Roots $x_k(y)$ of a formal power series $$f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$$ with applications to graph enumeration and q-series

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LECTURE #4

Higher roots and Hadamard-product formulae

Higher roots: The simplest situation (analytic approach)

- Consider, for concreteness, a power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n\, x^n\, y^{n(n-1)/2}$$

where $\alpha_0 = 1$ and $\alpha_n \in \mathbb{C} \setminus \{0\}$ satisfy $\lim_{n\to\infty} |\alpha_n|^{1/n^2} \le 1$.

- **Examples:**
  - Partial theta function: $\alpha_n = 1$.
  - Deformed exponential function: $\alpha_n = 1/n!$.
  - Rogers–Ramanujan function: $\alpha_n = \dfrac{(1-q)^n}{(q;q)_n}$ with $|q| < 1$.

- For $0 < |y| < 1$, $f(\,\cdot\,, y)$ is a nonpolynomial entire function of order 0.

- It therefore has infinitely many zeros $x_k(y)$ $(k = 0, 1, 2, \ldots)$ and a Hadamard factorization

$$f(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right)$$

where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$.

- For now the $x_k(y)$ have no special ordering, and need not be smooth in y.

- But wherever a root $x_k(y)$ is *simple*, it is analytic in y.

Higher roots at small $|y|$ (analytic approach)

- Let $f(x,y) = \sum_{n=0}^{\infty} \alpha_n\, x^n\, y^{n(n-1)/2}$ with $\alpha_0 = 1$ and all $\alpha_n \neq 0$
- Leading root $x_0(y)$: write $f(x,y) = (\alpha_0 + \alpha_1 x) +$ small corrections
  $\Longrightarrow\ x_0(y) = -(\alpha_0/\alpha_1)\, \xi_0(y)$ where $\xi_0(y) = 1 + O(y)$
- Root $x_k(y)$: write $f(x,y) = (\alpha_k x^k y^{k(k-1)/2} + \alpha_{k+1} x^{k+1} y^{k(k+1)/2}) +$ small corrections
  $\Longrightarrow\ x_k(y) = -y^{-k}\,(\alpha_k/\alpha_{k+1})\, \xi_k(y)$ where $\xi_k(y) = 1 + O(y)$
- Therefore expect to write f as a Hadamard product

$$f(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,+\, x y^k\, \frac{\alpha_{k+1}}{\alpha_k}\, \eta_k(y)\right)$$

  where $\eta_k(y) = 1/\xi_k(y) = 1 + O(y)$ are *analytic* for small $|y|$.
- Can prove this when $|y| \lesssim 0.207875 \Big/ \sup_{n\geq 1} \left| \frac{a_{n-1}\, a_{n+1}}{a_n^2} \right|$.
- Proof uses a Rouché argument:
  - There exist radii $0 = R_0 < R_1 < R_2 < \ldots$ with $\lim_{k\to\infty} R_k = \infty$ (these radii depend on $|y|$) such that when $|x| = R_k$ the series is dominated by the term $n = k$ and hence $f(x,y) \neq 0$.
  - Then Rouché implies that there is precisely one root $x_k(y)$ in the annulus $R_k < |x| < R_{k+1}$.
  - Since $\lim_{k\to\infty} R_k = \infty$, there are no other roots.
  - Hence all the roots are simple and satisfy $|x_0(y)| < |x_1(y)| < \ldots$, and they vary analytically with y.
  - All this holds when $|y|$ lies in the stated disc, and can fail for larger $|y|$.

The general situation for *formal* power series

- Consider a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n(y)\, y^{\lambda_n}\, x^n$$

  where the $\alpha_n(y)$ are formal power series with invertible constant term (coefficients lying in a commutative ring-with-identity-element R) and $(\lambda_n)_{n=0}^{\infty}$ is a *strictly convex* sequence of integers.

- Then I expect to be able to prove the following:
  - There exists a unique formal Laurent series $x_k(y)$ with leading term of order $y^{-(\lambda_{k+1}-\lambda_k)}$ that is a root of $f(x,y)$, and it is of the form

$$x_k(y) \;=\; -\frac{\alpha_k(0)}{\alpha_{k+1}(0)}\, y^{-(\lambda_{k+1}-\lambda_k)}\, \xi_k(y)$$

    where $\xi_k(y)$ is a formal power series with constant term 1.

  - For $m \in \mathbb{Z}$ not of the form $\lambda_{k+1}-\lambda_k$, there does not exist any formal Laurent series with leading term of order y^{-m} that is a root of $f(x,y)$.

  - $f(x,y)$ has a Hadamard factorization

$$f(x,y) \;=\; y^{\lambda_0} \prod_{k=0}^{\infty} \left(1 + x y^{\lambda_{k+1}-\lambda_k}\, \frac{\alpha_{k+1}(0)}{\alpha_k(0)}\, \eta_k(y)\right)$$

    where $\eta_k(y) = 1/\xi_k(y) = 1 + O(y)$.

Computational use of Hadamard factorization

- Consider for simplicity $f(x,y) = \sum_{n=0}^{\infty} \alpha_n\, x^n\, y^{n(n-1)/2}$ with $\alpha_0 = 1$
- Recall from Lecture #2: Define $\{\widetilde{c}_n(y)\}_{n=1}^{\infty}$ by

$$\frac{x\, f'(x,y)}{f(x,y)} \;=\; \sum_{n=1}^{\infty} \widetilde{c}_n(y)\, x^n$$

where $'$ denotes $\partial/\partial x$. Can be computed by the recursion

$$\widetilde{c}_n(y) \;=\; n\alpha_n\, y^{n(n-1)/2} \;-\; \sum_{k=1}^{n-1} \widetilde{c}_k(y)\, \alpha_{n-k}\, y^{(n-k)(n-k-1)/2}$$

- Now insert Hadamard factorization

$$f(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,+\, xy^k\, \frac{\alpha_{k+1}}{\alpha_k}\, \xi_k(y)^{-1}\right)$$

where $\xi_k(y) = 1 + O(y)$.

- Computing logarithmic derivative and taking $[x^n]$ yields

$$(-1)^{n-1}\widetilde{c}_n(y) \;=\; \sum_{k=0}^{\infty} (\alpha_{k+1}/\alpha_k)^n\, y^{kn}\, \xi_k(y)^{-n}$$

- Taking only the $k = 0$ term implies

$$(-1)^{n-1}\widetilde{c}_n(y) \;=\; (\alpha_1/\alpha_0)^n\, \xi_0(y)^{-n} \;+\; O(y^n)\;,$$

which allows us to compute $\xi_0(y)$ through order y^{n-1}
(as we saw in greater generality in Lecture #2).

Computational use of Hadamard factorization (continued)

- But now we can go farther, using

$$(-1)^{n-1}\widetilde{c}_n(y) \;=\; \sum_{k=0}^{\infty} (\alpha_{k+1}/\alpha_k)^n \, y^{kn} \, \xi_k(y)^{-n}$$

  to compute higher $\xi_k(y)$:

  - First use $\widetilde{c}_n(y)$ to compute $\xi_0(y)$ through order y^{n-1}.
  - Then use $\widetilde{c}_{n/2}(y)$ and $\xi_0(y)$ to compute $\xi_1(y)$ through order $y^{n/2-1}$.
  - Then use $\widetilde{c}_{n/4}(y)$, $\xi_0(y)$ and $\xi_1(y)$ to compute $\xi_2(y)$ through order $y^{n/4-1}$.
  - And so forth ...

- This computes $\xi_k(y)$ but only up to $k \approx \log_2 n_{\max}$.
- Can we do better by using the *complete* set of $\{\widetilde{c}_n(y)\}_{n=1}^{n_{\max}}$???
- And how can this calculation be organized most efficiently???
- It is like trying to calculate the eigenvalues of a matrix M given $\operatorname{tr} M^n$ for $n = 1, 2, 3, \ldots$.

The partial theta function $\Theta_0(x,y) = \sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2}$

We have proven that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) \;&=\; 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ &\quad +948y^9 + 2610y^{10} + \ldots + \text{terms through order } y^{6999}\end{aligned}$$

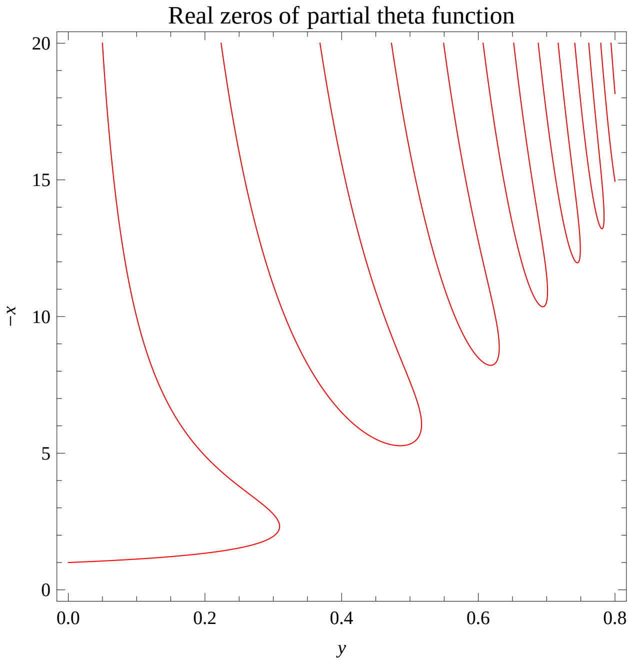
and more strongly that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} \;&=\; 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ &\quad -178y^9 - 490y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

And we have conjectured that $\xi_0(y) \in \mathcal{S}_{-2}$:*

$$\begin{aligned}\xi_0(y)^{-2} \;&=\; 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ &\quad -138y^9 - 386y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

What about higher roots?



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* **Note Added (13 April 2011):** I have now proven this, using an extension of the argument employed in Lecture #3 to prove $\xi_0(y) \in \mathcal{S}_{-1}$.

Higher roots for the partial theta function

- It seems that ξ_1 has the *reverse* behavior:

$$\begin{aligned}\xi_1(y) &= 1 - y^3 - 3y^4 - 9y^5 - 23y^6 - 60y^7 - 153y^8 - 397y^9 \\ &\quad -1043y^{10} - 2796y^{11} - \ldots - \text{terms through order } y^{3499}\end{aligned}$$

  But I don't know how to prove it.

- ξ_2 has *no* fixed sign:

$$\begin{aligned}\xi_2(y) &= 1 + y^6 + 3y^7 + 9y^8 + 22y^9 + 50y^{10} + \ldots + 1467y^{17} \\ &\quad -192y^{18} - \ldots - 2749396y^{28} + 2493265y^{29} + \ldots\end{aligned}$$

  with sign alternations at period ≈ 23. This suggests that the singularity of $\xi_2(y)$ closest to the origin has phase $\approx \pm 2\pi/23$. Indeed one finds a double root of $\Theta_0(x, y)$ at $y \approx 0.452374\, e^{2\pi i/22.8092}$, which is closer to the origin than the real root $y \approx 0.516959$.

- ξ_3 seems to behave like ξ_1:

$$\begin{aligned}\xi_3(y) &= 1 - y^{10} - 3y^{11} - 9y^{12} - 22y^{13} - 51y^{14} - 107y^{15} \\ &\quad -218y^{16} - 420y^{17} - \ldots - \text{terms through order } y^{874}\end{aligned}$$

- ξ_4 again has no fixed sign.
- And so forth: ξ_5 and ξ_7 behave like ξ_1 and ξ_3, while ξ_6 has no fixed sign.
- How to prove this???
- And what is pattern of crossing of roots in the complex y-plane?

Partially explicit formulae for $\xi_k(y)$

- From G.E. Andrews, Ramanujan's "lost" notebook. IX. The partial theta function as an entire function, *Adv. Math.* **191**, 408–422 (2005).

- Translated to my notation, we have

$$\xi_k(y) \;=\; 1 - \frac{A_k(y)}{(y;y)_\infty^3} - \frac{A_k(y)\, B_k(y)}{(y;y)_\infty^6} + O(y^{3(k+1)(k+2)/2})$$

  where

$$A_k(y) \;=\; \sum_{j=k+1}^{\infty} (-1)^j \, y^{j(j+1)/2}$$

$$B_k(y) \;=\; \sum_{j=k+1}^{\infty} (-1)^j \, j \, y^{j(j+1)/2}$$

  each start at order $y^{(k+1)(k+2)/2}$.

- Proof is based on perturbation around the full theta function, whose roots are known from the Jacobi triple product formula.

- Can this method be pushed to higher order? To all orders???[†]

[†] **Note Added (13 April 2011):** In discussion after my lecture, Thomas Prellberg asked whether we might have $(-1)^{k+1} A_k(y)/(y;y)_\infty^3 \succeq 0$ and $(-1)^{k+1} A_k(y)\, B_k(y)/(y;y)_\infty^6 \succeq 0$, and whether this might be used to prove the conjectured behavior $1 - \xi_k(y) \succeq 0$ for k odd. The answer to the first question appears to be yes; indeed, it appears that we have the stronger inequalities $(-1)^{k+1} A_k(y)/(y;y)_\infty \succeq 0$ and $(-1)^{k+1} B_k(y)/(y;y)_\infty \succeq 0$. Perhaps this can be proven using the identities for the partial theta function shown in Lecture #3. The second suggestion is a promising idea, but first we will need to extend this expansion to all orders.

Partially explicit formulae for $\xi_k(y)$, continued

- For the Rogers–Ramanujan function $A(x,y) = \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)}}{(y;y)_n}$, similar results can be found in
  - G.E. Andrews, Ramanujan's "lost" notebook. VIII. The entire Rogers–Ramanujan function, *Adv. Math.* **191**, 393–407 (2005)
  - T. Huber, Hadamard products for generalized Rogers–Ramanujan series, *J. Approx. Theory* **151**, 126–154 (2008)

  But I don't yet understand these papers very well!

Another approach to higher roots

- Let $f(x,y) = \sum_{n=0}^{\infty} \alpha_n\, x^n\, y^{n(n-1)/2}$ with $\alpha_0 = 1$ and all $\alpha_n \neq 0$
- Substitute $x = (\alpha_k/\alpha_{k+1})\, X\, y^{-k}$, and extract prefactors:

$$f_k(X,y) \;=\; \sum_{n=-k}^{\infty} \alpha_n^{(k)}\, X^n\, y^{n(n-1)/2}$$

where $\alpha_n^{(k)} = \dfrac{\alpha_{k+n}}{\alpha_k} \left(\dfrac{\alpha_k}{\alpha_{k+1}}\right)^n$.

- Root $\xi_k(y)$ for f is the *leading* root $\xi_0(y)$ of the *Laurent* series f_k.
- General theory of leading root extends to *bilateral* series

$$f(x,y) \;=\; \sum_{n=-\infty}^{\infty} a_n(y)\, x^n$$

where $a_n(y) \in R[[y]]$ with

  (a) $a_0(0) = a_1(0) = 1$;
  (b) $a_n(0) = 0$ for $n \in \mathbb{Z} \setminus \{0,1\}$; and
  (c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\pm\infty} \nu_n = +\infty$.

- Explicit implicit function formula also extends:
  - Might this help to understand $\xi_k(y)$ in the partial theta function?
  - For deformed exponential function, $\alpha_n^{(k)}$ is a *rational* function of k for each n, so can do calculations *symbolically in k* (see Lecture #1).
- Does method based on exponential formula extend? I'm not sure … If it did, we could push calculations to large k and learn more.

- Finally, bilateral series should also have a Hadamard-product formula: prototype is Jacobi triple product formula for theta function.