Problems from the Seventeenth British Combinatorial Conference

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Abstract

Most of these problems were presented at the 17th British Combinatorial Conference in Canterbury, 12–16 July 1999. I have added some problems given to me after the sessions.

This is the first official release: BCC numbers will now be stable and may be quoted.

Please address comments to the proposers, with copy to the editor.

Problem 1 (BCC17.1) Drawing configurations in the plane.
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The projective plane $PG(2, 2)$ has a familiar drawing in the euclidean plane with six of its seven lines drawn as Euclidean straight lines. What is the maximum number of straight lines in a Euclidean drawing of $PG(2, 3)$? (A drawing with eight straight lines is known.)

Problem 2 (BCC17.2) Nested BIBDs. Proposed by D. A. Preece and R. A. Bailey.

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Let Δ be a balanced incomplete block design (BIBD) on v letters, having b blocks of size k. Let Γ be a BIBD on the same v letters having bs blocks of size k/s (with $s > 1$), its blocks obtained by splitting the blocks of Δ into s subblocks. Then Γ is *nested* in Δ.

(a) Find examples with $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma) = 1$.

(b) Do almost all pairs of nested BIBDs satisfy this?

Problem 3 (BCC17.3) Intersecting families and $S(4, 7, 23)$.
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The family $\mathcal{F}$ of 7-subsets of a 23-set has the property that any two members of $\mathcal{F}$ intersect in 1 or 3 elements. (Hence $|\mathcal{F}| \leq 253$.) How large must $|\mathcal{F}|$ be to guarantee that $\mathcal{F}$ can be embedded in $S(4, 7, 23)$?

Problem 4 (BCC17.4) An extremal problem related to biplanes.
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Given n, what is the smallest m such that there exist m subsets (called blocks) of the point set $\{1, \ldots, n\}$ such that

(a) any two points lie in *at least* two blocks,

(b) any two blocks meet in *at most* two points?

Remark: It is known that $n \le m \le (2+o(1))n$; the lower bound comes from a simple counting argument, and the upper bound is obtained by taking blocks to be the translates of D and $-D$, where D is a planar difference set in C_n, with $n = q^2 + q + 1$.

Problem 5 (BCC17.5) Some 1-factorizations. Proposed by Chris Rodger.

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Let G be the multigraph whose vertex set is $\mathbb{Z}_{2n}$, the integers modulo $2n$, in which i and j are joined by two edges if $j = i + n$ and one edge otherwise. Give an easy way to find a 1-factorization of G in which each of the $2n$ 1-factors contains one of the doubled edges $\{i, i+n\}$.

This 1-factorization is equivalent to a symmetric Latin square with holes of size 2, so can be constructed reasonably easily using design-theoretic techniques.

Editor's note: There is a simple solution when n is odd. Take the vertices to be those of a regular $2n$-gon, the doubled edges being the long diagonals. Now let one 1-factor containing a long diagonal $\{i, i+n\}$ contain all edges and short diagonals parallel to $\{i, i+n\}$, and the other contain all short diagonals perpendicular to $\{i, i+n\}$.

A solution for $n \equiv 0 \pmod 4$ is given by Bailey and Monod [1]; but this perhaps does not qualify as "easy".

Problem 6 (BCC17.6) Unions of random matchings. Proposed by Nick Wormald.

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Choose k perfect matchings of the complete graph K_n (for n even) at random. Denote them by $M_1, \ldots, M_k$. Let $p_k(n)$ denote the probability that for every $i \neq j$, the union of M_i and M_j forms a Hamilton cycle. How does this probability behave as $n \to \infty$ with k fixed? In particular, is it true that

$$p_k(n) \sim (p_2(n))^{\binom{k}{2}}?$$

Remark: The answer to the last question is known to be "yes" for $k = 3$ (shown in joint work of the proposer and J. H. Kim [9]).

Problem 7 (BCC17.7) Edge-colouring nearly complete graphs.
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Let G be a graph obtained from K_n, where $n = 2t + 1$, by deleting any t edges. Let α be a proper χ'-edge-colouring of G, where $\chi' = \chi'(G) = n - 1$.

Problems: (a) Does there exist a subgraph of G having $n - 1$ edges and maximum degree $\Delta = 2$, whose edges all have different colours in the colouring α?

(b) For every edge $e \in E(G)$, does there exist a subgraph of G containing e which has $n - 2$ edges and has maximum degree $\Delta = 2$, whose edges all have different colours in the colouring α?

Problem 8 (BCC17.8) Caterpillar-arboricity of planar graphs.
Proposed by Y. Roditty.
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The *arboricity* $a(G)$ of a graph G is defined to be the smallest number of forests containing all the edges of G. In a similar way, the *linear arboricity* $la(G)$ (resp., *star arboricity* $sa(G)$, *caterpillar arboricity* $ca(G)$) of G is the smallest number of forests containing all edges of G such that each component of each forest is a path (resp., a star, a caterpillar). (A *caterpillar* is a tree with the property that removal of all the end vertices and the edges containing them yields a path.)

Nash-Williams [13] proved that any planar graph G satisfies $a(G) \leq 3$. Hakimi, Mitcham and Schmeichel [7] showed that a planar graph satisfies $sa(G) \leq 5$.

Conjecture: A planar graph G satisfies $ca(G) \leq 4$.

Problem 9 (BCC17.9) Binding functions for graphs. Proposed by Ingo Schiermeyer and Bert Randerath.

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As introduced by Gyárfás [6], a family $\mathcal{G}$ of graphs is called χ-*bound* with χ-*binding function* f if $\chi(G') \leq f(\omega(G'))$ holds whenever G' is an induced subgraph of $G \in \mathcal{G}$. (Here as usual $\omega(G')$ and $\chi(G')$ are the clique number and chromatic number of G'.)

Let $\mathcal{G}^I(3,4)$ denote the class of graphs whose induced cycles have length 3 or 4 only.

Problem: Determine a χ-binding function for $\mathcal{G}^I(3,4)$.

Remark: The authors [14] have shown that this class does not have a linear χ-binding function.

Problem 10 (BCC17.10) Group analogues of graph problems. Proposed by Frank Harary.

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Find and solve problems about finite groups (or finite abelian groups) motivated by results in graph theory.

For example, Harary and Hawthorn [8] found the minimum number of elements in a set $S \subseteq A \setminus \{0\}$ such that $A \setminus S$ has no subgroup isomorphic to B, where A and B are finite abelian groups; this is an analogue of Turán's Theorem for graphs.

Problem 11 (BCC17.11) Neighbourhood-symmetric graphs.
Proposed by Dalibor Fronček.

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The *neighbourhood* of a vertex x in a graph G, denoted $N_G(x)$, is the *subgraph* of G induced on the set of all neighbours of x. We say that G has *constant neighbourhood* H if $N_G(x) \cong H$ for all $x \in V(G)$. We say that G is a *neighbourhood-symmetric graph* (or NSG) if $N_G(x) \cong N_{\overline{G}}(x) \cong H$, for some H, and all $x \in V(G)$. Clearly any vertex-transitive self-complementary graph is a NSG.

Problem: Construct a counterexample to the converse assertion; that is, find a NSG which fails to be vertex-transitive and self-complentary.

Problem 12 (BCC17.12) Semiregular automorphisms. Proposed by Peter Cameron and John Sheehan.

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Marušič and Scapellato [12] proved that a vertex-transitive connected cubic simple graph has a non-trivial semiregular automorphism (one for which all cycles have the same length). Is it true that there exists such an automorphism having order at least $f(n)$, where n is the number of vertices and f is a function for which $f(n) \to \infty$ as $n \to \infty$? Easy examples show that $f(n)$ cannot exceed $O(n^{1/3})$.

Problem 13 (BCC17.13) Regular graphs admitting a given group.
Proposed by Peter J. Cameron.
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It is known (see [2]) that, for any finite group Γ, there exists a rational number $a(\Gamma) \in [0, 1]$ such that, if G denotes a random graph on the vertex set $\{1, \ldots, n\}$ (with all graphs equally likely), then

$$\mathrm{Prob}(\mathrm{Aut}(G) = \Gamma \mid |\, \mathrm{Aut}(G) \geq \Gamma) \to a(\Gamma) \text{ as } n \to \infty.$$

Does a similar result hold for other random graph models, in particular for random regular graphs of degree $d > 2$ (as described by Nick Wormald [15] at the conference)?

Problem 14 (BCC17.14) Symmetry groups of boolean functions.
Proposed by Andrzej Kisielewicz.
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Let f be a *k-valued boolean function* of n variables, that is, a function from $\{0,1\}^n$ to $\{0,1,\ldots,k-1\}$. If $k=2$, we simply call f a *boolean function.* The *symmetry group* $G(f)$ of f is the group of all permutations $\sigma \in S_n$ such that

$$f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = f(x_1,\ldots,x_n)$$

for all $(x_1,\ldots,x_n) \in \{0,1\}^n$.

We say that a subgroup G of S_n is *k-representable* if $G = G(f)$ for some k-valued boolean function f. Clote and Kranakis [4] proved that, if G is k-representable for some $k \geq 2$, then G is 2-representable. However, the Klein group

$$V_4 = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \leq S_4$$

is a counterexample: it is 3-representable but not 2-representable. The proof in [4] has a gap, which does not seem to be fixable. For more information see [10].

Problem: Are there any other counterexamples?

Problem 15 (BCC17.15) Non-zero-sum sequences. Proposed by J. Schönheim.

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Let $a_i \in \mathbb{Z}_{2m+1}$ (the integers modulo $2m+1$) for $i = 1,\ldots,t$. We say that the sequence $a = (a_1,\ldots,a_t)$ has the *k-NZS property* if no subsequence of k or fewer consecutive terms has sum zero (mod $2m+1$).

In these problems, $a = (a_1,\ldots,a_m)$ denotes a permutation of $(1,\ldots,m)$, and $\epsilon = (\epsilon_1,\ldots,\epsilon_m)$ a sequence of signs ($\epsilon_i = \pm 1$ for all i).

It is known that there exist such sequences a, ϵ such that the sequence $(\epsilon_1 a_1,\ldots,\epsilon_m a_m)$ has sum zero (mod $2m+1$) but has the $(m-1)$-NZS property.

Problem: Given $b_1,\ldots,b_m$, show that there exist $c_1,\ldots,c_m$ such that

(a) $(c_1,\ldots,c_m)$ is a permutation of $(\epsilon'_1 b_1,\ldots,\epsilon'_m b_m)$ for some choice of signs $\epsilon' = (\epsilon'_1.\ldots,\epsilon'_m)$;

(b) each row in the array

$$\begin{array}{ccccc} b_1 & b_2 & \dots & b_{m-1} & c_m \\ b_2 & b_3 & \dots & b_m & c_1 \\ b_3 & b_4 & \dots & b_1 & c_2 \\ \vdots & & & & \\ b_m & b_1 & \dots & b_{m-2} & c_{m-1} \end{array}$$

is m-NZS.

Problem: Prove that given ϵ, there exists a such that the sequence ϵa has the m-NZS property.

Problem 16 (BCC17.16) Acceptable cross-sections. Proposed by Nigel Martin.

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A *cross-section* of a sequence $(S_1, \dots, S_k)$ of sets of integers is a sequence $(x_1, \dots, x_k)$ with $x_i \in S_i$ for $i = 1, \dots, k$. A cross-section is *acceptable* if for all i, j with $i \neq j$,

$$x_j - x_i \not\equiv j - i \pmod{k}.$$

Fix $p = 2r + 1$, and consider the sets $S_1, \dots, S_p$ given by

$$\begin{aligned} S_i &= \{j : i \leq j \leq r + i - 1\} \quad \text{for} \quad 1 \leq i \leq r, \\ S_{r+1} &= \{j : 1 \leq j \leq r\}, \\ S_{r+i+1} &= \{j : i \leq j \leq r + i\} \quad \text{for} \quad 1 \leq i \leq r - 1, \\ S_{2r+1} &= \{j : r \leq j \leq r - 1\} \end{aligned}$$

Find $p - 2$ acceptable cross-sections for this sequence so that, in aggregate, every number in the range $\{1, \dots, p - 2\}$ occurs $p + 1$ times.

Solutions are known for $p = 2^n \pm 1$, $p = 6 \cdot 2^n - 1$ and finitely many other values.

Problem 17 (BCC17.17) Evaluating inversion numbers. Proposed by Timothy R. Walsh.

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The number $M(n,r)$ of permutations of $\{1,2,\ldots,n\}$ with r inversions is the coefficient of x^r in

$$(1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}).$$

To find a single value of $M(n,r)$ by evaluating $M(n',r')$ for all $n' \leq n$ and $r' \leq r$ takes $O(nr) = O(n^3)$ arithmetic operations.

Problem: Find a non-recursive formula for $M(n,r)$ which can be evaluated in at most the same time.

Donald E. Knuth [11], page 16 gave a simple formula in the case $r \leq n$. The proposer has given a monstrously complicated formula for the general case.

Problem 18 (BCC17.18) Sets of permutations with given minimum distance. Proposed by Wendy Myrvold.

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What is the largest number of permutations of n symbols in a set with the property that any two agree in at most r columns? This number is at most $n!/(n-r-1)!$, with equality if and only if the set is sharply $(r+1)$-transitive.

For further details on this problem, see Section 5 of the survey [3].

Problem 19 (BCC17.19) Covering radius and Tutte polynomial. Proposed by Carrie G. Rutherford, Fuad Shareef.

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Associated with any matrix A over a field F, there is a matroid representable over F (whose elements are the columns of A, and in which dependence is linear dependence) and a linear code (spanned by the rows of F. Greene [5] showed that the weight enumerator of the code is a specialisation of the Tutte polynomial of the matroid.

Do there exist two binary linear codes which have the same Tutte polynomials but different covering radii? (There are codes with the same weight enumerators but different covering radii.)

Editor's Note: At the conference, the following problem was presented:

Problem 20 (BCC17.20) Derangements in the alternating group.
Proposed by Ömer Eceğlioğlu.

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Let d_n be the number of derangements (fixed-point-free permutations) in the symmetric group S_n, and a_n the number of derangements in the alternating group A_n. It is known that

$$a_n = \binom{n}{2} d_{n-2} + (-1)^{n-1}(n-1).$$

Problem: Find a bijective proof of this fact.

The problem is equivalent to the statement that the numbers $d_n^+ = a_n$ and d_n^- of derangements which are even and odd permutations satisfy

$$d_n^+ - d_n^- = (-1)^{n-1}(n-1).$$

This was subsequently solved by Robin Chapman, Department of Mathematics, University of Exeter, North Park Road, Exeter EX4 4QE, U.K., `rjc@maths.ex.ac.uk`. His elegant solution follows.

Let D_n the be set of derangements. We can split this set into $n-1$ equal parts according to where the permutation sends n. We look therefore only at $D'_n = \{\sigma \in D_n : \sigma(n) = n-1\}$. Define a sign-reversing involution on the set $D'_n \setminus \{(1\ 2\ 3\ \ldots\ n-2\ n\ n-1)\}$. This will give us what we want.

For $\sigma \in D'_n$ let $a(\sigma)$ be the least number a with $\sigma(a) < a$. Then $a(\sigma) < n-1$ with the sole exception $\sigma = (1\ 2\ 3\ \ldots\ n-2\ n\ n-1)$. For all $\sigma \in D'_n$ apart from this one permutation, let $f(\sigma) = (a(\sigma)\ n) \circ \sigma$.

It suffices to show that $\tau = f(\sigma)$ satisfies $\tau \in D'_n$ and $a(\tau) = a(\sigma)$.

First of all $\sigma(n) = n-1$ which is not in $\{a, n\}$, so $\tau(n) = n-1$. Suppose $\tau(j) = j$ for some j. Then $\tau(j) \neq \sigma(j)$, so that $\sigma(j) = a$ or n and $\tau(j) = j = n$ or a respectively. But $\sigma(a) < a$, so $j = a$ and $\sigma(j) = n$ is impossible. Also $\tau(n) = n-1 > a$, so that $j = n$ and $\sigma(j) = a$ is impossible. Hence τ lies in D'_n.

If $j < a$, then $\sigma(j) > j$ and $\tau(j)$ equals one of $\sigma(j)$, a and n, all of which exceed j. But $\sigma(a) < a < n$ and so $\tau(a) = \sigma(a) < a$. Hence $a(\tau) = a$.

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