Cage on zero

> Curiously enough, the twelve-tone system has no zero in it. Given a series: 3, 5, 2, 7, 10, 8, 11, 9, 1, 6, 4, 12 and the plan of obtaining its inversion by numbers which when added to the corresponding ones of the original series will give 12, one obtains 9, 7, 10, 5, 2, 4, 1, 3, 11, 6, 8 and 12. For in this system 12 plus 12 equals 12. There is not enough of zero in it.
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> John Cage, "Eric Satie", *Silence: Lectures and Writings*, Calder and Boyars, 1968.

I contend that Cage is confusing two different zeros, the zero element of the real numbers and the zero element of the integers mod 12.

Real numbers

Cage was very much attracted to the Zen concept of emptiness. One of his most famous compositions, entitled $4'33''$, involves a pianist sitting at the keyboard of a piano for 4 minutes and 33 seconds without striking a note; the audience notices the background noise (since no emptiness is truly empty). The real numbers represent sound intensity, so zero is the absence of sound.

Integers mod 12

Musical notation is based on the fact that notes an octave apart (that is, when the frequency of one is double that of the other) have a very similar subjective effect in melodic terms. So we regard such notes as 'equivalent'. More generally, two notes are equivalent if they are a whole number of octaves apart.

In Western music, only a discrete set of notes is used. The octave is divided into twelve intervals called *semitones*. Thus, the semitones appear (on a keyboard, say), stretching to infinity in both directions like the integers. As above, two semitones are equivalent if they differ by a whole number of octaves; that is, if (as integers) they are congruent mod 12. So the musical scale, for thematic purposes, has the structure of the integers mod 12. Various musical operations fit into this framework. For example, transposition just involves adding a fixed constant to each note. Inversion involves replacing each equivalence class by its negative. (This is what Cage describes.)

Two kinds of zeros

The equivalence classes referred to are the congruence classes mmod 12, that is, the cosets of $12\mathbf{Z}$ in $\mathbf{Z}$. We can make any choice of coset representatives we like. Mathematicians usually use $0, 1, 2, \ldots, 11$. Musicians use 12 instead of 0 as the representative of the class $12\mathbf{Z}$, so that their semitones are labelled $1, 2, 3, \ldots, 12$.

Now Cage's arithmetic checks, since $-3 = 9$, $-5 = 7$, etc., in $\mathbf{Z}/12\mathbf{Z}$ (the integers mod 12). The mathematician says $-0 = 0$, the musician $-12 = 12$; it is exactly the same, just involving a different choice of coset representative.

So, contrary to what Cage says, there is a zero in the twelve-tone scale (but musicians call it 12); and it has nothing to do with the real number zero, the zero of intensity or absence of sound when the pianist is not striking the keys.

Footnote

Non-mathematicians often have trouble with the concept of zero. For example, when the QAA inspectors visited the School of Mathematical Sciences last semester, they gave the School a score in the range $\{1, 2, 3, 4\}$ on each of six heads related to undergraduate teaching. So, even if we had been completely unsatisfactory on every head, we would still have scored 6. Far more logical to use the range $\{0, 1, 2, 3\}$, so that a score of zero would have meant zero. But that would have required the designers of the scheme to understand zero.

In fact, the use of positive integers goes back to the mists of time, but zero was not invented until about 1500 years ago. So perhaps it is just too new-fangled!