Problems from the Fourteenth British Combinatorial Conference

Edited by Peter J. Cameron, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, U.K.

These problems were presented at the Problem Session held at the conference. The originator of the problem is given, where known and different from the proposer. If no originator is given, either it is the proposer's problem, or my ignorance is to blame. Problems are grouped roughly by subject matter. Since an edition of these problems has been circulated to participants already, I have included the original numbers in the form BCCn.

Two of the problems have now been solved, those of D. B. West on the bandwidth of a triangular lattice graph (solved by R. Hochberg, C. McDiarmid and M. Saks), and P. J. Cameron on a bijection between permutations with even and odd cycles (solved by R. Lewis and S. Norton). Papers containing these solutions have been submitted for publication in the Conference proceedings; the problems in question have been left in the list for reference. In addition, R. Häggkvist and J. Janssen have solved part of Problem 8; their solution will appear elsewhere.

Problem 1 (BCC14.1). Total colourings of hypergraphs. Posed by P. Cowling.

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A *total colouring* of a hypergraph is a colouring of vertices and edges such that

(a) the restrictions to vertices and edges are strong colourings;
(b) an incident vertex and edge have different colours.

The *total chromatic number* is the least number of colours required for a total colouring.

Conjecture. If $H = (V, \mathcal{E})$ is a linear hypergraph (two vertices on at most one edge) with total chromatic number $\chi_T(H)$, then

$$\chi_T(H) \leq \max_{x \in V} \left| \bigcup_{\substack{E \in \mathcal{E} \\ x \in E}} E \right| + 1.$$

Problem 2 (BCC14.2). Critical K_l-free graphs. Posed by J. Schöhheim.

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It is known that k-chromatic critical graphs on n vertices have at least $\left(\frac{k-1}{2}\right)n$ edges. Gallai showed that a better lower bound holds for graphs containing no K_k. Can this bound be further improved for graphs containing no K_l, for fixed l with $3 \leq l \leq k$?

For example, with $k = 10$, $l = 9$, can the bound $4.5n$ be improved to $5n - 10$?

Problem 3 (BCC14.3). Bandwidth of a graph. Posed by D. B. West.

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The *bandwidth* of an n-vertex graph G is

$$\min_f \max_{x \sim y} |f(x) - f(y)|,$$

where the minimum is over all bijections from the vertex set to $\{1, \ldots, n\}$.

What is the bandwidth of the "triangular lattice" graph whose vertices are all triples of non-negative integers with sum l, vertices (x, y, z) and (x', y', z') being adjacent whenever $|x - x'| + |y - y'| + |z - z'| = 2$? (A lower bound of $l/2$ is known, and an upper bound of $l + 1$ is obtained by numbering the vertices in layers.)

Problem 4 (BCC14.4). How small is Tutte's wheel? Posed by A. Shastri.

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W. T. Tutte proved that any 3-connected graph can be obtained from a wheel by repeatedly adding an edge or splitting the central vertex (keeping the minimum degree at least 3).

Conjecture. Any 3-connected cubic graph on n vertices may be obtained by this procedure from a wheel on k vertices, where $k \geq cn$ (for some absolute constant c).

Problem 5 (BCC14.5). Characteristic polynomials of graphs. Posed by R. Häggkvist.

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How many distinct characteristic polynomials of (adjacency matrices of) n-vertex graphs are there?

The proposer conjectures that a typical n-vertex graph has n^2 cospectral mates, so that the answer to the problem is $O(2^{n(n-1)/2}/n^2 n!)$.

Problem 6 (BCC14.6). Cliques and cocliques in Cayley graphs. Posed by N. Alon.

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Conjecture. There is a constant c such that, for every finite group G of order $n > 1$, there is a symmetric (i.e., inverse-closed) generating set S for G such that the Cayley graph $\Gamma(G, S)$ has neither a clique nor an independent set of size $c \log n$.

This is not known for any infinite sequence of finite groups; but it is true with $\log^2 n$ replacing $\log n$.

Problem 7 (BCC14.7). Local structure in 2-transitive graphs. Posed by A. A. Ivanov.

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Problem. Determine the vertex and edge stabilizers in all locally finite 2-transitive graphs in which $G_1(x) = 1$.

(A graph is 2-transitive if it admits a group G acting transitively on 2-arcs. The condition $G_1(x) = 1$ means that a vertex stabilizer acts faithfully (and 2-transitively) on its neighbours. The answer to this problem would be a list of pairs (H, t), where H is a finite 2-transitive group (the vertex-stabiliser $G(x)$) and t an outer automorphism of order 2 of the stabilizer H_y (so that $H_y\langle t\rangle = G(e)$, where $e = \{x, y\}$), along with the trivial possibility that $G(e) = H_y \times 2$.)

Problem 8 (BCC14.8). The rows of a Latin square. Posed by P. J. Cameron.

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(a) It is known that, for almost all Latin squares of order n (i.e., a proportion tending to 1 as $n \to \infty$), the rows of the square (regarded as permutations) generate S_n or A_n. Is this statement still true if the squares are normalized so that the first row is the identity permutation?

(b) Is it true that the distribution of the number of rows of a random Latin square which are odd permutations is "approximately" binomial $B(n, \frac{1}{2})$?

(c) Let $M(n)$ and $m(n)$ denote the maximum and minimum numbers of extensions of a $2\times n$ Latin rectangle to an $n\times n$ Latin square. Find a good upper bound for $M(n)/m(n)$.

(d) How do you choose a random Latin square of order n?

Editor's note: In connection with (b), R. Häggkvist and J. Janssen have shown that the proportion of Latin squares in which all rows are even permutations is exponentially small. This was the form asked at the Conference; the strengthened version was suggested by J. Janssen.

Problem 9 (BCC14.9). A bijective proof of the Dyson conjecture. Posed by R. Lewis. Originated by F. Dyson and G. Andrews.

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Let $R(r,m,n)$ denote the set of partitions of n whose rank is congruent to r modulo m, where the *rank* of a partition is the largest part minus the number of parts. Freeman Dyson conjectured, and Atkin and Swinnerton-Dyer proved, that

$$|R(0,5,5n+4)| = |R(1,5,5n+4)| = \ldots = |R(4,5,5n+4)|.$$

The problem is to find a bijective proof.

Problem 10 (BCC14.10). Even and odd permutations. Posed by P. J. Cameron.

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For even n, the number of permutations of $\{1,\ldots,n\}$ with all cycles of even length is equal to the number of permutations with all cycles of odd length. Find a bijective proof of this fact.

Problem 11 (BCC14.11). How many sum-free sets? Posed by P. J. Cameron. Originated by P. J. Cameron and P. Erdős.

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Let $s(n)$ be the number of sum-free subsets of $\{1, \ldots, n\}$ (i.e., containing no solution to $x + y = z$). Show that there exist constants c_o and c_e such that $s(n)/2^{n/2} \to c_o$ or c_e as $n \to \infty$ through odd or even values respectively.

It is known only that $s(n) = 2^{(\frac{1}{2}+o(1))n}$.

Problem 12 (BCC14.12). Non-crossing queens. Posed by G. B. Khosrovshahi.

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What is the maximum number of non-crossing n-queens? It is known that the maximum is n if n is prime.

Problem 13 (BCC14.13). Block-transitive designs. Posed by P. J. Cameron. Originated by P. J. Cameron and C. E. Praeger.

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A t-(v, k, λ) design has v points and a collection of blocks of size k, any t points lying in exactly λ blocks. Terms such as "block-transitive" apply to the action of the automorphism group.

(a) Show that there is no block-transitive 6-design.
(b) Show that a block-transitive, point-imprimitive 3-design satisfies $v \leq \binom{k}{2} + 1$.
(c) Is there a block-transitive 2-$(\infty, 4, 1)$ design which is not point-transitive?

Problem 14 (BCC14.14). Blocking-set-free configurations. Posed by H. Gropp. Originated by J. W. DiPaola and H. Gropp.

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A configuration n_3 has n points and n lines, with three points on each line and three lines through each point, such that two points lie on at most one line. A *blocking set* is a set of points meeting every line but containing none. For which $n \geq 7$ does there exist a connected n_3 configuration with no blocking sets?

It is known that such configurations exist for all but finitely many values of n. They do not exist for the values 8–12 or 14. The values in doubt are 15–18, 20, 23, 24, 26. (The

value 15 may now be settled). See J. W. DiPaola and H. Gropp, *Mitt. Math. Sem. Gießen* (to appear).
Editor's note: This problem is repeated in updated form from the Problem Session at the 13^{th} BCC.

Problem 15 (BCC14.15). Arranging rows and columns. Posed by D. B. West.

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A matrix of zeros and ones is said to be "zero-partitionable" if its rows and columns can be permuted independently so that the zeros of the resulting matrix can be labeled R or C such that

- every position to the right of an R is a 0 labeled R, and
- every position below a C is a 0 labeled C.

What is the complexity of recognizing zero-partitionable matrices?

(This is equivalent to recognition of interval digraphs. If a 0 is allowed to receive both R and C, this becomes recognition of digraphs with Ferrers dimension 2, which runs in polynomial time.)