Problems from the 15th British Combinatorial Conference

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Most of these problems were presented at the problem session at the conference on 7 July 1995. A few others, given to me before or after the session, have been added.

Problem 1 (BCC15.1). A generalization of Hadwiger's conjecture. Posed by Ding, Oporowski, Sanders, and Vertigan.

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A *vertex partition* of G is a set $\{A_1, \ldots, A_k\}$ of induced subgraphs such that $V(G)$ is the disjoint union $V(A_1) \cup \ldots \cup V(A_k)$.

Conjecture. Every graph with no K_n minor has a vertex partition into $n-m+1$ graphs with no K_m minor.

For $m = 2$, this is Hadwiger's conjecture. It is known to be true for $n \leq 5$ (Wagner [3]; Ding, Oporowski, Sanders, Vertigan); for $n = 6, m = 2$ (Robertson, Seymour, Thomas [2]), and for $6 \leq n \leq 8$, $m = 3$ (Jørgensen [1]).

1. L. K. Jørgensen, Contractions to K_8, *J. Graph Theory* **18** (1994), 431–448.

2. N. Robertson, P. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.

3. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.

Problem 2 (BCC15.2). Uniquely total colourable graphs. Posed by M. Behzad and E. S. Mahmoodian.

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A *total colouring* of a graph is a colouring of the vertices and edges in such a way that no two adjacent or incident elements have the same colour.

Problem. Show that, apart from empty graphs, paths, and cycles C_{3k}, there is no graph which has a unique total colouring (in the minimum number of colours).

A prize of 500000 Iranian rials is offered for this problem.

1. E. S. Mahmoodian, Defining sets and uniqueness in graph colourings: a survey, submitted.

Problem 3 (BCC15.3). 1-track-less orientations. Posed by Jörg Zuther.

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A 1-*track* is a one-way infinite directed path (which may be directed either in or out).

Problem. Characterize those graphs which admit a 1-track-less orientation.

Note that every locally finite graph, and every m-partite graph (for finite m) has a 1-track-less orientation, but the countable complete graph does not.

Problem 4 (BCC15.4). Continuous maps between graphs. Posed by Anthony Hilton.

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A map is k-*to*-1 if the inverse of every point in the codomain has cardinality k.

Problem. Determine the triples (k, m, n) for which there is a k-to-1 continuous map from $K_{m,m}$ to $K_{n,n}$, where these graphs are regarded as 1-dimensional simplicial complexes in the usual way (with edges homeomorphic to $[0, 1]$).

1. A. J. W. Hilton, Initial and threshold values for exactly k-to-1 continuous maps between graphs, *Congressus Numerantium* **91** (1992), 254–270.

Problem 5 (BCC15.5). A generalization of Tarsi's problem. Posed by R. Klein and J. Schönheim.

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A graph is m-*degenerate* if every subgraph has a vertex of valency at most m.

Problem. Prove or disprove that a graph which is the edge-disjoint union of subgraphs $G_1, \ldots, G_s$, where G_i is m_i-degenerate, can be coloured with

$$\sum_{i=1}^{s} m_i + \left\lfloor \tfrac{1}{2} \left(1 + \sqrt{1 + 8 \sum_{1 \le i < j \le s} m_i m_j} \right) \right\rfloor$$

colours.

For $s = 2$, $m_1 = 1$, $m_2 = 2$, this is M. Tarsi's problem [1].

1. R. Klein, On the colorability of m-composed graphs, *Discrete Math.* **133** (1994), 181–190.

Problem 6 (BCC15.6). Common vertices on longest paths. Posed by T. Gallai.

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Let G be a finite connected graph. Do any three longest paths in G have a common vertex? It is trivially true that every two longest paths have a common vertex; but there are graphs in which no vertex lies on all the longest paths.

1. T. Gallai, Problem 4, in: P. Erdős, G. Katona, Eds., “Theory of graphs”, Proc. Colloq. Tihany, Hungary, Sept. 1966, Academic Press, New York, 1968, p. 362.

2. T. Zamfirescu, A two-connected planar graph without concurrent longest paths, *J. Combinatorial Theory* Ser. B **13** (1972), 116–121.

3. S. Klavžar and M. Petkovšek, Graphs with nonempty intersection of longest paths, *Ars Combinatoria* **29** (1990), 43–52.

Problem 7 (BCC15.7). Edge-colourings of complete graphs. Posed by Peter Johnson.

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Suppose that the edges of the complete graph K_n $(n > 1)$ are coloured with four colours R, G, B, Y such that each colour-class gives a connected subgraph on n vertices. It is easy to see from Satz 1.2(3) of Gallai [1] that at least three of the four triangles with edge colourings RGB, RGY, RBY, GBY occur.

Questions.

(a) Do all four occur?
(b) If not, how small can n be?
(c) What happens with more than four colours?

1. T. Gallai, Transitiv Orientbare Graphen, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 25–66.

Problem 8 (BCC15.8). On the probability of connectedness. Posed by Peter Cameron.

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Which graphs G have the property that, in the class $\mathcal{X}(G)$ of graphs having no induced subgraph isomorphic to G, the limiting probability of connectedness is strictly between zero and one (in either the unlabelled or the labelled case)? (The smallest G with this property is the path of length 3; the probability of connectedness in $\mathcal{X}(G)$ is $\frac{1}{2}$ if the number of vertices is greater than one.)

Problem 9 (BCC15.9). Characteristic and chromatic polynomials. Posed by Roland Häggkvist.

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The *characteristic polynomial* of a graph G is the polynomial $\det(xI - A(G))$, where $A(G)$ is the adjacency matrix of G. Its roots are the *eigenvalues* of G.

Question. Are there more characteristic polynomials than chromatic polynomials of graphs on n vertices?

Problem 10 (BCC15.10). Graphs with three eigenvalues. Posed by Willem Haemers.

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Let G be a connected graph with just three distinct eigenvalues. Such a graph, if regular, must be strongly regular; and any strongly regular graph has this property. Non-regular examples include the complete bipartite graphs, and one further example on 36 vertices constructed by M. Muzychuk.

Questions.
(a) Is it true that G has at most two distinct valencies?
(b) Is G switching-equivalent to a null or strongly regular graph?
(c) Find more examples.

(The operation of *switching* a graph with respect to a set X of vertices replaces each edge from X to its complement by a non-edge and each such non-edge by an edge, leaving edges within or outside X unaltered.)

Editor's Note. M. Klin and M. Muzychuk [2] have pointed out that a family of examples were found in 1981 by Bridges and Mena [1], and have also constructed some 'sporadic' examples and re-formulated and analysed the question.

1. W. G. Bridges and R. A. Mena, Multiplicative cones — a family of three eigenvalue graphs, *Aequat. Math.* **22** (1981), 208–214.
2. M. Muzychuk and M. Klin, On graphs with three eigenvalues, in preparation.

Problem 11 (BCC15.11). Spectra of K_{s+1}-free graphs. Posed by Stephan Brandt.

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Let λ_1 and λ_n be the greatest and smallest eigenvalues of a graph on n vertices.

Conjecture. $(\lambda_1 + \lambda_n)/n \leq 4/25$ for any regular triangle-free graph on n vertices.

This conjecture would be true if any of the following two old Erdős conjectures holds (see e.g. [2]): Let G be a triangle-free graph on n vertices. Then (a) G contains a set of $\lfloor n/2 \rfloor$ vertices which span at most $n^2/50$ edges, and (b) G can be made bipartite by the omission of at most $n^2/25$ edges.

Problem. Let $\xi(s)$ be the supremum of $(\lambda_1 + \lambda_n)/n$ over the class of regular K_{s+1}-free graphs on n vertices. Determine or estimate $\xi(s)$.

The author [1] can show that

$$0.14 \leq \xi(2) \leq 3 - 2\sqrt{2} = 0.1715\ldots$$
$$(s-2)/s \leq \xi(s) \leq (s-2)/(s-1) \qquad \text{for } s \geq 3.$$

1. S. Brandt, The local density of triangle-free graphs, Preprint 1995.
2. P. Erdős, Problems and results in graph theory and combinatorial analysis, in: Proc. 5th British Combinatorial Conference (Aberdeen, 1975), Utilitas Math., Winnipeg, pp. 169–192.

Problem 12 (BCC15.12). Semiregular automorphism groups. Posed by Dragan Marušič, Mikhail Klin.

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A permutation group is *semiregular* if no non-identity group element fixes a point. It is *regular* if it is transitive and semiregular. A graph is a Cayley graph if and only if its automorphism group contains a regular subgroup. It is known that there are vertex-transitive graphs which are not Cayley graphs (the smallest such being the Petersen graph.)

Question. Is there a vertex-transitive graph whose automorphism group contains no non-identity semiregular subgroup?

More generally, is there a 2-closed transitive permutation group containing no non-identity semiregular subgroup? (A permutation group is 2-*closed* if it is the automorphism group, preserving the colours, of some edge-coloured directed graph.)

Editor's Note. Not every transitive permutation group contains a non-identity semiregular subgroup: the smallest counterexample has degree 12 (see [1]).

1. D. Marušič, On vertex-symmetric digraphs, *Discrete Math.* **36** (1981), 69–81.

Problem 13 (BCC15.13). A distance-regular graph. Posed by Leonard Soicher.

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Let C_{22} be the code obtained by puncturing the non-extended binary Golay code C_{23} in one coordinate. Then C_{22} is a [22, 12, 6] code with automorphism group $M_{22}:2$. Let M be the set of words of minimum non-zero weight in C_{22}, so that $|M| = 77$.

Let V be the set of pairs $\{v_1, v_2\}$ of words of C_{22} which satisfy $\mathrm{wt}(v_1) = \mathrm{wt}(v_2)$ and $v_1 + v_2 = \mathbf{1}$, where $\mathbf{1}$ is the all-1 word. Then $|V| = 672$. For $v = \{v_1, v_2\} \in V$, define

$$M(v) = \{m \in M | \mathrm{wt}(v_1 + m) = \mathrm{wt}(v_2 + m)\}.$$

Then $|M(v)| = 55$ for all $v \in V$.

Define a graph Γ with vertex set V, in which $v \sim w$ if and only if $|M(v) \cap M(w)| = 43$. Then Γ is a distance-regular, but not distance-transitive graph. Moreover, the distance function in Γ is given by

$$d_\Gamma(v, w) = \tfrac{1}{4}(47 - |M(v) \cap M(w)|)$$

for $v, w \in V$, $v \neq w$.

These facts have been proved using the package **GRAPE** (see [1]).

Problem. (a) Prove this by hand, to help understand Γ.

(b) Can a similar construction be applied to other codes with even length and minimum weight, to construct other distance-regular graphs?

1. L. H. Soicher, Yet another distance-regular graph related to a Golay code, *Electronic J. Combinatorics* **2** (1995), #N1.

Problem 14 (BCC15.14). Pasch configurations in 3-hypergraphs. Posed by G. B. Khosrovshahi.

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Let $X = \{1, 2, \ldots, v\}$. Denote the set of all 3-subsets of X by $P_3(X)$. Show that for $v \geq 6$, any $(\binom{v}{2}+1)$-subset of $P_3(X)$ must contain a *Pasch configuration*, that is, $\{abc, axy, bxz, cyz\}$ for some $a, b, c, x, y, z \in X$.

Editor's Note. A. Blokhuis (personal communication) has constructed hypergraphs with more than $\binom{v}{2}+1$ edges with no Pasch configuration. The question should be modified to read: how many edges can a hypergraph with no Pasch configuration have?

Problem 15 (BCC15.15). Critical sets in Latin squares. Posed by Ebad Mahmoodian.

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A *critical set* in an $n \times n$ array with entries from the set $\{1, \ldots, n\}$ is a set S of the positions of the array with the property that the entries in the positions of S have a unique extension to a Latin square of order n.

Problem. Show that any critical set in a Latin square of order n has cardinality at least $\lfloor n^2/4 \rfloor$.

Problem 16 (BCC15.16). Loops with conditions on area. Posed by Alain Valette.

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A *loop* γ is a closed trajectory in the square lattice. Its *algebraic area* is $A(\gamma) = \oint_\gamma x\mathrm{d}y$. Let $N(2k; \mathcal{P})$ be the number of loops based at 0, of length $2k$, which satisfy property $\mathcal{P}$. So, for example, $N(2k; \emptyset) = \binom{2k}{k}^2$.

Problem. Find either formulae or asymptotics for $N(2k; A = l)$ and $N(2k; A \equiv l \pmod q)$, for given l, q.

For example, $\lim_{k\to\infty} N(2k; A = 0)^{1/2k} = 4$. Also, if $n(2k, l, q)$ denotes $N(2k; A \equiv l \pmod q)$, then it is known that

$\lim_{k\to\infty}(n(2k; 0, 2) - n(2k; 1, 2))^{1/2k} = 2\sqrt{2}$,
$\lim_{k\to\infty}(n(2k; 0, 3) - n(2k; 1, 3))^{1/2k} = 1 + \sqrt{3}$,
$\lim_{k\to\infty}(n(2k; 0, 4) - n(2k; 2, 4))^{1/2k} = 2\sqrt{2}$.

This problem (secretly) deals with the walk generating function of the discrete Heisenberg group in its 2-generator presentation.

Problem 17 (BCC15.17). Proof of an identity. Posed by Richard Lewis.

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For complex numbers $z \neq 0$, $|w| < 1$, set $[z; w] = \prod_{n=1}^{\infty}(1 - zw^{n-1})(1 - z^{-1}w^n)$. It can be shown, using Cauchy's theorem, that for any non-zero complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $a_1 \ldots a_n = b_1 \ldots b_n$, and any q with $|q| < 1$,

$$\sum_{r=1}^{n} \frac{[a_1 b_r^{-1}; q][a_2 b_r^{-1}; q] \ldots [a_n b_r^{-1}; q]}{[b_1 b_r^{-1}; q][b_2 b_r^{-1}; q] \ldots \hat{} \ldots [b_n b_r^{-1}; q]} = 0,$$

where the $\hat{}$ means to omit the term $[b_r b_r^{-1}; q]$.

Problem. Find a combinatorial (bijective) proof of this inequality.

Problem 18 (BCC15.18). Counting classes of graphs. Posed by Peter Cameron.

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Find good asymptotic estimates for the numbers of

(a) line graphs,
(b) line graphs of bipartite graphs,
(c) comparability graphs of 2-dimensional posets

on n vertices? (The last class of graphs are defined as follows: Take a permutation π of $\{1, \ldots, n\}$, and join i to j whenever $(i - j)(i\pi - j\pi) > 0$.)

Problem 19 (BCC15.19). Antichains in products of chains. Posed by Jonathan D. Farley.

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Let $\theta(P)$ be the set of antichains of the poset P, and let $\underline{n}$ be the n-element chain. Dedekind's problem [2] asks for the value of $|\theta(\underline{2}^n)|$. It is easy to show that $|\theta(\underline{n})| = n + 1$ and $|\theta(\underline{m} \times \underline{n})| = \binom{m+n}{m}$. MacMahon, Stanley [3], [4], Berman and Köhler [1] showed that

$$|\theta(\underline{k} \times \underline{m} \times \underline{n})| = \prod_{j=0}^{k-1} \binom{m+n+j}{m} \Big/ \binom{m+j}{m}.$$

(Despite appearances, this function is symmetric!)

Problem. What is $|\theta(\underline{j} \times \underline{k} \times \underline{m} \times \underline{n})|$?

1. J. Berman and P. Kohler, Cardinalities of Finite Distributive Lattices, *Mitt. Math. Sem. Giessen* **121** (1976), 103–124.

2. D. J. Kleitman and G. Markovsky, On Dedekind's problem: the number of isotone boolean functions, II, *Trans. Amer. Math. Soc.* **213** (1975), 373–390.

3. R. P. Stanley, Ordered Structures and Partitions, *Mem. Amer. Math. Soc.* **119** (1972).

4. R. P. Stanley, *Enumerative Combinatorics, I*, Wadsworth & Brooks, Monterey, 1986.

Problem 20 (BCC15.20). Cycles of a permutation. Posed by Peter Cameron.

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As an example of a "typical" automorphism of the space of periodic integrable functions (acting on Fourier coefficients), W. Rudin [1] considered the permutation of the integers defined by

$$3n \mapsto 2n, \qquad 3n+1 \mapsto 4n+1, \qquad 3n-1 \mapsto 4n-1.$$

Problem. Describe the cycles of this permutation. In particular, does it have only finitely many finite cycles?

1. W. Rudin, The automorphisms and the endomorphisms of the group algebra of the unit circle, *Acta Math.* **95** (1956), 39–56.

Problem 21 (BCC15.21). Combinatorics and control theory. Posed by Holger Schellwat.

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In place of the Laplace transform, which is used to model continuous time control systems, in discrete control the $\mathcal{Z}$-transform is the basic tool. For a function $f : \mathbb{Z} \to \mathbb{C}$, its $\mathcal{Z}$–transform is defined by $\mathcal{Z}(f)(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$. One problem, for example, is the stability of an inert linear controller and is determined by the loci of poles of a fraction of polynomials in $\mathbb{R}[x]$, constituting the $\mathcal{Z}$-transform of the transfer function [1]. On the other hand, the method of generating functions [2] is used widely in combinatorics to solve enumeration problems. If $(a_i : i \in \mathbb{N})$ is a sequence of numbers, for instance counting the number of distinct combinatorial objects of a certain kind, its associated ordinary generating function is the formal power series $\sum_{n=0}^{\infty} a_n z^n$. But up to the sign of the exponent, this is the defining sum for the $\mathcal{Z}$-transform of the sequence, viewed as a function. Thus it seems natural to explore the implications of this correspondence. Is it even possible to use it to translate problems in control theory into problems in combinatorics and/or vice versa? Could representation theory help to establish such a correspondence?

1. K. J. Åström, B. Wittenmark, *Computer Control Systems*, Prentice Hall, Englewood Cliffs, NJ, 1990
2. H. S. Wilf, *generatingfunctionology*, Academic Press, San Diego, 1994.