Problems from the Thirteenth British Combinatorial Conference

Edited by Peter J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
London E1 4NS
U.K.
p.j.cameron@qmw.ac.uk

Abstract

These problems were presented at the thirteenth British Combinatorial Conference, held at the University of Surrey in 1991. I have updated the references and appended comments on the subsequent history of the problems.

Problem 1 (BCC13.1) Stable partitions of graphs. Proposed by Z. Füredi.

Correspondent: Z. Füredi
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W Green St.
Urbana, IL 61801-2917
USA

A partition of the vertex set V of a (simple) graph G into two non-empty parts V_1 and V_2 is called *stable* if, for $i = 1, 2$ and all vertices $x \in V_i$,

$$\deg_{G|V_i}(x) > \tfrac{1}{2}\deg_G(x).$$

A graph with a stable partition is called *stable*. The complete graph is not stable, but every 3-regular graph other than K_4 is stable.

What is the number of stable graphs? In particular, is the proportion of graphs on n vertices which are stable $o(1)$?

Problem 2 (BCC13.2) Ramsey perfect graphs. Proposed by J. Nešetřil.

Correspondent: J. Nešetřil
KAM
Department of Applied Mathematics
Charles University
Malostranské nám. 25
11800 Praha 1
Czechoslovakia

Is it true that, for every perfect graph G, there exists a perfect graph H such that, for every partition $E(H) = E_1 \cup E_2$, there is an induced subgraph G' of H such that G' is isomorphic to G and $E(G') \subseteq E_i$ for some i ($i = 1$ or 2)?

We expect a negative answer to this question, despite the fact that the analogous question for partitions of vertices has an affirmative answer (using the Lovász Multiplication Lemma).

Problem 3 (BCC13.3) Reconstructing spanning trees. Proposed by W. L. Kocay.

Correspondent: C. Wakelin
Department of Mathematics
University of Nottingham
University Park
Nottingham NG7 2RD
U.K.

Is the number of spanning trees with exactly d automorphisms reconstructible from the deck of vertex-deleted subgraphs of a graph, for each d?

Problem 4 (BCC13.4) 1 time 2-tough implies 2 times 1-tough?. Proposed by C. Hoede.

Correspondent: C. Hoede
Department of Applied Mathematics
Universityof Twente
P.O. Box 217
7500AE Enschede
The Netherlands

Prove that the edge-set of a 2-tough graph can be partitioned into two sets E_1 and E_2 such that each of E_1 and E_2 induces a 1-tough spanning subgraph of G.

If this conjecture is true, then a 4-regular 2-tough graph would consist of two edge-disssjoint Hamiltonian cycles. One could accordingly extend Thomassen's conjecture to the assertion that any 2-tough graph contains two edge-disjoint Hamiltonian cycles.

[A graph is *t-tough* is, for any vertex cutset Sm $G - S$ has at most $|S|/t$ components. Chvátal conjectured that, for $t > \frac{3}{2}$, a t-tough graph is Hamiltonian. This was refuted by Thomassen, who conjectured instead that a 2-tough graph is Hamiltonian.]

Problem 5 (BCC13.5) Path-tough graphs. Proposed by I. Schiermeyer.

Correspondent: I. Schiermeyer
Lehrstuhl C für Mathematik
Techniche Hochschule Aachen
Templergraben 55
W-5100 Aachen
Germany

Let G be a simple graph on n vertices, where $n \geq 3$, which is *path-tough* (that is, $G - v$ has a Hamiltonian path for any vertex v). Suppose that $d(u) = d(v) = d(w) \geq n - 2$ for any three independent vertices u, v, w. Prove that either G is Hamiltonian, or G is isomorphic to the Petersen graph.

Problem 6 (BCC13.6) Hamiltonian cycles and a bit more. Proposed by R. Häggkvist.

Correspondent: R. Häggkvist
Department of Mathematics
University of Umeå
S-90187 Umeå
Sweden

Prove that any graph of order n amd minimum degree at least $\frac{1}{2}(n+1)$ contains a subgraph which is a Hamiltonian cycle together with a longest diagonal.

Problem 7 (BCC13.7) The second longest cycle. Proposed by R. Häggkvist.

Correspondent: R. Häggkvist
Department of Mathematics
University of Umeå
S-90187 Umeå
Sweden

Give lower bounds for the length of the second longest cycle in a Hamiltonian 3-regular graph. (The best bound currently is $n-4\sqrt{n}$.) More generally, the same problem with "minimum degree 3" in place of "3-regular".

Problem 8 (BCC13.8) 4-chromatic covering graphs. Proposed by D. Youngs.

Correspondent: D. Youngs
Racal Research Ltd.
Worton Drive
Reading, Berks. RG2 0SB
U.K.

What is the smallest 4-chromatic covering graph? (A graph is a *covering graph* if its edges can be directed in such a way that it becomes the Hasse diagram of a poset.) The answer lies between 12 and 14 inclusive.

Problem 9 (BCC13.9) Special Hamiltonian paths. Proposed by T. Trotter and S. Felsner.

Correspondent: T. Trotter
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804
USA

Consider the diagram of the poset of all subsets of $\{1, 2, \ldots, n\}$. As a graph, this is the n-cube. Does the graph have a Hamiltonian path $A_1, A_1, \ldots, A_{2^n}$ starting at $A_1 = \emptyset$ and having the following property?

> If, at step i, you visit a set A_i, then you must previously have visited all subsets of A_i with at most one exception. If there is an exception, you must visit it next (that is, it is A_{i+1}).

For example, with $n = 4$,

$$\emptyset, 1, 12, 2, 23, 3, 34, 4, 24, 124, 14, 134, 13, 123, 1234, 234$$

is such a path.

Problem 10 (BCC13.10) Interval-regular graphs. Proposed by H. M. Mulder.

Correspondent: H. M. Mulder
Department of Mathematics
Erasmus Universiteit
Postbus 1738
3000DR Rotterdam
The Netherlands
`martyn@cvx.eur.nl`

Let G be a connected simple graph. The *interval* $I(u,v)$ between vertices u and v is the set of all vertices which lie on some shortest path form u to v. The graph is called *interval-regular* if, for any two vertices u and v, we have

$$||N(u) \cap I(u,v)| = d(u,v),$$

where $N(u)$ is the set of neighbours of u.

Examples of interval-regular graphs include hypercubes, and the 2-cube and 3-cube with added edges joining all vertices in $N(u)$ for some vertex u. The class of interval-regular graphs is closed under taking Cartesian products. See [9].

Conjecture: Let G be interval-regular. Then

$$x, y \in I(u,v) \Longrightarrow I(x,y) \subseteq I(u,v)$$

for any vertices u and v.

Problem 11 (BCC13.11) Permutations of a cube. Proposed by M. K. Siu.

Correspondent: M. K. Siu
Department of Mathematics
University of Hong Kong
`mathsiu@hkucc.earn`

Let C be the n-cube graph, d the graph metric. Is every permutation of the vertices of c the composite of at most n permutations s_i, each satisfying

$$d(x, s_i(x)) \leq 1$$

for all $x \in C$?

Editor's Note: More generally, which graphs of diameter n have this property?

Note that it is not true that a permutation s he vertices of C which satisfies $d(x, s(x)) \leq k$ can be written as the product of at most k permutations s_i as above.

Problem 12 (BCC13.12) Permutations of projective space.
Proposed by A.Gyárfás.

Correspondent: P. J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.
p.j.cameron@qmw.ac.uk

For which n and q does there exist a permutation π of the point set of $PG(n,q)$ with the property that, for any hyperplane H, there exists a hyperplane H' with $\pi(H) \cap H' = \emptyset$? It is known that the answer is "yes" for $n = 2$ (asymptotically, almost all permutations have this property), and "no" for $n > q$ (by a short argument due to A. Blokhuis).

Problem 13 (BCC13,13) q-polynomials. Proposed by A. Bonisoli.

Correspondent: A. Bonisoli
Dipartimento di Matematica
Università della Basilicata
via N. Sauro 85
85100 Potenza
Italy
bonisoli@pzvx85.cineca.it

A polynomial $c(z) = \sum c_i z^i$ ovber $GF(q)$ is said to be a *q-polynomial* if $c_i \neq 0$ only if i is a power of q. (See [8].) Let $m = 2^d - 1$ be a Mersenne prime. Does there exist a 2-polynomial

$$c(z) = \sum_{i=0}^{d} x_i z^{2^i}$$

of degree 2^d such that $c(z)/z$ is irreducible over $GF(2)$? The answer is "yes" for the first four Mersenne primes, viz., 3, 7, 31, 127.

Problem 14 (BCC13.14) Cyclic shifts of binary words. Proposed by P. J. Cameron.

Correspondent: P. J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.
p.j.cameron@qmw.ac.uk

For odd n, let W_n be the space of binary words of length n and even weight. Let $f(n)$ be the maximum codimension of a subspace U of W_n such that the union of all cyclic shifts of U is equal to W_n. It is known that $f(n) \geq 2$ for $n > 3$, but little more is known. Does $f(n) \to \infty$ as $n \to \infty$? Or is $f(n) = 2$ for infinitely many n?

Problem 15 (BCC13.15) A generalisation of arcs. Proposed by J. Bierbrauer.

Correspondent: J. Bierbrauer
Mathematisches Institut deer Universität
Im Neuenheimer Feld 288
69 Heidelberg
Germany

Let $(V, \mathcal{B})$ be an affine plane of odd order q. Given a function $w : V \to \mathbb{Z}$ such that

$$\sum_{p \in L} w(p) \leq 2$$

for all $L \in \mathcal{B}$, set

$$\mathrm{mass}(w) = \sum_{p \in V} w(p).$$

Then $\mathrm{mass}(w) \leq q + 2$ holds. The value $q + 1$ can be attained (by the characteristic function of a conic). Prove that necessarily $\mathrm{mass}(w) \leq q + 1$.

Problem 16 (BCC13.16) Perfect Steiner triple systems. Proposed by C. J. Colbourn and A. Rosa.

Correspondent: A. Rosa
Department of Mathematics
McMaster University
Hamilton, Ontario
Canada L8S 4K1
rosa@sscvax.cis.mcmaster.ca

Let $(V, \mathcal{B})$ be a Steiner triple system of order v (a STS(v)). For distinct points x, y contained in the block $\{x, y, z\}$, the *interlacing graph* G_{xy} is the 2-regular graph on the vertex set $V \setminus \{x, y, z\}$, in which a and b are adjacent whenever $\{a, b, x\}$ or $\{a, b, y\}$ is in $\mathcal{B}$. The STS is called *perfect* if G_{xy} is a Hamiltonian cycle for all $x, y \in V$.

Problem: Find more perfect STS. Are there infinitely many?

Only four are known, with orders 7, 9, 25 and 33. There is none of order 13 or 15. All known examples have point-transitive automorphism groups.

Problem 17 (BCC13.17) A generalisation of affine designs.

Proposed by P. J. Cameron and M. E. Kimberley.

Correspondent: P. J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.
p.j.cameron@qmw.ac.uk

An *affine design* is a resolvable 2-design in which any two non-parallel blocks meet in a constant number y of points. What happens if we replace "non-parallel" by "non-disjoint" in this definition? In addition to affine designs, there are resolvable designs with $\lambda = 1$ (and $y = 1$), for example Kirkman systems. No others are known: the first unknown case is a resolvable 2-$(70, 10, 6)$ design with $y = 2$. See [4].

Problem 18 (BCC13.18) Blocking-set-free configurations.

Proposed by J. W. DiPaola and H. Gropp.

Correspondent: H. Gropp
Muehlingstraße 19
D-6900 Heidelberg
Germany

A configuration n_3 has n points and n lines, with three points on each line and three lines through each point, so that two points lie on at most one line. A *blocking set* is a set of points meeting every line but containing none. For which $n \geq 7$ does there exist a connected n_3 configuration with no blocking sets?

It is known that such configurations exist for all but finitely many values of n. They do not exist for the falues 8–12 or 14. The values in doubt are 15–18, 20, 23, 24, 26, 29, 30, 32, 38, 44. See [6].

Problem 19 (BCC13.19) Permutations with few distances.
Proposed by P. J. Cameron.
Correspondent: P. J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.
`p.j.cameron@qmw.ac.uk`

For fixed s and large n, the best known upper and lower bounds for the maximum number of permutations of an n-set with s different distances are both roughly $(cn/s)^{2s}$ (with different values of c). (See [3].) Find the correct value.

Problem 20 (BCC13.20) Partial transversals of Latin rectangles.
Proposed by A. J. W. Hilton.
Correspondent: A. J. W. Hilton
Department of Mathematics
University of Reading
Whiteknights
Reading
U.K.

Let R be an $n \times 2n$ Latin rectangle on $2n$ symbols. A partial transversal T of size s of R is a collection of s cells, no two in the same row or column, and no two containing the same symbol. Is it true that R can be expressed as the union of $2n$ partial transversals of size n?

An equivalent formulation: Call two $n \times 2n$ Latin rectangles R, S on the same set of symbols *orthogonal* if the pairs (r_{ij}, s_{ij}), for $i = 1, \ldots, n$ and $j = 1, \ldots, 2n$, are all distinct. Does every $n \times 2n$ Latin rectangle have an orthogonal mate?

Problem 21 (BCC13.21) Semi-Latin squares. Proposed by R. A. Bailey.

Correspondent: R. A. Bailey
Department of Mathematics and Statistics
Goldsmiths College
Lewisham Way
London SE14 6NW
U.K.
`r.bailey@gold.lon.ac.uk`

A *semi-Latin square* is an $n \times n$ array with k symbols (chosen from a set of size nk) in each cell, such that each symbol occurs once in each row or column. We impose the further property:

No two symbols occur together in a cell more than once.

Such a structure clearly exists if there are k mutually orthogonal Latin squares of order n. on disjoint sets of symbols.

Problem: Find constructions for values (n, k) for which a set of k m.o.l.s. of order n does not exist (or is unknown). Examples are known for $(n, k) = (6, 2)$ or $(6, 3)$. What about $(6, 4)$ or $(10, 3)$?

Editor's Note: These designs are now known as SOMAs (see Phillips and Wallis [10]). For recent results on their structure and classification, including a construction for $(10, 3)$, see [11]. Further information on semi-Latin squares is available from [1].

Problem 22 (BCC13.22) Tiling the square. Proposed by D. Youngs.

Correspondent: D. Youngs
Racal Research Ltd.
Worton Drive
Reading RG2 0SB
U.K.

(a) What is the least *odd* number of congruent non-rectangular tiles needed to tile a square?

(b) Is there such a tiling in which the tiles are not polyominoes?

The best tiling known to the proposer uses 25 copies of the polyomino with two rows containing 6 and 3 squares, aligned at one end.

Problem 23 (BCC13.23) Covering the square. Proposed by F. Barnes.

Correspondent: F. Barnes
Department of Mathematics
Kenyatta University
P.O. Box 43844
Nairobi
Kenya

(a) Prove that, for any partition of the plane into sets (or regions) of diameter 1, the density must be at least $8/3\sqrt{3}$.

(b) A finite variation. What is the largest square which can be partitioned into n sets of diameter (at most) 1? The answer is known for $n \leq 5$. In general, we would expect a hexagonal honeycomb with some distortion at the edges.

An alternative formulation of (b) asks for the chromatic number of the graph whose vertices are the points of the unit square, two points adjacent if their distance exceeds d.

Problem 24 (BCC13.24) Sum-free sets containing 2. Proposed by P. J. Cameron.

Correspondent: N. J. Calkin
Department of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
USA
`calkin@math.gatech.edu`

What is the probability that, in a random sum-free set S of natural numbers, 2 is the only even number in S? (Is it zero or not?)

The probability measure is defined by the following rule. Consider the natural numbers in their usual order. If n is the sum of two numbers in S, then $n \notin S$; otherwise, decide on the toss of a fair coin. It is known that the probability that S contains no even number is non-zero, but the present problem seems a bit more delicate.

Editor's Note: This problem has been solved by the proposer and the presenter [2], who showed that the probability is strictly positive and gave a heuristic estimate for it.

Problem 25 (BCC13.25) Partitions of intersecting families.
Proposed by N. Alon, P. Seymour and Z. Füredi.

Correspondent: Z. Füredi
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green St.,
Urbana, IL 61801-2917
USA
zoltan@symcom.math.uiuc.edu

Let $\mathcal{F}$ be an intersecting family of k-subsets of the n-element set V, that is, $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. Let $p(\mathcal{F})$ be the minimum p for which one can find p pairs (2-subsets) $P_!, \ldots, P_p$ of V such that every member of $\mathcal{F}$ contains some P_i. Now let $f(n,k)$ be the maximum of $p(\mathcal{F})$, over all such intersecting families. Is it true that $f(n,k) \leq n$ for all n?

If true, this would imply a strengthened form of the case $t+1$ of Larman's conjecture, since, setting

$$\mathcal{F}_i = \{F \in \mathcal{F} : p_i \subseteq F, P_j \not\subseteq F \text{ for } j < i\},$$

then $\{\mathcal{F}_i : i = 1, \ldots, n\}$ is a decomposition of $\mathcal{F}$ into 2-intersecting families.

(*Larman's conjecture* [7] asserts that, if $\mathcal{F}$ is a t-intersecting family of k-subsets of the n-set V, that is, $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$, then $\mathcal{F}$ can be decomposed into n subfamilies each of which is $(t+1)$-intersecting.

Editor's note: A similiar question can be asked for arbitrary families of sets (not all of the same size).

Problem 26 (BCC13.26) Some families of sets. Proposed by N. J. Calkin.

Correspondent: N. J. Calkin
Department of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
USA
calkin@math.gatech.edu

What can be said about families $\mathcal{F}$ of subsets of an n-set V such that

(a) $F_1, F_2 \in \mathcal{F} \Longrightarrow F_1 \not\subset F_2$;

(b) $F_1, F_2 \in \mathcal{F} \Longrightarrow F_1 \cap F_2 \neq \emptyset$;

(c) $(\forall x \in V)(\exists F_1, F_2 \in \mathcal{F})(F_1 \cap F_2 = \{x\})$?

Problem 27 (BCC13.27) Some problems on perfect groups.
Proposed by J. Dénes and P. Yff.
Correspondent: P. Yff

(a) Prove the Feit–Thompson Theorem by elementary means. (Remark: either of the following equivalents of the Feit–Thompson Theorem may be more convenient:

- A finite perfect group can be generated by a self-inverse conjugacy class of elements of odd order (Heineken, unpublished);
- If G has odd order n, then some element of G is not the product of n distinct factors (Dénes and Hermann [5]).)

(b) Show that a perfect group can be generated by an involution and an element of odd order which is conjugate to its inverse.

(c) Characterise finite groups in which every element is a commutator. In particular, show that every non-abelian finite simple group has this peroperty (*Ore's conjecture*).

(d) Characterise finite groups G in which every element of the derived group is the product of k commutators. (This condition can be expressed in terms of the notion of k-conjugacy (Yff [12]).

References

[1] R. A. Bailey, Semi-Latin squares, available at
`http://www.maths.qmw.ac.uk/~rab/sls.html`

[2] N. J. Calkin and P. J. Cameron, Almost odd random sum-free sets, *Combinatorics, Probability and Computing* **7** (1998), 27–32.

[3] P. J. Cameron, Metric and geometric properties of sets of permutations, pp. 39–53 in *Algebraic, Extremal and Metric Combinatorics* (ed. M.-M. Deza, P. Frankl & I. G. Rosenberg), London Math. Soc. Lecture Notes **131**, Cambridge Univ. Press, Cambridge, 1988.

[4] P. J. Cameron, Quasi-symmetric designs possessing a spread, pp. 231–236 in *Combinatorics '88* (ed. A. Barlotti *et al.*), Mediterranean Press, Roma, 1992.

[5] J. Dénes and P. Hermann, On the product of all elements in a finite group, *Ann. Discrete Math.* **15** (1982), 105-109.

[6] J. W. DiPaola and H. Gropp, Symmetric configurations without blocking sets, *Mitt. Math. Semin. Giessen* **201** (1991), 49-54.

[7] D. G. Larman and N. K. Tamvakis, The decomposition of the n-sphere and the boundaries of plane convex domains, *Ann. Discrete Math.* **20** (1984), 209–214.

[8] R. Lidl and H. Niederreiter, *Finite fields*, Cambridge University Press, Cambridge, 1996.

[9] H. M. Mulder, *The interval function of a graph*, Mathematical Centre Tracts **132**, Mathematisch Centrum, Amsterdam, 1980.

[10] N. C. K. Phillips and W. D. Wallis, All solutions to a tournament problem. *Congressus Numerantium* **114** (1996), 193–196.

[11] L. H. Soicher, On the structure and classification of SOMAs: generalizations of mutually orthogonal Latin squares, *Electronic J. Combinatorics*, **6(1)** (1999), #R32 (15 pages). Available at `http://www.combinatorics.org`.

[12] P. Yff, On k-conjugacy in a group, *Proc. Edinburgh Math. Soc.* II **14** (1964), 1-4.