}(T_i)$ denotes the Jacobian determinant of T_i.

A more abstract analogue of these results is Theorem 11.1. Here we assume that D may be chosen to be invariant under complex conjugation, that it is mapped compactly inside itself (in a suitable sense) by the inverse branches T_i, and that the $\mathrm{Jac}(T_i)$ satisfy a suitable summability condition, either that

$$\sup_{z\in D} \sum_{i=n}^{\infty} |\mathrm{Jac}(T_i)(z)| \to 0 \text{ as } n \to \infty,$$

or the weaker condition (1.2). These abstract versions resemble a stronger theorem claimed by Mayer [**10**, Theorem, p. 12]. The proof presented in [**10**] is incorrect, however (see the comments in our §3, in particular Remark 3.2), and the claimed theorem should be considered an open problem[†].

Our methods owe much to the approach of Mayer. His idea was to check that the Perron-Frobenius operator $\mathcal{L}_T$, when acting on a suitable Banach space of analytic functions, is both compact and f_0-positive (see Definition 4.8) with respect to a certain cone. Work of Krasnosel'skiĭ [**8**] then implies that $\mathcal{L}_T$ has a unique fixed point ϱ such that $\varrho > 0$ and $\int \varrho \, dLeb = 1$. It follows that T has a unique acip μ, and that $d\mu/dLeb = \varrho$. Our approach, therefore, is to determine conditions on T which imply that $\mathcal{L}_T$ is both compact and f_0-positive. In fact this same strategy can be used to prove existence and uniqueness of invariant measures absolutely continuous with respect to certain more general reference measures (see §10): in this case $\mathcal{L}_T$ is replaced by a more general transfer operator $\mathcal{L}$ of the form $\mathcal{L}f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$, where the weight functions w_i are real analytic and strictly positive on X. An alternative approach is to view the transfer operator $\mathcal{L}$, defined in terms of the w_i (or, equivalently, in terms of a *potential function* φ, cf. §8), as the primary object of interest. Suitable hypotheses guarantee that $\mathcal{L}$ has an eigenmeasure m (the reference measure), and a strictly positive real analytic eigenfunction ϱ with $\int \varrho \, dm = 1$, and that the probability measure μ defined by $d\mu/dm = \varrho$ is T-invariant. In this generality our main results are Theorems 7.5 and 9.4. In this context, previous criteria for the existence of invariant measures have been formulated by various authors, notably Mauldin & Urbański [**12**], Sarig [**18**],

[†]Our Proposition 3.1 casts doubt on a key lemma in [**10**, Lem. 3, p. 11]. Even if this lemma is false, however, it may be possible to prove [**10**, Theorem, p. 12] by other means.

and Walters [**23**]. In Appendix B we present a simple example where the criteria of this paper are satisfied but those of [**12**], [**18**], [**23**] are not.

Compactness of the transfer operator $\mathcal{L}$ was first proved by Ruelle [**17**] in the case where T has finitely many branches. The situation is more delicate in the infinite branch case (this is where the incorrectness in [**10**] arises), and it turns out that there are two useful sufficient conditions for compactness of $\mathcal{L}$ (Propositions 2.7 and 2.8). These conditions are most naturally formulated in the context of holomorphic map-weight systems (see Definition 2.2), where the weight functions w_i are assumed to be holomorphic (and bounded) on the complex domain D, but the set X plays no role. Proposition 3.1 shows that a certain weakening of our two compactness criteria is *not* sufficient to guarantee compactness of $\mathcal{L}$.

The f_0-positivity of $\mathcal{L}$ follows from the strict positivity of the weight functions w_i on X. Our proof of this fact (Proposition 4.9) is simpler than the proof of the analogous result in [**10**], largely by virtue of working with a slightly different cone. Provided the domain D is invariant under complex conjugation, the spectral properties of $\mathcal{L}$ mentioned above can be deduced from [**8**] (see Proposition 5.4).

From a technical standpoint some of our methods may be of independent interest: for example the proofs of existence (Lemma 6.4) and uniqueness (Proposition 7.1) of the eigenmeasure hinge on general properties of positive operators, and differ from the classical proofs in thermodynamic formalism (see e.g. [**16**], [**12**], [**18**]).

It should be possible to generalise the main results of this paper to certain cases where the real analytic expanding map T is Markov, but not necessarily full branch. We do not pursue this generalisation here, however, preferring to present the main ideas in the simplest possible combinatorial setting.

Notation 1.1. All the results in this article are valid if the set $\mathcal{I}$ indexing the inverse branches of T is countable. Our principle interest, however, is in the case where $\mathcal{I}$ is infinite, and it will sometimes be notationally convenient to assume that $\mathcal{I} = \mathbb{N}$. For consistency we then adopt the convention that, if T has finitely many branches, the weight functions w_i (see Definition 2.2) and the norms $\|T_i'(z)\|$ are defined to be identically zero for all sufficiently large $i \in \mathbb{N}$.

The notation $\int f\, dm$ and $m(f)$ will be used interchangeably to denote the integral of a function f with respect to a measure m.

2. *Transfer operators for holomorphic map-weight systems*

Notation 2.1. If $(B, \|\cdot\|_B)$ is a Banach space, we write $\|\cdot\|$ instead of $\|\cdot\|_B$ whenever this does not lead to confusion. For X a compact metric space, and $(B, \|\cdot\|)$ a Banach space, let $C(X, B)$ denote the set of continuous functions from X to B. This is a Banach space when equipped with the norm $\|f\|_{C(X,B)} = \max_{x\in X} \|f(x)\|$. If $D \subset \mathbb{C}^d$ is a domain (a non-empty connected open subset of $\mathbb{C}^d$), let $H^\infty(D, B)$ denote the collection of functions $f : D \to B$ which are holomorphic on D with $\|f\|_{H^\infty(D,B)} := \sup_{z\in D} \|f(z)\| < \infty$. The space $(H^\infty(D, B), \|\cdot\|_{H^\infty(D,B)})$ is a complex Banach space. In the case where $(B, \|\cdot\|) = (\mathbb{C}, |\cdot|)$ we use $C(X)$ to denote $C(X, \mathbb{C})$, and $H^\infty(D)$ to denote $H^\infty(D, \mathbb{C})$. We use $L(B)$ to denote the space of bounded linear operators from a Banach space $(B, \|\cdot\|)$ to itself, which we always assume to be equipped with the induced operator norm.

Definition 2.2. Let $D \subset \mathbb{C}^d$ be a bounded domain, and let $\mathcal{I}$ be a non-empty countable set.

(i) A *holomorphic map system* (on D) is a collection $(T_i)_{i\in\mathcal{I}}$ (also denoted $(T_i, D)_{i\in\mathcal{I}}$) of holomorphic maps $T_i : D \to D$.

(ii) A *holomorphic weight system* (on D) is a collection $(w_i)_{i\in\mathcal{I}}$ (also denoted $(w_i, D)_{i\in\mathcal{I}}$) of functions $w_i \in H^\infty(D)$. The w_i are called *weight functions*.

(iii) If $(T_i)_{i\in\mathcal{I}}$ is a holomorphic map system and $(w_i)_{i\in\mathcal{I}}$ is a holomorphic weight system then $(T_i, w_i)_{i\in\mathcal{I}}$ (also denoted $(T_i, w_i, D)_{i\in\mathcal{I}}$) is called a *holomorphic map-weight system*.

For two subsets $D, D' \subset \mathbb{C}^d$, we write $D' \Subset D$ to mean that D' is *compactly contained* in D, i.e. that $\overline{D'}$ is a compact subset of D. For future reference we introduce the following possible conditions on a map system $(T_i)_{i\in\mathcal{I}}$:

$$\text{(D1)} \qquad T_i(D) \Subset D \quad \text{for all } i \in \mathcal{I}\,,$$

$$\text{(D2)} \qquad \bigcup_{i\in\mathcal{I}} T_i(D) \Subset D\,.$$

The following summability conditions on the weight system $(w_i)_{i\in\mathcal{I}}$ will also be used in the sequel:

$$\text{(S1)} \qquad \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)| < \infty\,,$$

$$\text{(S2)} \qquad \sup_{z\in D} \sum_{i=n}^{\infty} |w_i(z)| \to 0 \quad \text{as } n \to \infty\,.$$

Remark 2.3.

(a) Clearly (D2) $\Rightarrow$ (D1) and (S2) $\Rightarrow$ (S1).

(b) If $\mathcal{I}$ is finite then both (S1) and (S2) are trivially satisfied.

(c) If $\mathcal{I} = \mathbb{N}$ then we have the following equivalences (here $\ell^1(\mathbb{N})$ denotes the space of absolutely summable sequences with its usual topology):

$$\text{(S1)} \Leftrightarrow \{\, (w_n(z))_{n\in\mathbb{N}} \mid z \in D \,\} \text{ is a bounded subset of } \ell^1(\mathbb{N})\,,$$

$$\text{(S2)} \Leftrightarrow \{\, (w_n(z))_{n\in\mathbb{N}} \mid z \in D \,\} \text{ is a relatively compact subset of } \ell^1(\mathbb{N})\,,$$

The first equivalence is obvious. The second is by [**5**, IV.13, Ex. 3, pp. 338–9]).

With each holomorphic map-weight system we wish to associate a linear operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$, called a *transfer operator*, defined by the formula

$$\mathcal{L}f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i\,. \tag{2.1}$$

Operators of this kind were first considered by Ruelle in the case where $\mathcal{I}$ is finite (see [**16**] for their first introduction, and [**17**] for their first use in a holomorphic context). If $\mathcal{I}$ is infinite, it is not obvious whether (2.1) produces a well-defined endomorphism of $H^\infty(D)$. The following result gives a sufficient condition for this to be the case, at the same time ensuring the continuity of $\mathcal{L}$.

PROPOSITION 2.4. *If the holomorphic map-weight system $(T_i, w_i, D)_{i\in\mathcal{I}}$ satisfies (S1) then (2.1) defines a bounded linear operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$.*

Proof. Let $S := \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)| < \infty$. First we show that $\mathcal{L}$ maps $H^\infty(D)$ to $H^\infty(D)$. Fix $f \in H^\infty(D)$. Without loss of generality let $\mathcal{I} = \mathbb{N}$ (cf. Notation 1.1). If $g_k(z) := \sum_{i=1}^k w_i(z)f(T_i z)$ for $k \in \mathbb{N}$, then $g_k \in H^\infty(D)$. Since

$$|g_k(z)| \le \sum_{i=1}^{k} |w_i(z)||f(T_i z)| \le S\,\|f\|_{H^\infty(D)} \tag{2.2}$$

for all $z \in D$, we see that the sequence $\{g_k\}$ is uniformly bounded on D. Moreover, $\lim_{k\to\infty} g_k(z) =: g(z)$ exists for every $z \in D$. By Vitali's convergence theorem (see e.g. [**14**, Prop. 7]) g_k thus converges uniformly on compact subsets of D. Hence g is analytic on D. Moreover, by (2.2) we see that $|g(z)| \le S\|f\|$ for any $z \in D$. Thus $\mathcal{L}f = g \in H^\infty(D)$ and $\|\mathcal{L}f\|_{H^\infty(D)} \le S\,\|f\|_{H^\infty(D)}$, and the assertion follows. □

NOTATION 2.5. Let $(T_i, w_i, D)_{i\in\mathcal{I}}$ be a holomorphic map-weight system. For $\underline{i} \in \mathcal{I}^n$, $n \in \mathbb{N}$ we write $T_{\underline{i}} := T_{i_n} \circ \cdots \circ T_{i_1}$ and $w_{\underline{i}} := \prod_{k=1}^n w_{i_k} \circ T_{P_{k-1}\underline{i}}$, where for $k \in \mathbb{N}$, $P_k : \mathcal{I}^n \to \mathcal{I}^k$ denotes the projection $P_k\underline{i} = (i_1, \ldots, i_k)$ onto the first k coordinates, with the convention that $T_{P_0\underline{i}} = \mathrm{id}$.

REMARK 2.6. If $(T_i, w_i, D)_{i\in\mathcal{I}}$ is a holomorphic map-weight system satisfying (S1) and $\mathcal{L}$ the corresponding transfer operator, then it is easily seen that, for n a positive integer, $(T_{\underline{i}}, w_{\underline{i}}, D)_{i\in\mathcal{I}^n}$ is a holomorphic map-weight system satisfying (S1) whose transfer operator is $\mathcal{L}^n$; or, put differently, $\mathcal{L}^n f = \sum_{\underline{i}\in\mathcal{I}^n} w_{\underline{i}} \cdot f \circ T_{\underline{i}}$ for every $f \in H^\infty(D)$, with the sum converging in $H^\infty(D)$.

We now consider the possible *compactness* of the transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$. If $\mathcal{I}$ is finite, it is well known that $\mathcal{L}$ is compact[†] whenever (D1), or equivalently (D2), is satisfied. When $\mathcal{I}$ is infinite, conditions (D1) and (D2) are not equivalent, because (D1) does not preclude the accumulation of the sets $T_i(D)$ on the boundary of D. Moreover, neither condition (D1) nor (D2) is alone sufficient to guarantee the compactness of $\mathcal{L}$. We shall see, however, that combining (D1) or (D2) with an appropriate summability condition on the weights w_i will yield a criterion for compactness.

PROPOSITION 2.7. *For a holomorphic map-weight system $(T_i, w_i, D)_{i\in\mathcal{I}}$ satisfying (D2) and (S1), the transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ is compact.*

Proof. Let D' be a domain containing $\cup_{i\in\mathcal{I}} T_i(D)$ and such that $D' \Subset D$. Arguments analogous to those of Proposition 2.4 show that $\mathcal{L}(H^\infty(D')) \subset H^\infty(D)$, and that $\hat{\mathcal{L}} : H^\infty(D') \to H^\infty(D)$ defined by $\hat{\mathcal{L}} = \mathcal{L}|_{H^\infty(D')}$ is bounded (since $\left\|\hat{\mathcal{L}}f\right\|_{H^\infty(D)} \le \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)|\,|f(T_i z)| \le (\sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)|)\,\|f\|_{H^\infty(D')}$). The canonical embedding $J : H^\infty(D) \hookrightarrow H^\infty(D')$ is compact, by Montel's Theorem [**14**, Chapter 1, Prop. 6]. Since $\mathcal{L} = \hat{\mathcal{L}}J$, it follows that $\mathcal{L}$ is compact. □

[†]This was first observed by Ruelle [**17**], who noted that in fact such an $\mathcal{L}$ is nuclear of order zero.

PROPOSITION 2.8. *For a holomorphic map-weight system $(T_i, w_i, D)_{i\in\mathcal{I}}$ satisfying (D1) and (S2), the transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ is compact.*

Proof. For each $i \in \mathcal{I}$, the operator $\mathcal{L}_i f = w_i \cdot f \circ T_i$ is compact, by Proposition 2.7 applied to the index set $\mathcal{I}_i = \{i\}$. Since the space of compact linear operators $H^\infty(D) \to H^\infty(D)$ is closed in $L(H^\infty(D))$ (see e.g. [**21**, V.7.1, V.7.2]), it suffices to prove that the sum $\sum_{i\in\mathcal{I}} \mathcal{L}_i$ is convergent with respect to this norm.

We may write $\mathcal{I} = \mathbb{N}$ (cf. Notation 1.1). Then

$$\left\|\sum_{i=n}^{\infty} \mathcal{L}_i\right\| = \sup_{\|f\|_{H^\infty(D)}=1} \sup_{z\in D} \left|\sum_{i=n}^{\infty} w_i(z) f(T_i z)\right| \le \sup_{z\in D} \sum_{i=n}^{\infty} |w_i(z)| ,$$

and (S2) implies that this quantity tends to 0 as $n \to \infty$. □

3. *A non-compact transfer operator*

In view of Propositions 2.7 and 2.8, it is natural to wonder whether the transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ might be compact if the holomorphic map-weight system $(T_i, w_i, D)_{i\in\mathcal{I}}$ merely satisfies (D1) and (S1). In [**10**, Lem. 3] a claim to this effect is made†, but the proof is flawed (see Remark 3.2). The following result shows that (D1) and (S1) are not sufficient to guarantee the compactness of $\mathcal{L}$.

PROPOSITION 3.1. *There is a holomorphic map-weight system $(T_i, w_i, D)_{i\in\mathcal{I}}$ satisfying (D1) and (S1), but whose transfer operator is not compact.*

Proof. Let D be the unit disc. Below we shall construct $w_n \in H^\infty(D)$ and $z_n \in D$, where $n \in \mathbb{N}$, such that

$$w_n(z_n) = 1 \quad \text{for every } n \in \mathbb{N}, \tag{3.1}$$

$$\sum_{n\in\mathbb{N}\setminus\{m\}} |w_n(z_m)| \le \frac{1}{2} \quad \text{for every } m \in \mathbb{N}, \tag{3.2}$$

$$\sup_{z\in D} \sum_{n\in\mathbb{N}} |w_n(z)| < \infty . \tag{3.3}$$

Defining $T_n(z) = z_n$ for each $n \in \mathbb{N}$, it is not difficult to see that $(T_n, w_n, D)_{n\in\mathbb{N}}$ satisfies (D1) and (S1). We shall now show that the associated transfer operator $\mathcal{L}$ is not compact. To see this let $\ell^\infty(\mathbb{N})$ denote the space of bounded complex sequences, equipped with its usual norm. The map $A : H^\infty(D) \to \ell^\infty(\mathbb{N})$ defined by $A(f) = (f(z_n))_{n\in\mathbb{N}}$ is bounded, with norm 1. The map $B : \ell^\infty(\mathbb{N}) \to H^\infty(D)$ defined by $B((b_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} b_n w_n$ is bounded by (3.3). The transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ can be factorised as $\mathcal{L} = BA$. A calculation shows that

$$\|AB - I\|_{\ell^\infty(\mathbb{N})} \le \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}\setminus\{m\}} |w_n(z_m)| \le \frac{1}{2} ,$$

†The claim is made for a particular choice of weight system, though the purported proof only uses the fact that the weight functions all lie in a suitable space of analytic functions.

by (3.1) and (3.2), so AB has a continuous inverse on $l^\infty(\mathbb{N})$, hence the same is true for $(AB)^2$. So $(AB)^2$ is not compact (for otherwise $I = (AB)^2(AB)^{-2}$ would be compact), hence neither is $BA = \mathcal{L}$ (or else $(AB)^2 = A(BA)B$ would be compact).

It remains to construct the weights w_n and the points z_n. For convenience, we shall first work in the upper half-plane $H^+ = \{ z : \Im z > 0 \} \subset \mathbb{C}$.

For $n \in \mathbb{N}$, let $a_n = n + i \cdot 2^{-(n+2)} \in H^+$ and define

$$\tilde{w}(z) = \frac{a_n - \overline{a_n}}{z - \overline{a_n}}\,.$$

Then $|\tilde{w}(z)| \le |(a_n - \overline{a_n})/\Im a_n| = 2$ for any $z \in H^+$ and

$$\tilde{w}_n(a_n) = 1\,. \tag{3.4}$$

If $n, m \in \mathbb{N}$, $m \neq n$, we have $|\tilde{w}_n(a_m)| \le |a_n - \overline{a_n}| = 2^{-(n+1)}$, and thus

$$\sum_{n\in\mathbb{N}\setminus\{m\}} |\tilde{w}_n(a_m)| \le 1/2\,. \tag{3.5}$$

For $z \in H^+$ let $p \in \mathbb{Z}$ denote the integer part of $\Re z$, that is, $\Re z \in [p, p+1)$. Then $|\tilde{w}_n(z)| \le 2$ for $n = p$ or $n = p+1$ and $|\tilde{w}_n(z)| \le 2^{-(n+1)}$ otherwise, and hence

$$\sum_{n\in\mathbb{N}} |\tilde{w}_n(z)| \le 9/2 \quad \text{for any } z \in H^+. \tag{3.6}$$

Defining $w_n = \tilde{w}_n \circ M^{-1}$ and $z_n = M(a_n)$, where $M : H^+ \to D$ denotes the Möbius transformation $M(z) = (z - i)/(z + i)$, the desired properties (3.1), (3.2), and (3.3) now follow from the corresponding relations (3.4), (3.5), and (3.6). □

Remark 3.2.

(a) In the above counter-example the T_n are constant mappings. With some extra effort it possible to construct counter-examples where the T_n are open.

(b) In [**10**, Lem. 3, p. 11] it is claimed[†] that if (D1) and (S1) hold then $\mathcal{L}$ is nuclear of order zero, and hence compact. The argument for this is incorrect, however, as it hinges on the assertion that $\mathcal{L}$ maps the space $\mathcal{H}(D)$ of holomorphic functions on D into the space $H^\infty(D)$. This is false in general: for example if D is the unit disc, $w_n \equiv 2^{-n}$, and

$$T_n(z) = \frac{1 - 4^{-n}}{1 - 4^{-(n+1)}}\, z\,,$$

then the associated transfer operator $\mathcal{L}$ satisfies (D1) and (S1), but the image under $\mathcal{L}$ of the function $f(z) = 1/(1-z)$ does not belong to $H^\infty(D)$.

A second argument in [**10**, Lem. 3, pp. 11–12] is used to support the claim that some iterate of a transfer operator acting on a space of vector-valued holomorphic functions is compact. This argument is also false due to an incorrect application of a version of Montel's theorem.

[†]The claim is made in the case where w_i is the Jacobian determinant of T_i, and $(T_i)_{i\in\mathcal{I}}$ are inverse branches of an expanding map (i.e. the same context as in our §11), though only the analytic properties of the w_i are actually used in the purported proof. There is an additional hypothesis (see [**10**, (A1), p. 4]) that the w_i are holomorphic on some domain D' which compactly contains D, and that (S1) holds on D', though this hypothesis is not used anywhere in [**10**].

4. *Real analytic dynamics*

Having considered holomorphic map-weight systems, we now look at the interplay with the *real* structure.

Definition 4.1. Henceforth X will always denote a compact connected subset of $\mathbb{R}^d$ with non-empty interior. Let $\{X_i\}_{i\in\mathcal{I}}$ be a finite or countably infinite family of non-empty pairwise disjoint subsets of X such that each X_i is open in $\mathbb{R}^d$ and $\overline{\cup_{i\in\mathcal{I}}X_i} = X$. Suppose that $T: X \to X$ is Borel measurable, and such that for all $i \in \mathcal{I}$, $T(X_i)$ is open in $\mathbb{R}^d$, and $T|_{X_i} : X_i \to T(X_i)$ is a C^1 diffeomorphism which can be extended to a C^1 map on $\overline{X_i}$.

We shall say that T is *full branch* if $\overline{T(X_i)} = X$ for all $i \in \mathcal{I}$.

The map T is called *(uniformly) expanding* if there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$, and $\beta > 1$, such that, for any x, y which lie in the same partition element X_i,

$$\|T(x) - T(y)\| \geq \beta \|x - y\| \ . \tag{4.1}$$

For any partition element X_i, the restriction $T|_{X_i}$ is called a *branch* of T. If T is a full branch expanding map then each branch $T|_{X_i}$ has an inverse T_i such that $T \circ T_i$ is the identity map on the interior of X, and $T_i \circ T$ is the identity map on X_i. The maps T_i will be referred to as *inverse branches*. Condition (4.1) implies

$$\sup_{x\in int(X)} \|T_i'(x)\|_{L(\mathbb{R}^d)} \leq \beta^{-1} \quad \text{for all } i \in \mathcal{I}\,, \tag{4.2}$$

where $T_i'(x)$ denotes the derivative of T_i at the point x, and $\|\cdot\|_{L(\mathbb{R}^d)}$ denotes the induced operator norm on $L(\mathbb{R}^d) = L((\mathbb{R}^d, \|\cdot\|))$.

If $n \geq 1$, and $\underline{i} = (i_1, \dots, i_n) \in \mathcal{I}^n$, we write $X_{\underline{i}} := T_{\underline{i}}(int(X))$.

Lemma 4.2. *Let $T : X \to X$ be a full branch expanding map. For any non-empty open subset U of X, there exists $n \geq 1$ and $\underline{i} \in \mathcal{I}^n$ such that $T_{\underline{i}}(int(X)) \subset U$.*

Proof. Fix $\varepsilon > 0$ such that there is an open ball B of radius ε contained in U. Now $diam(X_{\underline{i}}) \leq \beta^{-n} diam(X)$ for all $\underline{i} \in \mathcal{I}^n$, and all $n \geq 1$, so we may fix $n \geq 1$ such that $diam(X_{\underline{i}}) < \varepsilon/2$ for all $\underline{i} \in \mathcal{I}^n$. The union $\cup_{i\in\mathcal{I}}X_i$ is open and dense in X, and therefore so is $\cup_{\underline{i}\in\mathcal{I}^n}X_{\underline{i}}$. So there exists $\underline{i} \in \mathcal{I}^n$ and $x \in X_{\underline{i}}$ such that the distance of x to the centre of B is less than $\varepsilon/2$. But since $diam(X_{\underline{i}}) < \varepsilon/2$, all of $X_{\underline{i}}$ must belong to B. Therefore $X_{\underline{i}} \subset B \subset U$, as required. □

Definition 4.3. A full branch expanding map $T : X \to X$ will be called *real analytic* if there is a bounded domain $D \subset \mathbb{C}^d$ (in particular D is connected), with $X \subset D$, such that each inverse branch T_i has a holomorphic extension to D. A real analytic full branch expanding map will often be denoted by (T, X, D).

For a real analytic full branch expanding map $T : X \to X$, the holomorphic extensions of the T_i to D, and in particular to X, will also be denoted T_i. With this convention, we have the following obvious strengthening of Lemma 4.2.

Corollary 4.4. *Let $T : X \to X$ be a real analytic full branch expanding map. For any non-empty open $U \subset X$, there exists $n \geq 1$, $\underline{i} \in \mathcal{I}^n$ such that $T_{\underline{i}}(X) \subset U$.*

DEFINITION 4.5. Let $\mathcal{I}$ be a non-empty countable set. A collection $(w_i)_{i\in\mathcal{I}}$ of functions $w_i : X \to \mathbb{R}$ is called a *real analytic weight system* if there is a bounded domain $D \subset \mathbb{C}^d$ with $X \subset D$, such that $(w_i, D)_{i\in\mathcal{I}}$ is a holomorphic weight system. If D is any such domain then the system will sometimes be denoted by $(w_i, X, D)_{i\in\mathcal{I}}$.

DEFINITION 4.6. Define† $H^\infty_{\mathbb{R}}(D) := \{ f \in H^\infty(D) : f(x) \in \mathbb{R} \text{ for } x \in X\}$, a real Banach space when equipped with the norm $\|f\|_{H^\infty_{\mathbb{R}}(D)} = \sup_{z\in D} |f(z)| = \|f\|_{H^\infty(D)}$. Define $K := \{ f \in H^\infty_{\mathbb{R}}(D) : f(x) \geq 0 \text{ for } x \in X \}$. Now X has non-empty interior as a subset of $\mathbb{R}^d$, so is a set of uniqueness in $\mathbb{C}^d$ (i.e. a holomorphic function on D which vanishes on X is identically zero on D), which implies that $K \cap -K = \{0\}$. Moreover K is closed, and is such that $\alpha f + \beta g \in K$ whenever $f, g \in K$ and $\alpha, \beta \geq 0$. So K is a *cone* (see [**8**, p. 17]). For $f, g \in H^\infty_{\mathbb{R}}(D)$ we write $f \leq g$ to mean that $g - f \in K$, and this defines a partial order on $H^\infty_{\mathbb{R}}(D)$.

If (T, X, D) is a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and $(w_i, X, D)_{i\in\mathcal{I}}$ is a real analytic weight system satisfying (S1), then the transfer operator $\mathcal{L}$ defined by (2.1) is an endomorphism of $H^\infty(D)$, by Proposition 2.4. Since each w_i is real-valued on X, $\mathcal{L}$ is also an endomorphism of $H^\infty_{\mathbb{R}}(D)$.

Henceforth we shall require some kind of positivity assumption on $(w_i)_{i\in\mathcal{I}}$.

DEFINITION 4.7. A real analytic weight system $(w_i)_{i\in\mathcal{I}}$ is *positive* if $w_i(x) \geq 0$ for all $x \in X$, $i \in \mathcal{I}$, and *strictly positive* if $w_i(x) > 0$ for all $x \in X$, $i \in \mathcal{I}$.

These positivity assumptions on the weight system will lead to positivity properties, defined below, of the transfer operator.

DEFINITION 4.8. Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic weight system satisfying (S1). The transfer operator $\mathcal{L} : H^\infty_{\mathbb{R}}(D) \to H^\infty_{\mathbb{R}}(D)$ is said to be *positive* if $\mathcal{L}(K) \subset K$. It is called *f_0-positive* if there is a non-zero $f_0 \in K$ such that for every $f \in K \setminus \{0\}$ there exist $\beta > \alpha > 0$ and $n \in \mathbb{N}$ such that $\alpha f_0 \leq \mathcal{L}^n f \leq \beta f_0$.

PROPOSITION 4.9. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic weight system satisfying (S1). If $\mathcal{L} : H^\infty_{\mathbb{R}}(D) \to H^\infty_{\mathbb{R}}(D)$ is the corresponding transfer operator then*

(i) *$(w_i)_{i\in\mathcal{I}}$ positive $\Rightarrow$ $\mathcal{L}$ positive,*

(ii) *$(w_i)_{i\in\mathcal{I}}$ strictly positive $\Rightarrow$ $\mathcal{L}$ f_0-positive.*

Proof. (i) Since $w_i(x) \geq 0$ for all $x \in X$ and $i \in \mathcal{I}$, if $f \in K$ then $\mathcal{L}f(x) = \sum_{i\in\mathcal{I}} w_i(x) f(T_i x) \geq 0$ for all $x \in X$. So $\mathcal{L}f \in K$.

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†This definition differs from the real Banach space $\{f \in H^\infty(D) : f(x) \in \mathbb{R} \text{ for } x \in D \cap \mathbb{R}^d\}$ used in [**10**, p. 10], and leads to a shorter proof of f_0-positivity of $\mathcal{L}$ (see Prop. 4.9 (ii)). The two spaces coincide in the case where $D \cap \mathbb{R}^d$ is connected.

(ii) Let $f_0 \equiv 1$ be the function which is constant and equal to 1 on X. To prove f_0-positivity it is sufficient [**8**, Thm. 2.2] to check that $\mathcal{L}$ is both f_0-*bounded below* (i.e. for all $f \in K \setminus \{0\}$ there exist $\gamma > 0$ and $l \in \mathbb{N}$ such that $\mathcal{L}^l f \geq \gamma f_0$) and f_0-*bounded above* (i.e. for all $f \in K \setminus \{0\}$ there exist $\delta > 0$ and $m \in \mathbb{N}$ such that $\mathcal{L}^m f \leq \delta f_0$). $\mathcal{L}$ is clearly f_0-bounded above. To see that $\mathcal{L}$ is f_0-bounded below, fix $f \in K \setminus \{0\}$ and note that since f is not identically zero, there must be a non-empty set $U \subset X$, open in X, such that $f(x) > 0$ for $x \in U$. By Corollary 4.4 there exists $\underline{i} \in \mathcal{I}^n$ such that $T_{\underline{i}}(X) \subset U$. But then $(\mathcal{L}^n f)(x) > 0$ for $x \in X$, since each w_i is *strictly* positive on X. Choosing $\gamma := \min_{x\in X}(\mathcal{L}^n f)(x) > 0$ yields

$$\mathcal{L}^n f \geq \gamma f_0 \,, \tag{4.3}$$

thus $\mathcal{L}$ is f_0-bounded below as well, and hence f_0-positive. □

5. *Positive compact transfer operators*

Definition 5.1. A domain $D \subset \mathbb{C}^d$ is said to be *conjugation-invariant* if it equals $\{\,\overline{z} : z \in D\,\}$, where $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_d)$ denotes the complex conjugate of $z = (z_1, \ldots, z_d)$.

The usefulness of a conjugation-invariant domain D stems from the fact that the complexification (see e.g. [**8**, pp. 73–74]) of $H^\infty_{\mathbb{R}}(D)$ is precisely $H^\infty(D)$:

Lemma 5.2. *If $D \subset \mathbb{C}^d$ is a conjugation-invariant domain then the complexification $H^\infty_{\mathbb{R}}(D) + iH^\infty_{\mathbb{R}}(D)$ of $H^\infty_{\mathbb{R}}(D)$ equals $H^\infty(D)$.*

Proof. The norm in $H^\infty(D)$ clearly coincides with the norm in the complexification of $H^\infty_{\mathbb{R}}(D)$. It now suffices to prove that $H^\infty(D) \subset H^\infty_{\mathbb{R}}(D) + iH^\infty_{\mathbb{R}}(D)$, since the reverse inclusion obviously holds. Note that since D is conjugation-invariant, $z \mapsto \overline{f(\overline{z})}$ is analytic on D whenever f is analytic on D as a consequence of the Cauchy-Riemann equations. Fix $f \in H^\infty(D)$ and define

$$f_1(z) := \frac{1}{2}\left(f(z) + \overline{f(\overline{z})}\right), \qquad f_2(z) := \frac{1}{2i}\left(f(z) - \overline{f(\overline{z})}\right).$$

Clearly $f_1, f_2 \in H^\infty_{\mathbb{R}}(D)$ and $f = f_1 + if_2$, so $H^\infty(D) \subset H^\infty_{\mathbb{R}}(D) + iH^\infty_{\mathbb{R}}(D)$. □

Remark 5.3. It is claimed in [**10**, p. 10] that $H^\infty(D) = H^\infty_{\mathbb{R}}(D) + iH^\infty_{\mathbb{R}}(D)$ for arbitrary domains, but this is not the case. For example if $f(z) = 1/(z-i)$ then $f = f_1 + if_2$ where $f_1(z) = z/(z^2+1)$, $f_2(z) = 1/(z^2+1)$. While the only singularity of f is the pole at $z = i$, the functions f_1, f_2 also have a pole at $z = -i$. So if D is the disc of radius $3/2$ centred at $-i$, say, then $f \in H^\infty(D)$ but $f_1, f_2 \notin H^\infty_{\mathbb{R}}(D)$.

Proposition 5.4. *Let (T, X, D) be a real analytic full branch expanding map, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic weight system. Suppose that D is conjugation-invariant, and that either (S2) and (D1) are satisfied, or (S1) and (D2) are satisfied. Let $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ denote the corresponding transfer operator.*

If $(w_i)_{i\in\mathcal{I}}$ is strictly positive, then there exists a real analytic function $\varrho \in H^\infty_{\mathbb{R}}(D)$ such that $\varrho > 0$ on X, and $\mathcal{L}\varrho = \lambda\varrho$ for some $\lambda > 0$. The eigenvalue λ is simple, with modulus strictly larger than any other eigenvalue of $\mathcal{L}$.

Proof. $\mathcal{L}$ is compact by either Proposition 2.7 or 2.8, and f_0-positive by Proposition 4.9 (ii). Clearly K is a *reproducing* cone (i.e. $H^\infty_{\mathbb{R}}(D) = K - K$), so we can apply Theorems 2.5, 2.10 and 2.13 of [**8**], which assert that the compact f_0-positive operator $\mathcal{L}$, acting on the complexification of $H^\infty_{\mathbb{R}}(D)$, has a positive simple maximal eigenvalue with corresponding eigenvector $\varrho \in K$. Moreover ϱ is strictly positive on X, since by (4.3) it is bounded below by $\gamma > 0$. The result follows because Lemma 5.2 implies that the complexification of $H^\infty_{\mathbb{R}}(D)$ is precisely $H^\infty(D)$. □

6. *Eigenmeasures for the transfer operator*

We start by extending $\mathcal{L}$ to a continuous endomorphism of $C(X)$.

PROPOSITION 6.1. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic weight system satisfying (S1). The transfer operator $\mathcal{L}$ given by (2.1) defines a bounded linear operator $C(X) \to C(X)$.*

Proof. Let $f \in C(X)$. Polynomials are dense in $C(X)$, by the Stone-Weierstrass theorem, so let $\{f_k\}$ be a sequence of polynomials such that $f_k \to f$ in $C(X)$. Each $f_k \in H^\infty(D)$, since D is bounded, so $\mathcal{L}f_k \in H^\infty(D)$, by Proposition 2.4. Now $S := \sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)| \le \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)| < \infty$ by (S1), and

$$\sup_{x\in X} |\mathcal{L}f(x) - \mathcal{L}f_k(x)| \le \sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)|\, |f(T_i x) - f_k(T_i x)| \le S\, \|f - f_k\|_{C(X)}$$

so $\mathcal{L}f_k \to \mathcal{L}f$ in $C(X)$, from which it follows that $\mathcal{L}f \in C(X)$. The operator $\mathcal{L} : C(X) \to C(X)$ is bounded because $\|\mathcal{L}f\|_{C(X)} \le \sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)|\, |f(T_i x)| \le S\, \|f\|_{C(X)}$. □

REMARK 6.2. The weaker summability assumption $\sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)| < \infty$ is not enough to guarantee that the transfer operator $\mathcal{L}$ is an endomorphism of $C(X)$. For example if $X = [0, 1]$ and $w_i(x) := x^i(1 - x)$ for $i \in \mathbb{N} \cup \{0\}$ then $\mathcal{L}1 = \chi_{[0,1)} \notin C(X)$, yet $\sup_{x\in X} \sum_{i=0}^\infty |w_i(x)| = \sup_{x\in X} \sum_{i=0}^\infty w_i(x) = 1 < \infty$.

DEFINITION 6.3. An *eigenmeasure* (for $\mathcal{L}$) is a finite Borel measure m on X for which there exists $\lambda \ge 0$ (the corresponding *eigenvalue*) such that

$$\int \mathcal{L}f\, dm = \lambda \int f\, dm \quad \text{for all } f \in C(X)\,. \tag{6.1}$$

Every positive transfer operator has an eigenmeasure†:

LEMMA 6.4. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic weight system satisfying (S1). Let $\mathcal{L} : C(X) \to C(X)$ denote the corresponding transfer operator.*

If $(w_i)_{i\in\mathcal{I}}$ is positive then there exists an eigenmeasure.

†See [**16**] for an alternative proof of this fact, which could be applied in the case where $(w_i)_{i\in\mathcal{I}}$ is strictly positive.

Proof. Let $K_C := \{ f \in C(X) : f(x) \geq 0 \text{ for } x \in X \}$, a cone with interior in the real Banach space $C_{\mathbb{R}}(X)$ of real-valued continuous functions on X. Since $(w_i)_{i\in\mathcal{I}}$ is positive, $\mathcal{L} : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$ is positive with respect to K_C. By the Riesz representation theorem, the topological dual $C_{\mathbb{R}}(X)'$ of $C_{\mathbb{R}}(X)$ can be identified with the space of signed Borel measures on X, and the dual cone K_C' with the set of Borel measures on X. Let $\mathcal{L}' : C_{\mathbb{R}}(X)' \to C_{\mathbb{R}}(X)'$ denote the adjoint of $\mathcal{L} : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$, given by $(\mathcal{L}'\nu)(f) = \nu(\mathcal{L}f)$ for all $f \in C_{\mathbb{R}}(X)$. Since $\mathcal{L} : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$ is positive and K_C has interior, [**20**, Corollary in Appendix 2.6] now implies that $\mathcal{L}'$ has an eigenvalue $\lambda \geq 0$ with corresponding eigenvector m in the dual cone K_C'. Thus $\mathcal{L}'m = \lambda m$, so m is an eigenmeasure. □

It will often be convenient to iteratively apply the transfer operator $\mathcal{L}$ to functions which are integrable with respect to an eigenmeasure. The following result guarantees that this is possible.

Lemma 6.5. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic positive weight system satisfying (S1). For any eigenmeasure m, the transfer operator $\mathcal{L}$ given by (2.1) defines a bounded linear operator $L^1(X, m) \to L^1(X, m)$. Moreover, $\int \mathcal{L}f\, dm = \lambda \int f\, dm$ for all $f \in L^1(X, m)$.*

Proof. If $f : X \to \mathbb{C}$ then $|\mathcal{L}f| = \big|\sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i\big| \leq \sum_{i\in\mathcal{I}} w_i \cdot |f| \circ T_i = \mathcal{L}(|f|)$, so combining with the eigenmeasure equation (6.1) gives

$$\int |\mathcal{L}f|\, dm \leq \int \mathcal{L}|f|\, dm = \lambda \int |f|\, dm \quad \text{for all } f \in C(X)\,. \tag{6.2}$$

Since $C(X)$ can be canonically identified with a subspace of $L^1(X, m)$, the transfer operator $\mathcal{L} : C(X) \to C(X)$ determines a linear endomorphism $C(X) \to L^1(X, m)$. This endomorphism is bounded with respect to the (incomplete) norm $\|\cdot\|_{L^1(X,m)}$ on $C(X)$, by (6.2). By the B.L.T. theorem [**15**, Thm. I.7] it can therefore be extended (uniquely) to a bounded linear operator from the completion of $(C(X), \|\cdot\|_{L^1(X,m)})$ to $L^1(X, m)$. To finish the proof we now observe that $C(X)$ is dense in $L^1(X, m)$ (see [**4**, Prop. 7.4.2]). We thus conclude that the completion of $(C(X), \|\cdot\|_{L^1(X,m)})$ is precisely $L^1(X, m)$, which, by the preceding argument, implies that $\mathcal{L} : L^1(X, m) \to L^1(X, m)$ is continuous, and that the eigenmeasure equation (6.1) in fact holds for all $f \in L^1(X, m)$. □

The following iteration formula will be very useful.

Lemma 6.6. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches $(T_i)_{i\in\mathcal{I}}$ form a holomorphic map system on D, and let $(w_i, X, D)_{i\in\mathcal{I}}$ be a real analytic positive weight system satisfying (S1). Let m be any eigenmeasure. Suppose that B is a Borel subset of int(X) and that $n \in \mathbb{N}$. Then*

$$(\mathcal{L}^n f) \cdot \chi_B = \mathcal{L}^n (f \cdot \chi_{B_n})$$

holds m-almost everywhere for all $f \in L^1(X, m)$, where $B_n := \cup_{\underline{j}\in\mathcal{I}^n} T_{\underline{j}}(B)$.

Proof. Let $B(X)$ denote the Banach space of all bounded functions, equipped with the sup-norm. If $f \in B(X)$ then

$$\|\mathcal{L}f\|_{B(X)} \le \sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)|\,|f(T_i x)| \le \|f\|_{B(X)} \sup_{x\in X} \sum_{i\in\mathcal{I}} |w_i(x)| < \infty\,.$$

Thus the transfer operator $\mathcal{L}$ given by (2.1) extends to a well-defined continuous endomorphism of $B(X)$. This, together with Remark 2.6, implies that for $n \in \mathbb{N}$ the operator $\mathcal{L}^n$ is given by the series

$$\mathcal{L}^n f = \sum_{\underline{i}\in\mathcal{I}^n} w_{\underline{i}} \cdot f \circ T_{\underline{i}}\,, \tag{6.3}$$

convergent in $B(X)$ for every $f \in B(X)$. Now $\chi_B = \chi_{B_n} \circ T_{\underline{i}}$ for all $\underline{i} \in \mathcal{I}^n$, so (6.3) implies that

$$(\mathcal{L}^n f) \cdot \chi_B = \mathcal{L}^n(f \cdot \chi_{B_n}) \tag{6.4}$$

everywhere for all $f \in B(X)$. But since bounded Borel measurable functions are dense in $L^1(X, m)$, and since $\mathcal{L}$ and multiplication by bounded Borel measurable functions are continuous operators on $L^1(X, m)$, a simple approximation argument shows that (6.4) holds m-almost everywhere for all $f \in L^1(X, m)$. □

7. *Invariant measures*

We saw in §6 that, provided the positive weight system $(w_i)_{i\in\mathcal{I}}$ satisfies (S1), the corresponding transfer operator has an eigenmeasure m. Since any positive multiple of m is also an eigenmeasure, we shall slightly abuse terminology by saying that the eigenmeasure is *unique* if all eigenmeasures are positive multiples of each other. Throughout this section we shall make the following Standing Hypothesis:

Standing Hypothesis for §7. *(T, X, D) is a real analytic full branch expanding map, $(w_i, X, D)_{i\in\mathcal{I}}$ is a strictly positive real analytic weight system, D is conjugation-invariant, and either (S2) and (D1), or (S1) and (D2), are satisfied.*

PROPOSITION 7.1. *Under the Standing Hypothesis, there is a unique eigenmeasure m, and its corresponding eigenvalue is precisely the (strictly positive) maximal eigenvalue of $\mathcal{L} : H^\infty(D) \to H^\infty(D)$. Moreover,*
(i) *m is non-atomic, and its support is the whole of X;*
(ii) *$m(X \setminus \cup_{\underline{i}\in\mathcal{I}^n} X_{\underline{i}}) = 0$ for every $n \in \mathbb{N}$.*

Proof. Let m be an eigenmeasure, with corresponding eigenvalue $\lambda \ge 0$. If $\widetilde{\lambda} > 0$ is the eigenvalue for $\mathcal{L}$ with corresponding eigenfunction $\varrho > 0$ on X (cf. Proposition 5.4), then $\lambda \int \varrho\, dm = \int \mathcal{L}\varrho\, dm = \widetilde{\lambda} \int \varrho\, dm$. But $\int \varrho\, dm \ne 0$, so $\lambda = \widetilde{\lambda}$.

In order to show that there is only one eigenmeasure, write $\mathcal{L}_C$ for $\mathcal{L} : C(X) \to C(X)$, and $\mathcal{L}_H$ for $\mathcal{L} : H^\infty(D) \to H^\infty(D)$. Note that $J\mathcal{L}_H = \mathcal{L}_C J$, where $J : H^\infty(D) \hookrightarrow C(X)$ denotes the canonical embedding. Taking adjoints gives

$$\mathcal{L}'_H J' = J' \mathcal{L}'_C\,. \tag{7.1}$$

Suppose now that m_1 and m_2 are two linearly independent eigenvectors of $\mathcal{L}'_C$ corresponding to the eigenvalue λ, that is, $\mathcal{L}'_C m_i = \lambda m_i$ for $i = 1, 2$. Using (7.1) we see that $J'm_1$ and $J'm_2$ are two eigenvectors of $\mathcal{L}'_H$ corresponding to the eigenvalue

λ. Since J has dense range, J' is injective. Thus $J'm_1$ and $J'm_2$ are also linearly independent, and consequently the eigenspace of $\mathcal{L}'_H$ corresponding to λ must have dimension strictly larger than one. This, however, is a contradiction: the eigenspace of $\mathcal{L}'_H$ corresponding to the maximal eigenvalue λ has dimension 1, because the maximal eigenvalue of $\mathcal{L}_H$ is simple by Proposition 5.4.

(i) If U is a non-empty open subset of X then, by Corollary 4.4, there exists $n \geq 1$ and $\underline{i} \in \mathcal{I}^n$ such that $T_{\underline{i}}(X) \subset U$. But $(w_i)_{i\in\mathcal{I}}$ is strictly positive, so $\mathcal{L}^n \chi_U(x) \geq w_{\underline{i}}(x)\chi_U(T_{\underline{i}}(x)) > 0$ for all $x \in X$, thus $m(U) = \int 1_U \, dm = \lambda^{-n} \int \mathcal{L}^n \chi_U \, dm > 0$, so m is fully supported. To show that m is non-atomic we can introduce an operator $\mathcal{M} : f \mapsto \sum_{i\in\mathcal{I}} (g\cdot f)\circ T_i$, for a certain strictly positive real analytic function g, with the property that m is the eigenmeasure for $\mathcal{M}$, with corresponding eigenvalue 1, and then proceed exactly as in [**23**, Cor. 12 (2), p. 134]. (More precisely, in the language of §8, $g := w\varrho/(\lambda\varrho\circ T)$ is a g-function whose g-measure is m, where w is the w-function corresponding to the weight system $(w_i)_{i\in\mathcal{I}}$).

(ii) The proof of this part is inspired by [**23**, Lem. 9, p. 131]. Fix $k \in \mathbb{N}$. Let μ be the probability measure, equivalent to m, with $d\mu/dm = \varrho$, where $\varrho = \lambda^{-1}\mathcal{L}\varrho > 0$ is as in Proposition 5.4. First we will show that if B is a Borel subset of $int(X)$ then $\mu(B) = \mu(\cup_{\underline{j}\in\mathcal{I}^k} T_{\underline{j}}(B))$. Defining $B_k := \cup_{\underline{j}\in\mathcal{I}^k} T_{\underline{j}}(B)$, Lemmas 6.5 and 6.6 imply that

$$\begin{aligned}
\mu(B) &= \int_B \varrho \, dm = \lambda^{-k} \int_B \mathcal{L}^k \varrho \, dm = \lambda^{-k} \int (\mathcal{L}^k \varrho)\cdot \chi_B \, dm \\
&= \lambda^{-k} \int \mathcal{L}^k (\varrho \cdot \chi_{B_k}) \, dm = \int \varrho \cdot \chi_{B_k} \, dm \\
&= \int_{B_k} \varrho \, dm = \mu(\cup_{\underline{j}\in\mathcal{I}^k} T_{\underline{j}}(B)) \,,
\end{aligned}$$

as required. Setting $B = int(X)$ gives $\mu(int(X)) = \mu(\cup_{\underline{j}\in\mathcal{I}^k} X_{\underline{j}})$. Hence

$$m(int(X)) = m(\cup_{\underline{j}\in\mathcal{I}^k} X_{\underline{j}}) \quad \text{for every } k \in \mathbb{N} \,. \tag{7.2}$$

It therefore remains to show that $m(\partial X) = 0$. Let U be an open ball in X whose closure is disjoint from ∂X. By Lemma 4.2 there exists $n \geq 1$ and $\underline{i} \in \mathcal{I}^n$ such that $T_{\underline{i}}(int(X)) \subset U$, and therefore $T_{\underline{i}}(X) = \overline{T_{\underline{i}}(int(X))} \subset \overline{U}$ is disjoint from ∂X.

In particular, $T_{\underline{i}}(\partial X) \subset int(X)$. Moreover, $T_{\underline{i}}(\partial X)$ is disjoint from $\cup_{\underline{j}\in\mathcal{I}^n} T_{\underline{j}}(int(X))$, because $T_{\underline{i}}(\partial X) = \partial X_{\underline{i}} = \partial(T_{\underline{i}}X)$ is disjoint from $T_{\underline{i}}(int(X)) = int(T_{\underline{i}}X)$, and $\overline{X}_{\underline{i}}$ is disjoint from $X_{\underline{j}} = T_{\underline{j}}(int(X))$ for all $\underline{j} \in \mathcal{I}^n \setminus \{\underline{i}\}$. So

$$T_{\underline{i}}(\partial X) \subset int(X) \setminus \cup_{\underline{j}\in\mathcal{I}^n} T_{\underline{j}}(int(X)) \,. \tag{7.3}$$

Combining (7.2) and (7.3) gives $m(T_{\underline{i}}(\partial X)) = 0$. Therefore, using the formula (6.3) for the n-th iterate of $\mathcal{L}$,

$$\begin{aligned}
0 = \lambda^n m(T_{\underline{i}}(\partial X)) &= \lambda^n \int \chi_{T_{\underline{i}}(\partial X)} \, dm = \int \mathcal{L}^n \chi_{T_{\underline{i}}(\partial X)} \, dm \\
&\geq \int w_{\underline{i}} \cdot \chi_{T_{\underline{i}}(\partial X)} \circ T_{\underline{i}} \, dm \geq \int_{\partial X} w_{\underline{i}} \cdot \chi_{T_{\underline{i}}(\partial X)} \circ T_{\underline{i}} \, dm \\
&= \int_{\partial X} w_{\underline{i}} \, dm \geq W_{\underline{i}} \; m(\partial X) \,,
\end{aligned}$$

where we define $W_{\underline{i}} := \min_{x\in X} w_{\underline{i}}(x) > 0$. So $m(\partial X) = 0$, as required. □

REMARK 7.2. An alternative proof of the uniqueness of m is to show that for all $f \in C(X)$, $\lambda^{-n}\mathcal{L}^n f \to m(f)\varrho$ in $C(X)$ as $n \to \infty$ (see [**16**], [**12**], [**18**]).

A consequence of Lemma 6.6 and Proposition 7.1 is the following transformation property of the eigenmeasure m.

LEMMA 7.3. *Under the Standing Hypothesis, if m denotes the unique eigenmeasure, and λ the corresponding eigenvalue, then for all Borel subsets A of X,*

$$\int_A \mathcal{L}^n f\, dm = \lambda^n \int_{T^{-n}A} f\, dm \quad \text{for all } f \in L^1(X,m),\ n \in \mathbb{N}\,. \tag{7.4}$$

Proof. If A is a Borel subset of X and $n \in \mathbb{N}$ then it is easily seen that

$$T^{-n}A \cap (\cup_{\underline{i}\in\mathcal{I}^n} X_{\underline{i}}) = \cup_{\underline{i}\in\mathcal{I}^n} T_{\underline{i}}(A \cap int(X))\,.$$

Therefore, writing $B := A \cap int(X)$ and $B_n := \cup_{\underline{i}\in\mathcal{I}^n} T_{\underline{i}}(A\cap int(X))$, Proposition 7.1 shows that $m(A) = m(B)$ and $m(T^{-n}A) = m(B_n)$. Thus, for any $f \in L^1(X,m)$,

$$\int_A \mathcal{L}^n f\, dm = \int (\mathcal{L}^n f)\cdot \chi_B\, dm = \int \mathcal{L}^n(f\cdot\chi_{B_n})\, dm = \lambda^n \int f\cdot \chi_{B_n} = \lambda^n \int_{T^{-n}A} f\, dm\,,$$

where the second and third equalities follow from Lemma 6.6 and Lemma 6.5. □

REMARK 7.4. As is clear from the above proof, formula (7.4) is in fact valid whenever (S1) holds and m is an eigenmeasure such that $m(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$.

It is now possible to interpret Proposition 5.4 in terms of T-invariant probability measures absolutely continuous with respect to the eigenmeasure.

THEOREM 7.5. *Under the Standing Hypothesis, there is a unique T-invariant probability measure μ absolutely continuous with respect to the eigenmeasure m. The corresponding Radon-Nikodym derivative $d\mu/dm$ is the eigenvector ϱ given by Proposition 5.4, so in particular is strictly positive on X, real analytic, and extends holomorphically to an element of $H^\infty_{\mathbb{R}}(D)$. The dynamical system (T,μ) is exact.*

Proof. If ϱ is the eigenfunction of $\mathcal{L}$ guaranteed by Proposition 5.4, and m the unique eigenmeasure with $\int \varrho\, dm = 1$, setting $d\mu/dm = \varrho$ defines a probability measure μ. Lemma 7.3 now implies the T-invariance of μ, since for any Borel subset A of X we have $\mu(T^{-1}A) = \int_{T^{-1}A} \varrho\, dm = \lambda^{-1}\int_A \mathcal{L}\varrho\, dm = \int_A \varrho\, dm = \mu(A)$.

The proof of the remaining assertions relies on the fact that

$$\lim_{n\to\infty} \left\|\lambda^{-n}\mathcal{L}^n f - m(f)\varrho\right\|_{L^1(X,m)} = 0 \quad \text{for all } f \in L^1(X,m)\,. \tag{7.5}$$

For this, observe that the spectral properties of $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ imply that

$$\lim_{n\to\infty} \left\|\lambda^{-n}\mathcal{L}^n f - m(f)\varrho\right\|_{H^\infty(D)} = 0 \quad \text{for all } f \in H^\infty(D)\,. \tag{7.6}$$

Now the canonical identification $H^\infty(D) \to L^1(X,m)$ is continuous, and the image of $H^\infty(D)$ is a dense subset of $L^1(X,m)$. This is because the canonical embedding $J_1 : H^\infty(D) \hookrightarrow C(X)$, and the canonical identification $J_2 : C(X) \to L^1(X,m)$, are both contractions, hence continuous, and J_1 has dense range by the Stone-Weierstrass theorem, while J_2 has dense range by [**4**, Prop. 7.4.2]. The desired limit (7.5) now follows from (7.6) using a simple approximation argument.

In order to prove uniqueness of μ, suppose that there is another T-invariant probability measure $\tilde{\mu}$ with $\tilde{\mu} \ll m$. If $\tilde{\varrho} \in L^1(X,m)$ denotes the Radon-Nikodym derivative of $\tilde{\mu}$ with respect to m then $\int_{T^{-1}A} \tilde{\varrho}\, dm = \int_A \tilde{\varrho}\, dm$ for all Borel subsets A of X. Thus, by Lemma 7.3, $\int_A \mathcal{L}\tilde{\varrho}\, dm = \lambda \int_{T^{-1}A} \tilde{\varrho}\, dm = \lambda \int_A \tilde{\varrho}\, dm$ for all Borel subsets A of X, and hence $\mathcal{L}\tilde{\varrho} = \lambda\tilde{\varrho}$ m-almost everywhere, which forces $\tilde{\varrho} = \varrho$ m-almost everywhere, by (7.5). Therefore $\tilde{\mu} = \mu$.

It remains to prove exactness, for which we adapt the argument of [**1**, Thm. 1.3.3]. Suppose to the contrary that (T, μ) is not exact, so that the tail σ-algebra of (T, μ) contains an element A with $\mu(A)\mu(X \setminus A) \neq 0$. Thus there are Borel sets $A_n \subset X$ with $T^{-n}A_n = A$, and $f \in L^1(X,m)$ with $m(f) = 0$ but $\int_A f\, dm > 0$. Hence for all $n \in \mathbb{N}$, $\|\lambda^{-n}\mathcal{L}^n f\|_{L^1(X,m)} \geq \lambda^{-n} \int_{A_n} \mathcal{L}^n f\, dm = \int_{T^{-n}A_n} f\, dm = \int_A f\, dm > 0$, by Lemma 7.3, thereby contradicting (7.5).

□

8. *Thermodynamic formalism*

Here the material of the preceding sections is related to some notions in thermodynamic formalism. In particular this will allow a comparison of the results in this paper to those already in the literature (see Appendix B).

Definition 8.1. $w : \cup_{i\in\mathcal{I}} X_i \to \mathbb{R}$ is called a *(strictly positive) real analytic w-function* if there is a (strictly positive) real analytic weight system $(w_i)_{i\in\mathcal{I}}$ such that $w|_{X_i} = w_i \circ T|_{X_i}$ for each $i \in \mathcal{I}$. A strictly positive real analytic w-function w is called a *real analytic g-function* if $\sum_{i\in\mathcal{I}} w_i(x) = 1$ for all $x \in int(X)$, where $(w_i)_{i\in\mathcal{I}}$ is as above. A function $\varphi : \cup_{i\in\mathcal{I}} X_i \to \mathbb{R}$ is called a *real analytic potential function* if $\varphi = \log w$ for some strictly positive real analytic w-function w, and a *normalised real analytic potential function* if this w is a real analytic g-function.

Remark 8.2.

(a) The terminology *g-function* was introduced by Keane [**7**]. The notion of a *potential function* is standard in thermodynamic formalism, originating from the analogous object in statistical mechanics. The terminology *w-function* is non-standard: functions w playing this role are sometimes called weight functions, but here this nomenclature is reserved for the members w_i of a weight system.

(b) There is a one to one correspondence between (strictly positive) real analytic w-functions w and (strictly positive) real analytic weight systems $(w_i)_{i\in\mathcal{I}}$. Note in particular that if w is a real analytic w-function then for every $i \in \mathcal{I}$, the restriction $w|_{X_i}$ has a holomorphic extension to the complex neighbourhood $D_i := T_i(D)$ of X_i, where D is such that $(w_i, D)_{i\in\mathcal{I}}$ is a holomorphic weight system.

Notation 8.3. Let φ be a real analytic potential function whose corresponding strictly positive real analytic weight system $(w_i)_{i\in\mathcal{I}}$ satisfies (S1) on some bounded domain $D \subset \mathbb{C}^d$ with $X \subset D$. The transfer operator $\mathcal{L} : H^\infty(D) \to H^\infty(D)$ (cf. Proposition 2.4) will be denoted by $\mathcal{L}_\varphi$. By Proposition 6.1 we know that $\mathcal{L} = \mathcal{L}_\varphi$ extends to a bounded linear operator $C(X) \to C(X)$. If either (S2) and (D1) are satisfied, or (S1) and (D2) are satisfied, then let ϱ_φ and λ_φ denote, respectively, the strictly positive eigenvector and its corresponding eigenvalue, for $\mathcal{L}_\varphi$, as guaranteed by Proposition 5.4. Let m_φ denote the unique eigenmeasure for

$\mathcal{L}_\varphi$ which is guaranteed by Proposition 7.1. Let μ_φ denote the measure equivalent to m_φ with $d\mu_\varphi/dm_\varphi = \varrho_\varphi$, and which is T-invariant by Theorem 7.5.

Remark 8.4.

(a) Notice that although the potential function φ is initially defined only on $\cup_{i\in\mathcal{I}} X_i$, it is still possible to make sense of $\mathcal{L}_\varphi$ as a continuous endomorphism of $C(X)$, subject to the hypothesis (S1). This should be compared to the different class of potential functions φ considered by Walters [**23**], which are also initially defined on $\cup_{i\in\mathcal{I}} X_i$, and where again $\mathcal{L}_\varphi$ makes sense as a continuous endomorphism of $C(X)$. The method of achieving this is different however, reflecting the different assumptions on φ: Walters [**23**, Lem. 1] first shows that $\mathcal{L}_\varphi$ determines an endomorphism of the space of uniformly continuous functions on $int(X)$, and then that it extends to a continuous endomorphism of $C(X)$.

(b) If φ is a normalised real analytic potential function then $\mathcal{L}_\varphi 1 = 1$, so that $\lambda_\varphi = 1$ and $\varrho_\varphi = 1$.

(c) If φ is any real analytic potential function whose corresponding strictly positive weight system satisfies either (S2) and (D1), or (S1) and (D2), then $\psi_\varphi = \varphi + \log \varrho_\varphi - \log \varrho_\varphi \circ T - \log \lambda_\varphi$ is a normalised real analytic potential function.

Normalised potential functions, and g-functions, are closely related to the notion of a *g-measure*, as introduced by Keane [**7**]:

Definition 8.5. If w is a real analytic g-function, or equivalently if φ is a normalised potential function, then the unique eigenmeasure m for $\mathcal{L}_\varphi = \mathcal{L}_{\log w}$ is called the corresponding *g-measure*. Note that $m = m_\varphi = \mu_\varphi$, so in particular the g-measure is T-invariant.

Note that every eigenmeasure is a g-measure: if it is an eigenmeasure for $\mathcal{L}_\varphi$ then it is a g-measure for ψ_φ (i.e. for the function $w = e^\varphi \varrho_\varphi/(\lambda_\varphi \varrho_\varphi \circ T)$). Thus the study of g-measures is equivalent to the study of eigenmeasures of transfer operators.

Note as well that if w is a real analytic g-function, so that $\sum_{i\in\mathcal{I}} w_i = 1$ on $int(X)$ for the corresponding strictly positive real analytic weight system $(w_i)_{i\in\mathcal{I}}$, it need not be the case that on some complex neighbourhood D the sum $\sum_{i\in\mathcal{I}} w_i$ is pointwise convergent, so in particular neither (S1) nor (S2) need hold. Therefore such a hypothesis is required in the following result, which is an immediate corollary of Proposition 7.1 formulated in the language of g-measures.

Theorem 8.6. *Let (T, X, D) be a real analytic full branch expanding map, and w a real analytic g-function with corresponding strictly positive real analytic weight system $(w_i)_{i\in\mathcal{I}}$. If D is conjugation-invariant, and either (S2) and (D1), or (S1) and (D2), are satisfied, then there is a unique g-measure for w.*

Now suppose that m is an eigenmeasure. For each $i \in \mathcal{I}$, the formula $m_i(A) = m(T(A \cap X_i))$ defines a finite measure on X whose total mass is the same as for m. Then defining $mT = \sum_{i\in\mathcal{I}} m_i$ gives a σ-finite measure mT (which is finite if and only if $\mathcal{I}$ is finite). We write mT^{-1} for the measure given by $(mT^{-1})(A) = m(T^{-1}A)$ and, for each $i \in \mathcal{I}$, we use mT_i for the measure defined by $(mT_i)(A) = m(T_i A)$.

Part (i) of the following proposition is adapted from [**23**, Lem. 3, Cor. 4].

Proposition 8.7. *Under the Standing Hypothesis of §7, let w be the corresponding w-function, φ the corresponding potential function, $m = m_\varphi$ the corresponding eigenmeasure, and $\lambda = \lambda_\varphi$ the corresponding maximal eigenvalue. Then*

(i) $mT \sim m$ *with* $\frac{dmT}{dm} = \frac{\lambda}{w} = \lambda e^{-\varphi}$;
(ii) $mT_i \sim m$ *for every* $i \in \mathcal{I}$, *with* $\frac{dmT_i}{dm} = \frac{w_i}{\lambda}$;
(iii) $mT^{-1} = \sum_{i\in\mathcal{I}} mT_i$, *and* $\frac{dmT^{-1}}{dm} = \lambda^{-1}\sum_{i\in\mathcal{I}} w_i$.

Proof. (i) First observe that $(X_i)_{i\in\mathcal{I}}$ is a partition (up to sets of mT-measure zero) of X into sets of finite mT-measure. Let A be a Borel subset of X_i for some $i \in \mathcal{I}$. Then $\chi_A/w \in L^1(X,m)$, since w_i is bounded away from zero on X and hence $\int_X \chi_A/w\,dm = \int_{X_i} \chi_A/(w_i \circ T)\,dm < \infty$. Thus, by Lemma 6.5,

$$\int \mathcal{L}_\varphi\left(\frac{\chi_A}{w}\right) dm = \lambda \int \frac{\chi_A}{w}\,dm\,. \tag{8.1}$$

By (8.1), the definition of mT, and the fact that m gives zero mass to ∂X,

$$\begin{aligned}(mT)(A) &= \int \chi_{T(A\cap X_i)}\,dm = \int \chi_A \circ T_i\,dm \\ &= \int \sum_{j\in\mathcal{I}} w\circ T_j \cdot \left(\frac{\chi_A}{w}\right)\circ T_j\,dm = \int \mathcal{L}_\varphi\left(\frac{\chi_A}{w}\right) dm \\ &= \int \lambda\,\frac{\chi_A}{w}\,dm = \int_A \frac{\lambda}{w}\,dm\,.\end{aligned}$$

So λ/w is the Radon-Nikodym derivative dmT/dm, and in particular $mT \ll m$. Moreover, since dmT/dm is bounded away from zero on each X_i then $m \ll mT$ as well. Note that $dmT/dm \in L^1(X_i,m)$ for any $i \in \mathcal{I}$; however, $dmT/dm \in L^1(X,m)$ if and only if $\mathcal{I}$ is finite.

(ii) Fix $i \in \mathcal{I}$. Let A be a Borel subset of X and let χ_i denote the characteristic function of X_i. Then $\mathcal{L}\chi_i = w_i$ on $int(X)$, and hence m-almost everywhere by Proposition 7.1, so by Lemma 7.3,

$$\lambda^{-1}\int_A w_i\,dm = \lambda^{-1}\int_A \mathcal{L}\chi_i\,dm = \int_{T^{-1}A} \chi_i\,dm = m(T^{-1}A\cap X_i)\,. \tag{8.2}$$

Observing that $T^{-1}A \cap X_i = T_i(A \cap int(X)) = T_iA \setminus T_i(A\cap\partial X)$, and that $m(T_i(A\cap\partial X)) = 0$, because $T_i(A\cap\partial X) \subset \partial X_i$ and $m(\partial X_i) = 0$ by Proposition 7.1, we conclude that

$$m(T^{-1}A\cap X_i) = m(T_iA)\,. \tag{8.3}$$

Combining (8.2) and (8.3) now yields

$$\lambda^{-1}\int_A w_i\,dm = m(T_iA)\,.$$

Thus $\lambda^{-1}w_i$ is the Radon-Nikodym derivative of mT_i with respect to m, and in particular $mT_i \ll m$. But $\lambda^{-1}w_i$ is strictly positive on X, so $m \ll mT_i$ as well.

(iii) It suffices to show that $mT^{-1} = \sum_{i\in\mathcal{I}} mT_i$; the formula for dmT^{-1}/dm then follows from (ii). Let A be a Borel subset of X. Then by Proposition 7.1 and (8.3), $m(T^{-1}A) = m(T^{-1}A\cap(\cup_{i\in\mathcal{I}}X_i)) = \sum_{i\in\mathcal{I}} m(T^{-1}A\cap X_i) = \sum_{i\in\mathcal{I}} m(T_iA)$. □

9. *The eventually complex contracting condition*

In view of Theorem 7.5, for a given real analytic full branch expanding map it is useful to find conditions guaranteeing the existence of a domain D which is conjugation-invariant and satisfies either (D1) or (D2). Definition 9.1 below provides such a sufficient condition. To formulate it we first require a norm $\|\cdot\|_{\mathbb{C}^d}$ on $\mathbb{C}^d$: we define $\|\cdot\|_{\mathbb{C}^d}$ to be the norm arising from the complexification (see e.g. [**8**, pp. 73–74]) of $(\mathbb{R}^d, \|\cdot\|)$, where $\|\cdot\|$ is the norm on $\mathbb{R}^d$ used in Definition 4.1. We then equip $L(\mathbb{C}^d)$ with the operator norm induced by $\|\cdot\|_{\mathbb{C}^d}$.

Definition 9.1. If (T, X, D) is a real analytic full branch expanding map with

$$\limsup_{i\to\infty} \|T_i'\|_{H^\infty(D,L(\mathbb{C}^d))} < 1\,, \tag{9.1}$$

we say that its inverse branches are *eventually complex contracting*.

Remark 9.2. If the real analytic full branch expanding map T has only finitely many inverse branches, then clearly they are eventually complex contracting (cf. the convention detailed in Notation 1.1).

In the following result we define $\operatorname{dist}(\zeta, z) := \|\zeta - z\|_{\mathbb{C}^d}$ for $\zeta, z \in \mathbb{C}^d$, and $\operatorname{dist}(\zeta, Z) := \inf_{z\in Z} \operatorname{dist}(\zeta, z)$ for $Z \subset \mathbb{C}^d$.

Proposition 9.3. *Let $T : X \to X$ be a real analytic full branch expanding map whose inverse branches are eventually complex contracting. If $\varepsilon > 0$ is sufficiently small then the ε-neighbourhood $D_\varepsilon = \{z \in \mathbb{C}^d : \operatorname{dist}(z, X) < \varepsilon\}$ satisfies (D2) and is conjugation-invariant.*

Proof. Since T is expanding there exists $\gamma' \in (0, 1)$ (see (4.2)) such that

$$\sup_{x\in int(X)} \|T_i'(x)\|_{L(\mathbb{R}^d)} \le \gamma' \quad \text{for all } i \in \mathcal{I}\,. \tag{9.2}$$

Since the inverse branches are eventually complex contracting, there exists a bounded domain D and $\gamma \in [\gamma', 1)$ such that $\|T_i'\|_{H^\infty(D,L(\mathbb{C}^d))} \le \gamma$ for all $i \in \mathcal{I} \setminus \mathcal{J}$, where $\mathcal{J}$ is some finite subset of $\mathcal{I}$. Since $\mathcal{J}$ is finite, and each T_i' is continuous on D, (9.2) implies that there exists $\varepsilon > 0$ such that $D_\varepsilon \subset D$ and $\|T_i'\|_{H^\infty(D_\varepsilon,L(\mathbb{C}^d))} \le \gamma$ for all $i \in \mathcal{J}$. Therefore in fact $\|T_i'\|_{H^\infty(D_\varepsilon,L(\mathbb{C}^d))} \le \gamma$ for all $i \in \mathcal{I}$.

From the several variables mean value theorem it follows that for each $i \in \mathcal{I}$, the inverse branch T_i is γ-Lipschitz on any convex subset of D_ε (see e.g. [**2**, Thm. 2.3]). We claim this implies that for each $i \in \mathcal{I}$,

$$\operatorname{dist}(T_i(z), X) \le \gamma \operatorname{dist}(z, X) \quad \text{for all } z \in D_\varepsilon\,. \tag{9.3}$$

To verify (9.3), let $z \in D_\varepsilon$ and choose $x \in X$ with $\operatorname{dist}(z, X) = \operatorname{dist}(z, x)$. In particular x, z both lie in the open ε-ball centred at x, a convex subset of D_ε, so $\operatorname{dist}(T_i(z), T_i(x)) \le \gamma \operatorname{dist}(z, x)$. Therefore $\operatorname{dist}(T_i(z), X) \le \operatorname{dist}(T_i(z), T_i(x)) \le \gamma \operatorname{dist}(z, x) = \gamma \operatorname{dist}(z, X)$, as required. So $T_i(D_\varepsilon) \subset D_{\gamma\varepsilon}$ for all $i \in \mathcal{I}$, and therefore $\cup_{i\in\mathcal{I}} T_i(D_\varepsilon) \Subset D_\varepsilon$, which is the condition (D2). The fact that D_ε is conjugation-invariant follows immediately from the fact that $X \subset \mathbb{R}^d$. □

Theorem 9.4. *Let $T : X \to X$ be a real analytic full branch expanding map whose inverse branches are eventually complex contracting, and suppose that $(w_i)_{i\in\mathcal{I}}$ is a strictly positive real analytic weight system satisfying (S1).*

Then there is a unique T-invariant probability measure μ which is absolutely continuous with respect to the eigenmeasure m. The corresponding Radon-Nikodym derivative $d\mu/dm$ is strictly positive on X, real analytic, and extends holomorphically to an element of $H^\infty_{\mathbb{R}}(D)$. The dynamical system (T,μ) is exact.

Proof. The eventually complex contracting hypothesis ensures the existence of a conjugation-invariant domain D for which condition (D2) is satisfied, by Proposition 9.3. The result then follows from Theorem 7.5. □

Similarly, combining Theorem 8.6 with Proposition 9.3 gives:

Theorem 9.5. *Let (T, X, D) be a real analytic full branch expanding map whose inverse branches are eventually complex contracting, and w a real analytic g-function whose corresponding strictly positive real analytic weight system $(w_i)_{i\in\mathcal{I}}$ satisfies (S1). Then there is a unique g-measure for w.*

10. *Invariant measures equivalent to a reference measure*

The character of this section differs from the preceding ones. Here we will start with some finite Borel measure m on X, and ask whether there exists a T-invariant probability measure μ which is equivalent to m. The single most interesting case, when m is equal to Lebesgue measure, will be considered in §11.

Definition 10.1. Let m be a finite measure on X. A map $T : X \to X$ is said to be *non-singular* with respect to m if $mT^{-1} \ll m$.

If $T : X \to X$ is a full branch map, and m is such that $m(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$, then $mT^{-1} = \sum_{i\in\mathcal{I}} mT_i$. This can be shown using the same arguments as in the proof of part (iii) of Proposition 8.7. Therefore T is non-singular with respect to m if and only if $\sum_{i\in\mathcal{I}} mT_i \ll m$, if and only if $mT_i \ll m$ for every $i \in \mathcal{I}$.

Lemma 10.2. *Let $T : X \to X$ be a full branch map which is non-singular with respect to a finite measure m. If $m(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$ then, for every $f \in L^1(X, m)$, the series $\sum_{i\in\mathcal{I}} \frac{dmT_i}{dm} \cdot f \circ T_i$ converges in $L^1(X, m)$ and*

$$\int \sum_{i\in\mathcal{I}} \frac{dmT_i}{dm} \cdot f \circ T_i \, dm = \int f \, dm \,. \tag{10.1}$$

Proof. Let $f \in L^1(X, m)$. Since $m(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$,

$$\begin{aligned}\sum_{i\in\mathcal{I}} \int \frac{dmT_i}{dm} \cdot f \circ T_i \, dm &= \sum_{i\in\mathcal{I}} \int_{int(X)} \frac{dmT_i}{dm} \cdot f \circ T_i \, dm \\ &= \sum_{i\in\mathcal{I}} \int_{T_i(int(X))} f \, dm = \int_{\cup_{i\in\mathcal{I}} X_i} f \, dm = \int f \, dm \,.\end{aligned}$$

But

$$\sum_{i\in\mathcal{I}}\int\left|\frac{dmT_i}{dm}\cdot f\circ T_i\right|\,dm=\sum_{i\in\mathcal{I}}\int\frac{dmT_i}{dm}\cdot|f|\circ T_i\,dm=\int|f|\,dm<\infty\,,$$

by the same argument as above. Thus $\sum_{i\in\mathcal{I}}\frac{dmT_i}{dm}\cdot f\circ T_i\,dm$ converges in $L^1(X,m)$, and

$$\int\sum_{i\in\mathcal{I}}\frac{dmT_i}{dm}\cdot f\circ T_i\,dm=\sum_{i\in\mathcal{I}}\int\frac{dmT_i}{dm}\cdot f\circ T_i\,dm=\int f\,dm\,.$$

□

THEOREM 10.3. *Let (T,X,D) be a real analytic full branch expanding map, where the domain D is conjugation-invariant. Suppose m is a finite measure on X with $m(X\setminus\cup_{i\in\mathcal{I}}X_i)=0$, and such that T is non-singular with respect to m. Suppose that $w_i:=\frac{dmT_i}{dm}\in H^\infty(D)$ for all $i\in\mathcal{I}$, and that each w_i is strictly positive on X. Suppose that either (S2) and (D1) are satisfied, or (S1) and (D2) are satisfied, for the weight system $(w_i)_{i\in\mathcal{I}}$.*

Then there is a unique T-invariant probability measure μ which is absolutely continuous with respect to m. The corresponding density function $d\mu/dm$ is real analytic and strictly positive on X. The dynamical system (T,μ) is exact.

Proof. The real analytic weight system $(w_i)_{i\in\mathcal{I}}$ is strictly positive, and m is the eigenmeasure for the associated transfer operator by Lemma 10.2, so the result follows from Theorem 7.5. □

11. *Invariant measures equivalent to Lebesgue measure*

Throughout this section, (non-normalised) Lebesgue measure on X will be denoted by *Leb*. For a given real analytic full branch expanding map $T:X\to X$, our aim is to derive a sufficient condition for the existence of a T-invariant probability measure μ on X which is absolutely continuous with respect to *Leb*. Such a μ will simply be referred to as an *acip*, and the Radon-Nikodym derivative $d\mu/dLeb$ will be called its *density function*. Let $\mathrm{Jac}(T)$ denote the Jacobian determinant of T, defined by $\mathrm{Jac}(T)(x)=|\det(T'(x))|$ for all $x\in\cup_{i\in\mathcal{I}}X_i$, where $T'(x)$ denotes the derivative of T at the point x. Analogously, the Jacobian determinant $\mathrm{Jac}(T_i)$ of any inverse branch T_i is defined by $\mathrm{Jac}(T_i)(x)=|\det(T_i'(x))|$ for all $x\in X$. The change of variables formula for integration with respect to *Leb* (see e.g. [**4**, Thm. 6.1.6]) implies that $dLeb\,T_i/dLeb=\mathrm{Jac}(T_i)$ for all $i\in\mathcal{I}$, and hence in particular T is non-singular with respect to Lebesgue measure.

If $D\subset\mathbb{C}^d$ is a domain such that $(T_i,D)_{i\in\mathcal{I}}$ is a holomorphic map system, then because $\det(T_i')$ does not change sign on X, each $\mathrm{Jac}(T_i)$ also has a holomorphic extension to D. If moreover $\sup_{z\in D}\sum_{i\in\mathcal{I}}|\mathrm{Jac}(T_i)(z)|<\infty$, then each $\mathrm{Jac}(T_i)\in H^\infty(D)$, and (S1) is satisfied for the weight system $(w_i)_{i\in\mathcal{I}}$ given by $w_i=\mathrm{Jac}(T_i)$.

Of all our results, the following is the one which most closely resembles the claimed theorem in [**10**].

THEOREM 11.1. *Let (T,X,D) be a real analytic full branch expanding map such that $Leb(X\setminus\cup_{i\in\mathcal{I}}X_i)=0$. Suppose that D is conjugation-invariant, and that* *either (S2) and (D1) are satisfied, or (S1) and (D2) are satisfied, for the weight system defined by* $w_i = \mathrm{Jac}(T_i)$.

Then T *has a unique acip* μ. *The corresponding density function is real analytic and strictly positive on* X. *The dynamical system* (T,μ) *is exact.*

Proof. As noted above $\sum_{i\in\mathcal{I}} Leb\, T_i \ll Leb$. Each function $w_i = dLeb\, T_i/dLeb = \mathrm{Jac}(T_i) = 1/|\det T' \circ T_i|$ is strictly positive on X, since the branch $T|_{X_i}$ is a C^1 diffeomorphism whose derivative has a continuous extension to $\overline{X_i}$ (see Definition 4.1). The result then follows from Theorem 10.3. □

In view of the strict positivity of the $\mathrm{Jac}(T_i)$ on X, the following result is an immediate corollary of Theorem 10.3 and Proposition 9.3.

THEOREM 11.2. *Let* (T,X,D) *be a real analytic full branch expanding map whose inverse branches are eventually complex contracting, with* $Leb(X\setminus \cup_{i\in\mathcal{I}} X_i) = 0$ *and* $\sup_{z\in D}\sum_{i\in\mathcal{I}} |\mathrm{Jac}(T_i)(z)| < \infty$.

Then T *has a unique acip* μ. *The corresponding density function is real analytic and strictly positive on* X. *The dynamical system* (T,μ) *is exact.*

DEFINITION 11.3. The real analytic full branch expanding map $T : X \to X$ has *uniformly summable derivatives* if there exists a domain D such that

$$\sup_{z\in D}\sum_{i=n}^{\infty} \|T_i'(z)\|_{L(\mathbb{C}^d)} \to 0 \quad \text{as } n\to\infty\,. \tag{11.1}$$

The usefulness of this definition is that, as we shall see in the proof of Theorem 11.4, (11.1) implies both that the inverse branches of T are eventually complex contracting *and* that the summability condition (S1) holds when $w_i = \mathrm{Jac}(T_i)$.

Note that (11.1) is implied by the absolute summability of the derivatives:

$$\sum_{i\in\mathcal{I}} \|T_i'\|_{H^\infty(D,L(\mathbb{C}^d))} < \infty\,. \tag{11.2}$$

Clearly (11.2), and hence (11.1), holds whenever $\mathcal{I}$ is finite.

THEOREM 11.4. *Let* $T : X \to X$ *be a real analytic full branch expanding map with uniformly summable derivatives, such that* $Leb(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$.

Then T *has a unique acip* μ. *The corresponding density function is real analytic and strictly positive on* X. *The dynamical system* (T,μ) *is exact.*

Proof. By Theorem 11.2 it is sufficient to verify that the inverse branches of T are eventually complex contracting, and that (S1) holds for the weight system $(w_i)_{i\in\mathcal{I}}$ defined by $w_i = \mathrm{Jac}(T_i)$. By (11.1), $\sup_{z\in D} \|T_n'(z)\|_{L(\mathbb{C}^d)} \to 0$ as $n\to\infty$, so $\lim_{n\to\infty} \|T_n'\|_{H^\infty(D,L(\mathbb{C}^d))} = 0 < 1$; thus the inverse branches of T are eventually complex contracting. Now $|\det(A)| \le \|A\|^d_{L(\mathbb{C}^d)}$ for any $A \in L(\mathbb{C}^d)$, so $|w_i(z)| = |\mathrm{Jac}(T_i)(z)| \le \|T_i'(z)\|^d_{L(\mathbb{C}^d)}$ for all $z \in D$. Since $\|T_i'\|_{H^\infty(D,L(\mathbb{C}^d))} < 1$ for all sufficiently large i, it follows that $|w_i(z)| \le \|T_i'(z)\|^d_{L(\mathbb{C}^d)} \le \|T_i'(z)\|_{L(\mathbb{C}^d)}$ for all sufficiently large i, and all $z \in D$. Combining this with condition (11.1) gives $\sup_{z\in D}\sum_{i=n}^{\infty} |w_i(z)| \to 0$ as $n\to\infty$, which is condition (S2), which in particular implies (S1). □

Remark 11.5. The proof of exactness of (T, μ) in Theorem 11.4 answers affirmatively a conjecture of Mayer [**10**, Remark 1, p. 13].

12. *Appendix A: The symbolic coding approach to invariant measures*

In this paper we have addressed the problem of finding a T-invariant probability measure μ which is absolutely continuous with respect to a suitable finite reference measure m, with particular emphasis on the case where m is Lebesgue measure. We proved that the transfer operator associated to the weight system $w_i = dmT_i/dm$ has, under appopriate hypotheses, a real analytic strictly positive eigenvector ϱ, and ϱ can be interpreted as the Radon-Nikodym derivative $d\mu/dm$.

The purpose of this appendix is to briefly describe an alternative, less direct, approach to solving this problem. The alternative method, which is well known, relies on setting up a symbolic dynamics which models the dynamical system $T : X \to X$; the initial problem is transferred to the symbolic setting, solved in this setting, and then the solution is transferred back to the original setting. More precisely, a full shift $\sigma : \Sigma \to \Sigma$ is introduced, together with a map $\pi : \Sigma \to X$ which in some sense conjugates σ and T (see below for more details). If m satisfies $m(X \setminus \cup_{i\in\mathcal{I}} X_i) = 0$ then $n := m\pi$ is a finite measure on Σ. The weight system $(w_i)_{i\in\mathcal{I}}$ on X induces a weight system $(W_i)_{i\in\mathcal{I}}$ on Σ, where $W_i := w_i \circ \pi$. There are various conditions on the W_i, typically formulated in terms of their continuity moduli, which imply the existence of a unique σ-invariant probability measure ν absolutely continuous with respect to n (see e.g. [**12**], [**13**], [**18**], [**19**], [**22**], [**23**], and §13). Usually it can be shown that ν is ergodic and fully supported, and in this case it follows that the probability measure $\mu := \nu \circ \pi^{-1}$ is T-invariant (see Lemma 12.1 below). It can be shown that the function $d\mu/dm = (d\nu/dn) \circ \pi^{-1}$ is well-defined m-almost everywhere, and can be interpreted as an element of $L^1(X, m)$, so that μ is absolutely continuous with respect to m, and the original problem is solved.

The main drawback of this symbolic coding method is that it does not provide any information on the analyticity, or even the continuity, of the density function $d\mu/dm$. On the other hand it applies to maps T which enjoy less regularity than real analyticity, and even in the real analytic category the conditions under which the approach works are genuinely different from the conditions we use in this paper (see §13 and [**3**]).

We now describe in greater detail the symbolic coding (which allows the setting of the problem to be transferred), and a key lemma guaranteeing that certain invariant measures on symbolic space yield T-invariant measures on X (thereby allowing symbolic solutions to be transferred back to the original setting).

Define Σ to be the set $\mathcal{I}^{\mathbb{N}}$ of sequences whose entries are elements of $\mathcal{I}$. If we equip $\mathcal{I}$ with the discrete topology, and $\Sigma = \mathcal{I}^{\mathbb{N}}$ with the product topology, then the left shift $\sigma : \Sigma \to \Sigma$ given by $\sigma(i_1, i_2, i_3, \ldots) = (i_2, i_3, \ldots)$ is continuous. Since T is expanding, for every $\underline{i} = (i_1, i_2, \ldots) \in \Sigma$, the limit $\pi(\underline{i}) = \lim_{n\to\infty} T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}(z)$ exists, and is independent of $z \in X$. This defines a continuous map $\pi : \Sigma \to X$, which is never injective (since there exist $i \neq j$ with $\overline{X}_i \cap \overline{X}_j \neq \emptyset$), and in general is not surjective (it is surjective if and only if $X = \cup_{i\in\mathcal{I}} \overline{X}_i$; this holds if $\mathcal{I}$ is finite, for example). For $i \in \mathcal{I}$, the i-th inverse branch $\sigma_i : \Sigma \to \Sigma$ of σ is defined by $\sigma_i(i_1, i_2, \ldots) = (i, i_1, i_2, \ldots)$. Then, for every $i \in \mathcal{I}$,

$$T_i \circ \pi = \pi \circ \sigma_i \quad \text{on } \Sigma, \tag{12.1}$$

because each T_i is continuous on X, and

$$\pi \circ \sigma = (T|_{X_i}) \circ \pi \quad \text{on } \pi^{-1}(X_i)\,, \tag{12.2}$$

because $T|_{X_i}$ is continuous, and $T \circ T_i$ is the identity on $int(X)$. From (12.1) and (12.2) it follows that for every $i \in \mathcal{I}$,

$$\sigma_i \pi^{-1}(B) = \pi^{-1} T_i(B) \quad \text{for all } B \subset int(X)\,. \tag{12.3}$$

LEMMA 12.1. *If ν is a fully supported Borel probability measure on Σ which is ergodic and σ-invariant, then $\mu = \nu \circ \pi^{-1}$ is a T-invariant Borel probability measure on X (and is itself ergodic and fully supported).*

Proof. The sets $U := \pi^{-1}(\cup_{i\in\mathcal{I}} X_i)$ and $V := \pi^{-1}(int(X))$ are easily seen to be non-empty, with $U \subset V$, and we claim that

$$\sigma^{-1}(V) \subset U\,. \tag{12.4}$$

To prove (12.4), fix $\underline{i} \in V$ and $i \in \mathcal{I}$. Then by (12.1), $\pi(\sigma_i(\underline{i})) = T_i(\pi(\underline{i})) \in T_i(int(X)) = X_i$. Thus $\sigma_i(\underline{i}) \in \pi^{-1}(X_i)$ for every $\underline{i} \in V$ and every $i \in \mathcal{I}$. Hence $\cup_{i\in\mathcal{I}} \sigma_i(V) \subset \pi^{-1}(\cup_{i\in\mathcal{I}} X_i) = U$, which proves (12.4), because $\sigma^{-1}(V) = \cup_{i\in\mathcal{I}} \sigma_i(V)$.

Next we show that

$$\nu(U) = 1\,. \tag{12.5}$$

To see this, note that since $\sigma^{-1}(U) \subset \sigma^{-1}(V) \subset U$, and ν is σ-invariant, $\sigma^{-1}(U) = U \pmod{\nu}$. But ν is ergodic, so $\nu(U) = 0$ or 1. As π is continuous, U is a non-empty open subset of Σ, and hence $\nu(U) > 0$, and (12.5) is proved. It follows that

$$\mu(\cup_{x\in\mathcal{I}} X_i) = 1\,. \tag{12.6}$$

The T-invariance of μ now follows: if A is a Borel subset of X then

$$\begin{aligned}
\mu(T^{-1}A) &= \mu((T^{-1}A) \cap (\cup_{i\in\mathcal{I}} X_i)) = \mu(\cup_{i\in\mathcal{I}} T_i(A \cap int(X))) \\
&= \sum_{i\in\mathcal{I}} \mu(T_i(A \cap int(X))) = \sum_{i\in\mathcal{I}} \nu(\pi^{-1}(T_i(A \cap int(X)))) \\
&= \sum_{i\in\mathcal{I}} \nu(\sigma_i(\pi^{-1}(A \cap int(X)))) = \nu(\cup_{i\in\mathcal{I}} \sigma_i \pi^{-1}(A \cap int(X))) \\
&= \nu(\sigma^{-1}\pi^{-1}(A \cap int(X))) = \nu(\pi^{-1}(A \cap int(X))) \\
&= \mu(A \cap int(X)) = \mu(A)\,,
\end{aligned}$$

using (12.6) twice, (12.3), and the σ-invariance of ν.

Since $\pi : \Sigma \to X$ is continuous and has dense range, and ν is fully supported on Σ, it follows that $\mu = \nu \circ \pi^{-1}$ is fully supported on X.

The ergodicity of μ follows from (12.3), (12.6), and the ergodicity of ν. □

13. *Appendix B: Comparison to other criteria for existence of invariant measures*

The purpose of this appendix is to describe some previously known criteria for the existence of invariant measures for expanding maps, and compare them to the results of this paper. These previous results neither require, nor exploit, the analyticity of the map T.

Before describing in detail the previously known criteria, let us summarise the ways in which they differ from those of this paper. Let $T : X \to X$ be a real analytic full branch expanding map with inverse branches $(T_i)_{i\in\mathcal{I}}$, and $(w_i)_{i\in\mathcal{I}}$ a strictly positive real analytic weight system. For countably infinite $\mathcal{I}$, both our criteria and the previously known criteria impose additional hypotheses on T and $(w_i)_{i\in\mathcal{I}}$ in order to guarantee the existence and uniqueness of μ. It turns out that the additional hypotheses arising from the differing approaches are *independent*: there exist examples which satisfy our hypotheses but not those of previous authors (see Example 13.1 below), but equally there are examples which fail to satisfy our hypotheses yet are covered by previously known results (one such example is detailed in [**3**]).

To describe the previous approaches, it will be convenient to work with the real analytic potential function φ (see Definition 8.1) associated to the strictly positive weight system $(w_i)_{i\in\mathcal{I}}$. Walters [**23**] has a criterion for existence and uniqueness of μ which in particular includes the condition† that for some fixed $\varepsilon_0 > 0$,

$$\sup_{\substack{x,y\in int(X)\\ \|x-y\|_{\mathbb{R}^d}<\varepsilon_0}} \sup_{n\geq 1} \sup_{\underline{i}\in\mathcal{I}^n} S_n\varphi(T_{\underline{i}}x) - S_n\varphi(T_{\underline{i}}y) < \infty\,, \tag{13.1}$$

where $S_n\varphi(z) := \sum_{i=0}^{n-1}\varphi(T^i z)$.

In fact a weaker‡ analogue of this condition arises by applying Walters' results to the symbolic dynamical system (a full shift on $\mathcal{I}$) obtained by the coding method described in Appendix A, namely that for some fixed $N \geq 0$,

$$\sup_{x,y\in int(X)} \sup_{n\geq 1} \sup_{\underline{i}\in\mathcal{I}^{N+n}} S_n\varphi(T_{\underline{i}}x) - S_n\varphi(T_{\underline{i}}y) < \infty\,. \tag{13.2}$$

Condition (13.2) is a kind of bounded distortion condition; for example if $\varphi = -\log \mathrm{Jac}(T)$ it asserts that for some $C > 1$,

$$\frac{1}{C} \leq \frac{\mathrm{Jac}(T^n)(T_{\underline{j}}x)}{\mathrm{Jac}(T^n)(T_{\underline{j}}y)} \leq C \quad \text{for all } \underline{j}\in\mathcal{I}^N, x,y\in int(X), n\in\mathbb{N}\,.$$

Work of Mauldin & Urbański [**12**], [**13**], and Sarig [**18**], [**19**], treats symbolic settings (namely, a rather general class of infinite alphabet subshifts of finite type) whose combinatorics is more complicated than the full shift, and where Walters' techniques do not necessarily apply (cf. [**18**, p. 1566]). However these authors do not strive for an optimally weak assumption on the potential function φ. Rather, they are usually content (see [**12**], [**13**], [**18**]) to assume that $\mathrm{var}_N(\varphi) < Cr^N$ for some $C > 0$, $r \in [0, 1)$, where for $N \geq 1$,

$$\mathrm{var}_N(\varphi) := \sup_{x,y\in int(X)} \sup_{\underline{i}\in\mathcal{I}^N} \left[\varphi(T_{\underline{i}}x) - \varphi(T_{\underline{i}}y)\right]\,,$$

though it is noted in [**19**] that this condition can be replaced by $\sum_{j=2}^{\infty}\mathrm{var}_j(\varphi) < \infty$. In fact if $\sum_{j=N}^{\infty}\mathrm{var}_j(\varphi) < \infty$ for any $N \geq 1$ then φ satisfies Walters' condition (13.2), because for all $x,y\in int(X)$, $n \geq 1$, $\underline{i}\in\mathcal{I}^{n+N}$,

$$S_n\varphi(T_{\underline{i}}x) - S_n\varphi(T_{\underline{i}}y) \leq \sum_{i=1}^{n}\mathrm{var}_{i+N}(\varphi) \leq \sum_{j=N+1}^{\infty}\mathrm{var}_j(\varphi) < \infty\,.$$

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†This is part of condition (iii) on [**23**, p. 125].

‡The condition is weaker because the natural topology on $\cup_{i\in\mathcal{I}}X_i$ is finer than the one induced by the coding map π and the topology of Σ (see §12 for the definitions of π and Σ). However the conclusions are also weaker: in particular one cannot conclude that the density function $d\mu/dm$ is continuous (cf. the discussion in §12).

Of the previously known criteria discussed above, we see that the weakest one guaranteeing existence and uniqueness of $\mu = \mu_\varphi$ is the symbolic interpretation of Walters' hypothesis, which involves in particular the condition (13.2). With this in mind, we now detail an example which satisfies the hypotheses of our Theorem 9.4, yet which fails to satisfy (13.2); in fact the potential function φ will satisfy $\mathrm{var}_N(\varphi) = +\infty$ for all $N \geq 1$.

EXAMPLE 13.1. Let $X = [0,1]$, and $X_i = (2^{-i}, 2^{1-i})$ for $i \in \mathbb{N}$. Define $T : X \to X$ to be the piecewise affine map with slope 2^i and $T(2^{1-i}) = 1$ on $(2^{-i}, 2^{1-i}]$, for $i \in \mathbb{N}$, and $T(0) = 0$. Clearly T is eventually complex contracting.

Let $\{\alpha_i\}_{i=1}^\infty$ be any sequence of positive reals such that $\sum_{i=1}^\infty \alpha_i < \infty$, and for $i \in \mathbb{N}$ define w_i to be the polynomial

$$w_i(x) = \alpha_i \left[(x/2)^{4^i} + (2^{-1} - 2^{-1-i})^{4^i} \right] .$$

Note that each w_i is strictly positive, and strictly increasing, on $[0,1]$. If D is the complex disc of radius 2 centred at $0 \in \mathbb{C}$, then

$$\|w_i\|_{H^\infty(D)} = w_i(2) = \alpha_i \left[1 + (2^{-1} - 2^{-1-i})^{4^i} \right] ,$$

and therefore $\sum_{i=1}^\infty \|w_i\|_{H^\infty(D)} < \infty$, so (S1) holds. Therefore the conditions of Theorem 9.4 are satisfied, and we deduce the existence of a unique T-invariant probability measure μ which is equivalent to the eigenmeasure m.

Let $w : \cup_{i=1}^\infty X_i \to \mathbb{R}$ be the real analytic w-function associated to $(w_i)_{i \in \mathcal{I}}$, and $\varphi = \log w$ the corresponding real analytic potential function. We claim that

$$\sup_{x,y \in int(X)} \sup_{\underline{i} \in \mathcal{I}^{N+1}} \left[\varphi(T_{\underline{i}} x) - \varphi(T_{\underline{i}} y) \right] = \infty \quad \text{for all } N \geq 0 . \tag{13.3}$$

Clearly (13.3) implies that Walters' condition (13.2) does not hold, and therefore nor does (13.1). It also implies that $\mathrm{var}_N(\varphi) = +\infty$ for all $N \geq 1$.

To prove (13.3) it suffices to show that

$$\sup_{j \in \mathbb{N}} \left[\varphi(T_{\underline{i}_j} 1) - \varphi(T_{\underline{i}_j} x_0) \right] = \infty , \tag{13.4}$$

where $\underline{i}_j := (j, 1, \ldots, 1) \in \mathcal{I}^{N+1}$, and $x_0 = \frac{1}{2}$,[†] since both points $\frac{1}{2}$ and 1 are accumulation points of $int(X)$ and $\phi \circ T_{\underline{i}_j}$ is continuous on $(0,1]$.

Now $T(T_{\underline{i}_j} 1) = 1$, so $\varphi(T_{\underline{i}_j} 1) = \log w_j(1)$, and $T(T_{\underline{i}_j} x_0) = 1 - 2^{-N-1}$, so $\varphi(T_{\underline{i}_j} x_0) = \log w_j(1 - 2^{-N-1})$. But w_j is an increasing function, so if $j \geq N+1$ then $w_j(1 - 2^{-N-1}) \leq w_j(1 - 2^{-j})$. Therefore

$$\varphi(T_{\underline{i}_j} 1) - \varphi(T_{\underline{i}_j} x_0) \geq \log \frac{w_j(1)}{w_j(1 - 2^{-j})} \quad \text{for all } j \geq N . \tag{13.5}$$

But

$$\frac{w_j(1)}{w_j(1-2^{-j})} = \frac{1}{2} \left[1 + (1 - 2^{-j})^{-4^j} \right] \to \infty \quad \text{as } j \to \infty , \tag{13.6}$$

so (13.5) and (13.6) together give (13.4), as required.

[†] In fact a similar argument would work for any choice of $x_0 \in X \setminus \{0,1\}$.

REMARK 13.2. In practice a condition such as (13.2), involving both φ and (iteration of) T, may be difficult to verify. The conditions of Theorem 9.4, where the assumption on φ (i.e. the summability condition (S1) on the associated weight system) is decoupled from the assumption on T (the eventually complex contracting condition), may be easier to check.

14. *Acknowledgments*

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