BOUNDED DISTORTION VERSUS UNIFORMLY SUMMABLE DERIVATIVES

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Abstract. Let $T : X \to X$ be a real analytic full branch expanding map, where X is a compact connected subset of $\mathbb{R}^d$ with non-empty interior. A well known sufficient condition for the existence of a T-invariant probability measure equivalent to Lebesgue measure is that T has bounded distortion. An alternative sufficient condition is that T has uniformly summable derivatives (see [BJ]). The purpose of this note is to show that the bounded distortion condition and the uniformly summable derivatives condition are independent.

1. Introduction

Let X be a compact connected subset of $\mathbb{R}^d$ with non-empty interior. A well-known folklore theorem in ergodic theory is:

Theorem 1. *Let $T : X \to X$ be a full branch expanding map. If T has bounded distortion then it has a (unique) acip.*

By an *acip (absolutely continuous invariant probability)* we mean a T-invariant Borel probability measure on X which is absolutely continuous with respect to Lebesgue measure on X. For a proof of Theorem 1, see for example the book by Mañé [Mañ, Thm. III.1.3].

The following analogue of Theorem 1 was proved in [BJ]:

Theorem 2. *Let $T : X \to X$ be a real analytic full branch expanding map. If T has uniformly summable derivatives then it has a (unique) acip.*

In particular, if the full branch expanding map is real analytic then to guarantee the existence of an acip it is sufficient to show that T *either* has bounded distortion, *or* has uniformly summable derivatives. Bounded distortion (see Definition 2) is a classical hypothesis, whose importance was first recognised by Rényi [Rén]. The uniformly summable derivatives condition (see Definition 3) was introduced in [BJ].

If the real analytic full branch expanding map T has only finitely many branches then it has both bounded distortion *and* uniformly summable derivatives. However if there are infinitely many branches then this is no longer the case: either one of the conditions may fail. The purpose of this article is to clarify that bounded distortion and uniformly summable derivatives are genuinely independent conditions. So there are real analytic full branch expanding maps for which Theorem 2 can be applied but Theorem 1 cannot (see

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Section 3), and there are real analytic full branch expanding maps for which Theorem 1 can be applied but Theorem 2 cannot (see Section 4)[1].

Clearly, the assumption that the branches of T are *real analytic* is more restrictive than the C^1 hypothesis in the usual definition of an expanding map (see Definition 1). Moreover, if T is *finite* branch then this assumption is unnecessarily restrictive: for example a consequence of Theorem 1 is that if every branch is merely $C^{1+\alpha}$ then there exists an acip (see Theorem 3 below), so in this case Theorem 1 is strictly stronger[2] than Theorem 2. The main purpose of this note is to emphasise that if T has infinitely many branches then Theorem 1 does *not* imply Theorem 2: the bounded distortion condition imposes a non-trivial uniform constraint which may fail even when each branch is highly regular (e.g. polynomial, cf. Theorem 4).

Real analytic infinite branch expanding maps occur naturally as induced maps for certain systems enjoying some hyperbolicity. For example they arise in the study of existence of absolutely continuous invariant measures for polynomial maps of the interval (see e.g. [Bru, BLS, Jak, MS]), for maps with an indifferent fixed point (see e.g. [Hol, PS, PY, Sch, Zwe]), and for non-uniformly expanding maps (see e.g. [ALP]). In these applications, a key step is to prove that the induced expanding map has an acip, invariably by checking the bounded distortion condition, then use this to deduce the existence of a (possibly infinite) absolutely continuous invariant measure for the original map. The results of this paper, clarifying the independence of bounded distortion and uniformly summable derivatives, suggest the possibility that in some situations it may be more convenient to check that the induced map satisfies the uniformly summable derivatives condition.

2. Bounded distortion and uniformly summable derivatives

Definition 1. Let X be a compact connected subset of $\mathbb{R}^d$ with non-empty interior and let $\{X_i\}_{i\in\mathcal{I}}$ be a finite or countably infinite family of non-empty pairwise disjoint subsets of X such that each X_i is open in $\mathbb{R}^d$, and $X \setminus \cup_{i\in\mathcal{I}} X_i$ has zero Lebesgue measure. Suppose that $T : X \to X$ is Borel measurable, and such that for all $i \in \mathcal{I}$, $T(X_i)$ is open in $\mathbb{R}^d$, and $T|_{X_i} : X_i \to T(X_i)$ is a C^1 diffeomorphism which can be extended to a C^1 map on $\overline{X_i}$. We say that T is *full branch* if $\overline{T(X_i)} = X$ for all $i \in \mathcal{I}$. The map T is called *expanding* if there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$, and $\beta > 1$, such that

$$\|T(x) - T(y)\| \geq \beta \, \|x - y\|$$

for any x, y which lie in the same partition element X_i.

For any partition element X_i, the restriction $T|_{X_i}$ is called a *branch* of T. If T is a full branch expanding map then each branch $T|_{X_i}$ has an inverse T_i, which will be referred to as an *inverse branch*.

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[1]Equally, there are real analytic full branch expanding maps for which both, or neither, of the theorems can be applied.

[2]As stated here, Theorem 1 is indeed stronger than Theorem 2 in the finite branch case. However, the methods used to prove Theorem 2 actually provide more information, namely that the density function for the acip is itself real analytic (see [BJ]). The methods used to prove Theorem 3 do not yield this sharper conclusion in the case where T is real analytic.

Definition 2. We say that a full branch expanding map T has *bounded distortion* if there exists $C > 0$ and $\alpha > 0$ such that

$$\left| \frac{\mathrm{Jac}(T)(x)}{\mathrm{Jac}(T)(y)} - 1 \right| \le C \, \|T(x) - T(y)\|^\alpha \tag{1}$$

whenever x, y lie in the same partition element X_i. Here

$$\mathrm{Jac}(T)(x) = \left|\det(T'(x))\right|$$

is the Jacobian determinant of T, where $T'(x)$ denotes the derivative of T at x.

In the case that T has a finite partition, a sufficient condition for bounded distortion to hold is that the derivative of each $T|_{X_i}$ extends to a *Hölder* continuous function on $\overline{X_i}$:

Theorem 3. *(Folklore Theorem: Finite Partition)*
Let $T : X \to X$ be a full branch expanding map with finite partition $\{X_i\}_{i\in\mathcal{I}}$, and suppose that the derivative of each $T|_{X_i}$ extends to a Hölder continuous function on $\overline{X_i}$. Then T has a (unique) acip.

Theorem 3 follows readily from Theorem 1, since there exist $\alpha, C_1, C_2, C_3 > 0$ such that if $x, y \in X_i$, $i \in \mathcal{I}$, then

$$\begin{aligned} \left|\det(T'(x)) - \det(T'(y))\right| &\le C_1 \left\|T'(x) - T'(y)\right\| \\ &\le C_2 \|x - y\|^\alpha \le C_3 \|T(x) - T(y)\|^\alpha , \end{aligned}$$

where the constants may be chosen independently of $i \in \mathcal{I}$ because $\mathcal{I}$ is finite; since $|\det(T')(\cdot)|$ is bounded away from zero we obtain the bounded distortion condition (1), and then use Theorem 1.

The Hölder hypothesis in Theorem 3 is a convenient one, though can be weakened slightly. On the other hand there exist full branch expanding maps with a finite partition, such that each branch $T|_{X_i}$ extends to a C^1 map on the closure $\overline{X_i}$, which nevertheless do not have an acip (see e.g. [BH, GS, Krz]); indeed this phenomenon is a generic one in the C^1 topology [CQ, Qua]. Henceforth, however, counterexamples of this kind shall not concern us, as we shall always assume that each branch $T|_{X_i}$ is highly regular:

Definition 3. A full branch expanding map $T : X \to X$ will be called *real analytic* if there is a domain $D \subset \mathbb{C}^d$, with $X \subset D$, such that each inverse branch T_i has a holomorphic extension to D. Such a domain D will be called a *common domain of holomorphy* for the inverse branches $\{T_i\}_{i\in\mathcal{I}}$.

A real analytic full branch expanding map $T : X \to X$ has *uniformly summable derivatives* if there exists a domain of holomorphy D such that[3]

$$\sup_{z\in D} \sum_{i=n}^{\infty} \left\|T_i'(z)\right\| \to 0 \quad \text{as } n \to \infty , \tag{2}$$

where $\|\cdot\|$ denotes any norm on $\mathbb{C}^{d\times d}$, and $T_i'(z)$ denotes the derivative of T_i at z.

[3]For the purposes of this definition we assume that the inverse branches of T are indexed by the natural numbers $\mathbb{N}$. If in fact T has only finitely many inverse branches then we define $\|T_i'(z)\| = 0$ for all sufficiently large i, and all $z \in D$, so that (2) is obviously satisfied.

The existence of an acip for real analytic full branch expanding maps with uniformly summable derivatives (Theorem 2) was proved in [BJ].

If the real analytic full branch expanding map T has only finitely many inverse branches then T clearly has uniformly summable derivatives. Such a T also has bounded distortion: the derivative of each branch $T|_{X_i}$ extends continuously to the closure $\overline{X_i}$, and T_i is holomorphic on a complex neighbourhood of X, so $T|_{X_i}$ has a holomorphic extension to a complex neighbourhood of $\overline{X_i}$, and in particular the derivative of $T|_{X_i}$ extends to a Lipschitz function on $\overline{X_i}$. By the proof of Theorem 3 we then see that T has bounded distortion.

If T has infinitely many inverse branches then it may fail to have uniformly summable derivatives, and may fail to have bounded distortion. In Sections 3 and 4 we shall show that the uniformly summable derivatives condition and the bounded distortion condition are independent. Before doing so, however, it is useful to point out a sufficient condition for a real analytic full branch expanding *interval* map to have bounded distortion, even when its partition is infinite. The key condition is that the T_i admit a *univalent* extension to a common neighbourhood of X.

Proposition 1. *Let $X \subset \mathbb{R}$ be a compact interval with non-empty interior, and $T : X \to X$ a real analytic full branch expanding map. Suppose there exists a common domain of holomorphy $D \subset \mathbb{C}$ for the inverse branches $\{T_i\}_{i\in\mathcal{I}}$ such that each T_i is univalent on D.*

Then T has bounded distortion, and hence an acip.

Proof. The bounded distortion condition (1) is equivalent to the existence of $C, \alpha > 0$ such that

$$\left|\frac{T_i'(x)}{T_i'(y)} - 1\right| \le C|x-y|^\alpha \quad \text{for all } i \in \mathcal{I},\ x, y \in X\,. \tag{3}$$

We may assume that D is simply connected. We claim that there exists $C > 0$ such that if $\varphi : D \to \mathbb{C}$ is univalent, then

$$\left|\frac{\varphi'(x)}{\varphi'(y)} - 1\right| \le C|x-y| \quad \text{for all } x, y \in X\,, \tag{4}$$

which in particular implies (3).

To prove this, choose a compact subset $K \subset D$ whose interior contains X and whose boundary ∂K is a rectifiable Jordan curve. For φ analytic on D and $x, y \in X$ we then have

$$\varphi'(x) - \varphi'(y) = \frac{1}{2\pi i}\oint_{\partial K} \varphi'(\zeta)\frac{x-y}{(\zeta-x)(\zeta-y)}\,d\zeta$$

by Cauchy's integral formula. Thus

$$|\varphi'(x) - \varphi'(y)| \le A|\varphi'(z_0)|\,|x-y|\,, \tag{5}$$

where $A > 0$ is a constant depending only on K and X, and $z_0 \in \partial K$ is chosen so as to maximise $|\varphi'|$ on ∂K, i.e. $|\varphi'(z_0)| = \sup_{z\in\partial K}|\varphi'(z)|$.

Since K is a compact subset of D, Koebe's distortion theorem [McM, Thm. 2.9] asserts that for any $\eta \in K$ there exists $B = B(K, \eta) > 0$ such that for all univalent maps $\varphi : D \to \mathbb{C}$,

$$B^{-1}|\varphi'(\eta)| \le |\varphi'(z)| \le B|\varphi'(\eta)| \quad \text{for all } z \in K\,.$$

Consequently

$$B^{-2} \leq \frac{|\varphi'(x)|}{|\varphi'(y)|} \leq B^2 \quad \text{for all } x, y \in K\,. \tag{6}$$

Now let φ be univalent on D and $x, y \in X$. Then

$$\left|\frac{\varphi'(x)}{\varphi'(y)} - 1\right| = \frac{|\varphi'(x) - \varphi'(y)|}{|\varphi'(y)|} \leq A\frac{|\varphi'(z_0)|}{|\varphi'(y)|}|x-y| \leq AB^2|x-y|$$

by (5) and (6), so (4) follows. □

3. Uniformly Summable derivatives but unbounded distortion

In this section we construct one-dimensional real analytic full branch expanding maps with uniformly summable derivatives but unbounded distortion. For such maps Theorem 1 does not guarantee the existence of an acip, but Theorem 2 does. Note that the map T in Theorem 4 below has the property that each inverse branch T_n has a single critical point, and the sequence of critical points increases to the value 0. In particular these critical points accumulate on $[0,1]$, so the T_n do not extend *univalently* to any complex neighbourhood of $[0,1]$ (cf. Proposition 1).

Theorem 4. *There exist real analytic full branch expanding maps $T : [0,1] \to [0,1]$ with uniformly summable derivatives, but without bounded distortion.*

Proof. Let $\{\delta_n\}_{n=1}^\infty$ and $\{\kappa_n\}_{n=1}^\infty$ be sequences of strictly positive real numbers such that

(a) $\sum_{n=1}^\infty \delta_n = 1$;
(b) $\kappa_n < 1$ for every n and $\lim_{n\to\infty} \kappa_n = 1$;
(c) $\sup_{n\in\mathbb{N}} \delta_n(1+\kappa_n) < 1$.

For example we could choose $\delta_n = 2^{-n}$ and $\kappa_n = 1 - n^{-1}$.

For $n \in \mathbb{N}$, define $T_n : [0,1] \to [0,1]$ by

$$T_n(x) = \kappa_n\delta_n x^2 + \delta_n(1-\kappa_n)x + \sum_{k=n+1}^\infty \delta_k\,.$$

Then $T_n(0) = T_{n+1}(1)$, and since

$$T_n'(x) = 2\kappa_n\delta_n x + \delta_n(1-\kappa_n)$$

we see that

$$\sup_{x\in[0,1]} T_n'(x) > 0 \quad \text{for all } n \in \mathbb{N}\,,$$

and

$$\sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} T_n'(x) < 1\,.$$

Therefore the T_n are indeed inverse branches of a real analytic full branch expanding map T.

Now

$$\frac{T_n'(1)}{T_n'(0)} = \frac{1+\kappa_n}{1-\kappa_n} \to \infty \quad \text{as } n \to \infty\,,$$

so (3) does not hold, and therefore T does not have bounded distortion[4].

[4]A similar calculation shows that no iterate of T has bounded distortion.

On the other hand

$$|T_n'(z)| \le 2\kappa_n\delta_n|z| + \delta_n(1-\kappa_n) \quad \text{for all } z \in \mathbb{C}, n \in \mathbb{N},$$

so if D is any bounded open subset of $\mathbb{C}$ then

$$\sum_{n=1}^\infty \sup_{z\in D} |T_n'(z)| \le \sum_{n=1}^\infty (2\kappa_n\delta_n R + \delta_n(1-\kappa_n)) < \infty,$$

where R is the radius of a ball centred at 0 containing D. Thus T has uniformly summable derivatives. □

4. Bounded distortion but not uniformly summable derivatives

In this section we construct a real analytic full branch expanding map T with bounded distortion but without uniformly summable derivatives. The idea behind the construction is the observation that bounded distortion is a property affecting only the first two derivatives of a map, while the uniformly summable derivatives condition influences all higher order derivatives.

Theorem 5. *There exist real analytic full branch expanding maps $T : [0,1] \to [0,1]$ with bounded distortion, but without uniformly summable derivatives.*

Proof. We shall construct a real analytic full branch expanding map T whose inverse branches $T_n : [0,1] \to [0,1]$ are all entire and satisfy

$$T_n'(x) \ge 0, \quad T_n''(x) \ge 0, \quad T_n'''(x) \ge 0 \quad \text{for all } n \in \mathbb{N},\ x \in [0,1], \tag{7}$$

$$\sup_{n\in\mathbb{N}} \frac{\sup_{x\in[0,1]} |T_n''(x)|}{\inf_{x\in[0,1]} |T_n'(x)|} \le 1, \tag{8}$$

$$\sup_{x\in[0,1]} |T_n'''(x)| \ge 1 \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

For any $x, y \in [0,1]$ and $n \in \mathbb{N}$, inequality (8) gives

$$\begin{aligned}\left|\frac{T_n'(x)}{T_n'(y)} - 1\right| &= \frac{1}{|T_n'(y)|}\left|T_n'(x) - T_n'(y)\right| \\ &\le \frac{\sup_{x\in[0,1]} |T_n''(x)|}{|T_n'(y)|}|x-y| \\ &\le \frac{\sup_{x\in[0,1]} |T_n''(x)|}{\inf_{y\in[0,1]} |T_n'(y)|}|x-y| \\ &\le |x-y|,\end{aligned}$$

and therefore T has bounded distortion.

On the other hand, we claim that T does not have uniformly summable derivatives. To see this, suppose to the contrary that T has uniformly summable derivatives. Then for some domain D,

$$\sup_{z\in D} \sum_{i=n}^\infty \left|T_i'(z)\right| \to 0 \text{ as } n \to \infty,$$

and in particular

$$\sup_{z\in D}\left|T_n'(z)\right| \to 0 \text{ as } n\to\infty\,. \tag{10}$$

Let D' be a domain such that $[0,1]\subset D'$ and $\overline{D'}\subset D$. By Cauchy's integral formula there is a constant $C>0$ such that

$$\sup_{x\in[0,1]}\left|\varphi''(x)\right| \le C\sup_{z\in D'}|\varphi(z)|$$

for all functions φ which are analytic on D. Using (9) and (10) we now see that

$$1\le \sup_{x\in[0,1]}\left|T_n'''(x)\right| \le C\sup_{z\in D'}\left|T_n'(z)\right| \le C\sup_{z\in D}\left|T_n'(z)\right| \to 0 \text{ as } n\to\infty\,,$$

which is absurd. So indeed T does not have uniformly summable derivatives[5].

We now turn to the construction of the map T, which will involve the functions

$$a(x) = \int_{-\infty}^{x} e^{-\pi t^2}\,dt\,,$$

$$b(x) = \int_{-\infty}^{x} a(t)\,dt\,,$$

$$c(x) = \int_{-\infty}^{x} b(t)\,dt\,.$$

Note that $a(x) = \frac{1}{2}(1+\operatorname{erf}(\sqrt{\pi}x))$, where erf denotes the *error function* (see [AAR, p. 196]), and in particular that

$$a(x)\in(0,1) \quad \text{for } x\in\mathbb{R}\,. \tag{11}$$

Clearly all of a, b, c extend to entire functions in the complex plane and $c'=b$ and $b'=a$. Moreover the relations

$$b(x) = \frac{1}{2\pi}(2\pi x\,a(x) + e^{-\pi x^2}) \tag{12}$$

$$c(x) = \frac{1}{4\pi}\left((1+2\pi x^2)a(x) + xe^{-\pi x^2}\right) \tag{13}$$

are easily verified.

The inverse branches $T_n:[0,1]\to[0,1]$ of the map T will all have the general form

$$T_n(x) = \varepsilon_n^3 c\left(\frac{2x-1}{2\varepsilon_n}\right) + A_n x + B_n\,, \tag{14}$$

where $\varepsilon_n, A_n > 0$ and $B_n\in\mathbb{R}$ are constants to be determined later. Note that all these T_n are entire, because the function c is.

[5] A similar argument shows that no iterate of T has uniformly summable derivatives.

The first three derivatives of T_n are

$$\begin{aligned} T_n'(x) =& \varepsilon_n^2 b\left(\frac{2x-1}{2\varepsilon_n}\right) + A_n \\ T_n''(x) =& \varepsilon_n a\left(\frac{2x-1}{2\varepsilon_n}\right) \\ T_n'''(x) =& e^{-\pi\left(\frac{2x-1}{2\varepsilon_n}\right)^2}\,, \end{aligned}$$

which are all positive because ε_n, A_n are positive constants and a, b are positive functions when restricted to the real line. So (7) is satisfied. Moreover (9) is satisfied because

$$T_n'''(1/2) = 1 \quad \text{for all } n \in \mathbb{N}\,. \tag{15}$$

We now show how to choose the parameters ε_n, A_n, B_n in such a way that $\{T_n\}_{n\in\mathbb{N}}$ are the inverse branches of a full branch expanding map, and that (8) is satisfied.

To do this let $\{\delta_n\}_{n=1}^\infty$ be any sequence of real numbers such that

$$0 < \delta_n \leq \frac{1}{2} \quad \text{and} \quad \sum_{n=1}^\infty \delta_n = 1\,.$$

The δ_n will be the lengths of the partition pieces X_n.

Next, for each $n \in \mathbb{N}$, choose $\varepsilon_n \in (0, 1/2]$ such that

$$\varepsilon_n^3\left(c\left(-\frac{1}{2\varepsilon_n}\right) - c\left(\frac{1}{2\varepsilon_n}\right)\right) + \delta_n = \varepsilon_n\,. \tag{16}$$

This is possible since the function $g : (0,\infty) \to (0,\infty)$ defined by

$$g(x) = x - x^3\left(c\left(-\frac{1}{2x}\right) - c\left(\frac{1}{2x}\right)\right)\,,$$

satisfies $\lim_{x\to 0} g(x) = 0$ and $g'(x) \geq 1$ for $x \in (0,\infty)$, as can be verified using (12) and (13).

Finally, for $n \in \mathbb{N}$ set

$$A_n = \varepsilon_n\,, \tag{17}$$

$$B_n = \sum_{k=n+1}^\infty \delta_k - \varepsilon_n^3 c\left(-\frac{1}{2\varepsilon_n}\right)\,. \tag{18}$$

We now claim that with these choices,

(i) $T_n(0) = \sum_{k=n+1}^\infty \delta_k$ for every $n \in \mathbb{N}$,
(ii) $T_n(1) = \sum_{k=n}^\infty \delta_k$ for every $n \in \mathbb{N}$,
(iii) there is a constant $\gamma \in (0,1)$ such that $0 < T_n'(x) \leq \gamma$ for every $x \in [0,1]$ and $n \in \mathbb{N}$.

These three facts imply that $\{T_n\}_{n\in\mathbb{N}}$ are the inverse branches of a real analytic full branch expanding map. The corresponding partition $\{X_n\}_{n\in\mathbb{N}}$ of $[0,1]$ is given by $X_n = (\sum_{k=n+1}^\infty \delta_k, \sum_{k=n}^\infty \delta_k)$.

Now (i) follows from (14) and (18), while (ii) follows from (14), (16), (17), and (18). For (iii) we have already checked that T_n' is positive when restricted to the real line. To show that there exists $\gamma \in (0,1)$ such that $T_n' \le \gamma$ on $[0,1]$ it is enough to show that

$$T_n'(1) \le \gamma < 1\,, \tag{19}$$

because $T_n'' > 0$. To prove (19), observe that the function $h(x) = x^2 b(\frac{1}{2x}) + x$ is strictly increasing on $(0,\infty)$ with $0 < h(1/2) < 1$, as can be seen using (12). So setting $\gamma = h(1/2)$ we have

$$T_n'(1) = \varepsilon_n^2 b\left(\frac{1}{2\varepsilon_n}\right) + \varepsilon_n \le \gamma < 1\,,$$

since $\varepsilon_n \le 1/2$.

It remains to check that (8) is satisfied. Now

$$\sup_{x\in[0,1]} |T_n''(x)| \le \varepsilon_n$$

because of (11), while since $T_n'' \ge 0$ we have

$$\inf_{x\in[0,1]} |T_n'(x)| = T_n'(0) = \varepsilon_n^2 b\left(-\frac{1}{2\varepsilon_n}\right) + \varepsilon_n\,.$$

Therefore $\sup_{x\in[0,1]} |T_n''(x)| \le \inf_{x\in[0,1]} |T_n'(x)|$ for every $n \in \mathbb{N}$, and (8) is proved. □

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